On a certain type of warped-twisted product submanifolds

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Abstract: We introduce a certain type of warped-twisted product submanifolds which is called warped-twisted product hemislant submanifolds of the form $f_2 M^\perp \times f_1 M^\theta$ with warping function $f_2$ on $M^\perp$ and twisting function $f_1$, where $M^\perp$ is a totally real and $M^\theta$ is a slant submanifold of a globally conformal Kaehler manifold. We prove that a warped-twisted product hemislant submanifold of a globally conformal Kaehler manifold is a locally doubly warped product. Then we establish a general inequality for doubly warped product mixed geodesic hemislant submanifolds and get some results for such submanifolds by using the equality sign of the general inequality.

Key words: Twisted product, warped product, totally real distribution, slant distribution, hemislant submanifold, globally conformal Kaehler manifold

1. Introduction

One of the ways to get a new Riemannian manifold (or, more generally, pseudo-Riemannian manifold) is to product two pseudo-Riemannian manifolds in the usual sense. The most general manifolds obtained in this way are the doubly twisted product manifolds [16]. In fact, this notion appeared in the literature a long time ago, under the name of conformally seperable spaces [23]. It is well known that the notion of doubly twisted product is a natural generalization of the notion of doubly warped product [11], twisted product [8], warped product [3] and direct product.

In [18], we defined two classes of doubly twisted products under the names of nearly doubly twisted products of type 1 and type 2. In this article, we rename the nearly doubly twisted products of type 1 as warped-twisted products.

One of the popular research areas in differential geometry is the theory of submanifolds. Actually, there are well known classes of submanifolds such as holomorphic (invariant), totally real (antiinvariant) [22], CR-(semiinvariant)[1, 2], slant [5], semislant [15], hemislant or antislant [4, 17]. All classes are determined by the behavior of the almost complex or almost product structure of the ambient manifold. On the other hand, the theory of warped product submanifolds has become a popular research area since Chen [6] studied the warped product CR-submanifolds in Kaehler structures. Afterwards, several classes of warped product submanifolds appeared in the literature. Also, the warped product submanifolds have been studied in different kind of structures. Most of the studies related to this theory can be found in the book [7] and its list of references.

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In this paper, we consider and study a certain type of warped-twisted products which is warped-twisted product hemislant submanifolds in globally conformal Kaehler manifolds. More precisely, we study warped-twisted product submanifolds which have the form \( f_2 M^1 \times f_1 M^\theta \) with warping function \( f_2 \) on \( M^\theta \) and twisting function \( f_1 \), where \( M^1 \) is a totally real and \( M^\theta \) is a slant submanifold of the globally conformal Kaehler manifold. We obtain necessary and sufficient conditions for such submanifolds to be twisted product, base conformal warped product and direct product submanifolds. In the main theorem, we prove that a warped-twisted product hemislant submanifold of a globally conformal Kaehler manifold is locally a doubly warped product. After the main theorem mentioned, we focus on the study of doubly warped product hemislant submanifolds and we get some results for doubly warped product mixed geodesic hemislant submanifolds when the Lee vector field of the globally conformal Kaehler manifold is tangent to them. We say that a warped-twisted product is nontrivial if it is neither twisted nor base conformal warped or direct product.

2. Preliminaries

In this section, we recall the fundamental definitions and notions needed for the further study. Actually, in subsection 2.1, we give the definition of doubly twisted and warped-twisted products and in subsection 2.2, we will recall the definitions of locally and globally conformal Kaehler manifolds. The basic background for submanifolds of Riemannian manifolds will be presented in subsection 2.3.

2.1. Warped-twisted products

Let \( M_1 \) and \( M_2 \) be Riemannian manifolds endowed with metric tensors \( g_1 \) and \( g_2 \), respectively and let \( f_1 \) and \( f_2 \) be positive smooth functions defined on \( M_1 \times M_2 \). Then the doubly twisted product manifold [16] \( f_2 M_1 \times f_1 M_2 \) is the product manifold \( \tilde{M} = M_1 \times M_2 \) equipped with metric \( g \) given by

\[
g = f_2^2 \pi_1^* g_1 + f_1^2 \pi_2^* g_2,
\]

where \( \pi_i : M_1 \times M_2 \to M_i \) is the canonical projection, for \( i = 1, 2 \). Each function \( f_i \) is called a twisting function of the doubly twisted product \( f_2 M_1 \times f_1 M_2 \). If the twisting functions \( f_1 \) and \( f_2 \) depend only on the points of \( M_1 \) and \( M_2 \), respectively, then \( f_2 M_1 \times f_1 M_2 \) becomes a doubly warped product manifold [11] and each function \( f_i \) is called a warping function of the doubly warped product manifold. In this case, if \( f_1 \equiv 1 \) or \( f_2 \equiv 1 \), then we get a warped product [3].

Let \( f_2 M_1 \times f_1 M_2 \) be doubly twisted product manifold. If \( f_1 \equiv 1 \) or \( f_2 \equiv 1 \), then we get a twisted product [8] with the twisting function \( f_2 \) or a twisted product with the twisting function \( f_2 \). In a warped or twisted product case, the notation \( f_2 M_1 \times f_1 M_2 \) is simplified to \( f_2 M_1 \times M_2 \) or \( M_1 \times f_1 M_2 \). In addition, if both \( f_1 \) and \( f_2 \) are constant, then we get a usual or direct product manifold [7].

Let us recall the definition of a warped-twisted product manifold. Let \( (M_1, g_1) \) and \( (M_2, g_2) \) be Riemannian manifolds and let \( f_2 : M_2 \to (0, \infty) \) and \( f_1 : M_1 \times M_2 \to (0, \infty) \) be smooth functions. The warped-twisted product \( f_2 M_1 \times f_1 M_2 \) [18] is the product manifold \( M_1 \times M_2 \) equipped with the metric tensor \( g \) defined by

\[
g = (f_2 \circ \pi_2)^2 \pi_1^* (g_1) + f_1^2 \pi_2^* (g_2). \tag{2.1}
\]

The function \( f_2 \in C^\infty(M_2) \) is called a warping function and the function \( f_1 \in C^\infty(M_1 \times M_2) \) is called a twisting function of \( f_2 M_1 \times f_1 M_2 \). In this case, if the function \( f_1 \) depends only on the points of \( M_2 \), then the warped-
twisted product $f_2 M_1 \times_{f_1} M_2$ becomes a base conformal warped product [9]. We say that a warped-twisted product is nontrivial if it is neither doubly warped product nor warped product or base conformal warped product.

Let $f_2 M_1 \times_{f_1} M_2$ be a warped-twisted product manifold with the Levi-Civita connection $\nabla$ of $g$, given in (2.1). Also, $\nabla^i$ denote by the Levi-Civita connection of $g_i$, for $i \in \{1, 2\}$, respectively. By usual convenience, we denote the set of lifts of vector fields on $M_i$ by $\mathcal{L}(M_i)$ and we use the same notation for a vector field and for its lift. On the other hand, each $\pi_i$ is a positive homothety, so it preserves the Levi-Civita connection. Thus, there is no confusion using the same notation for a connection on $M_i$ and for its pullback via $\pi_i$. Then, the covariant derivative formulas of the warped-twisted product manifold $f_2 M_1 \times_{f_1} M_2$ with warping function $f_2 \in C^\infty(M_2)$ and twisting function $f_1$ are given by

$$\nabla_X Y = \nabla^1_X Y - g(X, Y)\nabla \ln(f_2 \circ \pi_2),$$  

(2.2)

$$\nabla_Y X = \nabla_X Y = V(\ln(f_2 \circ \pi_2))X + X(\ln f_1)V,$$  

(2.3)

$$\nabla_U V = \nabla^1_U V + U(\ln f_1)V + V(\ln f_1)U - g(U, V)\nabla \ln f_1,$$  

(2.4)

for any $X, Y \in \mathcal{L}(M_1)$ and $U, V \in \mathcal{L}(M_2)$. These formulas immediately come from Lemma 2.1 of [12] with $X(\ln(f_2 \circ \pi_2)) = Y(\ln(f_2 \circ \pi_2)) = 0$.

**Remark 2.1** Until Section 5, we will use the same symbol for the warping function $f_2$ and its pullback $f_2 \circ \pi_2$, i.e. we will put $f_2 = f_2 \circ \pi_2$.

### 2.2. Locally and globally conformal Kaehler manifolds

Let $(\tilde{M}, J, g)$ be a Hermitian manifold of dimension $2m$. Then it is called a locally conformal Kaehler manifold (briefly l.c.K. manifold) [10], if each point of $p \in \tilde{M}$ has an open neighborhood $U$ with smooth function $\sigma : U \to R$ such that $\tilde{g} = e^{-\sigma} g|_U$ is a Kaehler metric on $U$. If one choose $U = M$, then $(\tilde{M}, J, g)$ is called a globally conformal Kaehler manifold (briefly g.c.K. manifold).

**Theorem 2.2** ([10]) Let $(\tilde{M}, J, g)$ be a Hermitian manifold and let $\Omega$ be a $2-$ form defined by $\Omega(\tilde{X}, \tilde{Y}) = g(\tilde{X}, J\tilde{Y})$ for all vector fields $\tilde{X}$ and $\tilde{Y}$ in $\tilde{M}$. Then $(\tilde{M}, J, g)$ is a l.c.K. manifold if and only if there exists a globally defined $1-$ form $\omega$ such that

$$d\Omega = \omega \wedge \Omega \quad \text{and} \quad d\omega = 0.$$  

(2.5)

The closed $1-$ form $\omega$ is called the Lee form of the l.c.K. manifold $(\tilde{M}, J, g)$. In addition, the manifold $(\tilde{M}, J, g)$ is g.c.K., if its Lee form $\omega$ is also exact. In this case, we have $\omega = d\sigma$ [21]. The Lee vector field $B$ is defined by

$$\omega(\tilde{X}) = g(B, \tilde{X}),$$  

(2.6)

for any vector fields $\tilde{X}$ on $\tilde{M}$. One can see that, the globally conformal Kaehler case is a special case of the locally conformal Kaehler case. We denote by $\nabla$ (resp. $\bar{\nabla}$) the Levi-Civita connection on a g.c.K. manifold $\tilde{M}$...
with respect to $\tilde{g} = e^{-\sigma} g$ (resp. $g$). Then we have [10]

$$\tilde{\nabla}_{\tilde{X}} \tilde{Y} = \nabla_{\tilde{X}} \tilde{Y} - \frac{1}{2} \left\{ \omega(\tilde{X}) \tilde{Y} + \omega(\tilde{Y})\tilde{X} - g(\tilde{X}, \tilde{Y}) B \right\},$$  

(2.7)

for any vector fields $\tilde{X}$ and $\tilde{Y}$ on $\tilde{M}$. The connection $\tilde{\nabla}$ is a torsionless linear connection on $\tilde{M}$ which is called the Weyl connection of $g$. It is easy to see that the Weyl connection $\tilde{\nabla}$ satisfies the condition

$$\tilde{\nabla} J = 0.$$  

(2.8)

**Remark 2.3** Throughout this paper, we denote by $(\tilde{M}, J, \omega, g)$ the g.c.K. manifold with the Lee form $\omega$.

### 2.3. Submanifolds of Riemannian manifolds

Let $M$ be an isometrically immersed submanifold in a Riemannian manifold $(\bar{M}, g)$. Let $\nabla$ is the Levi-Civita connection on $\bar{M}$ with respect to the metric $g$ and let $\nabla$ and $\nabla^\perp$ be the induced, and induced normal connection on $M$, respectively. Then, for all $X, Y \in T M$ and $Z \in T^\perp M$, the Gauss and Weingarten formulas are given respectively by

$$\nabla_X Y = \nabla_X Y + h(X, Y),$$  

(2.9)

$$\nabla_X Z = -A_Z X + \nabla_X^\perp Z,$$  

(2.10)

where $TM$ is the tangent bundle and $T^\perp M$ is the normal bundle of $M$ in $\bar{M}$. Additionally, $h$ is the second fundamental form of $M$ and $A_Z$ is the Weingarten endomorphism associated with $Z$. The second fundamental form $h$ and the shape operator $A$ are related by

$$g(h(X, Y), Z) = g(A_Z X, Y).$$  

(2.11)

The **mean curvature vector field** $H$ of $M$ is given by $H = \frac{1}{m} (\text{trace } h)$, where $\text{dim}(M) = m$. We say that the submanifold $M$ is **totally geodesic** in $\bar{M}$ if $h = 0$, and **minimal** if $H = 0$. The submanifold $M$ is called **totally umbilical** if $h(X, Y) = g(X, Y) H$ for all $X, Y \in TM$.

Let $M$ be any submanifold of a g.c.K. manifold $(\tilde{M}, J, \omega, g)$. Then the Gauss and Weingarten formulas with respect to $\tilde{\nabla}$ are given by

$$\tilde{\nabla}_X Y = \tilde{\nabla}_X Y + \tilde{h}(X, Y),$$  

(2.12)

$$\tilde{\nabla}_X Z = -\tilde{A}_Z X + \tilde{\nabla}_X^\perp Z,$$  

(2.13)

for $X, Y \in TM$ and $Z \in T^\perp M$. Thus, using (2.7), (2.9)–(2.13), we have the following relations

$$\tilde{\nabla}_X Y = \nabla_X Y - \frac{1}{2} \left\{ \omega(X) Y + \omega(Y) X - g(X, Y) B^T \right\},$$  

(2.14)

$$\tilde{A}_Z X = A_Z X + \frac{1}{2} \omega(Z) X,$$  

(2.15)
The function \( \tilde{h}(X, Y) = h(X, Y) + \frac{1}{2} g(X, Y) B^N \) (2.16) where \( X, Y \in TM \) and \( Z \in T^\perp M \), where \( B^T \) and \( B^N \) are the tangential and the normal part to \( M \) of \( B \), respectively.

3. Hemislant submanifolds of a g.c.K. manifold

In this section, we recall some fundamental properties of hemislant submanifolds of a g.c.K. manifold given in [19] and we give some auxiliary results to prove our main theorem.

Let \((\bar{M}, J, g)\) be an almost Hermitian manifold and let \( M \) be a Riemannian manifold isometrically immersed in \( \bar{M} \). A distribution \( D \) on \( M \) is called a slant distribution if for \( V \in D_p \), the angle \( \theta \) between \( JV \) and \( D_p \) is constant, i.e. independent of \( p \in M \) and \( V \in D_p \). The constant angle \( \theta \) is called the slant angle of the slant distribution \( D \). We know that holomorphic and totally real distributions on \( M \) are slant distributions with \( \theta = 0 \) and \( \theta = \frac{\pi}{2} \), respectively. A slant distribution is called proper if it is neither holomorphic nor totally real. A submanifold \( M \) of \( \bar{M} \) is said to be a slant submanifold [5] if the tangent bundle \( TM \) of \( M \) is slant. For examples and more details, (see [5]).

A hemislant submanifold \( M \) [15] of an almost Hermitian manifold \((\bar{M}, J, g)\) is a submanifold such that its tangent bundle \( TM \) admits two orthogonal complementary totally real distribution \( D^\perp \) (\( \forall X \in D^\perp, JX \in T^\perp M \)) and slant distribution \( D^\theta \), i.e. we have

\[ TM = D^\perp \oplus D^\theta. \] (3.1)

We say that a hemislant submanifold \( M \) is proper if \( \text{dim}(D^\perp) \neq 0 \) and \( \theta \neq 0, \frac{\pi}{2} \).

For any \( X \in TM \) we write

\[ JX = PX + FX, \] (3.2)

where \( PX \) is the tangential part of \( JX \), and \( FX \) is the normal part of \( JX \). Similarly, for any \( Z \in T^\perp M \), we put

\[ JZ = tZ + nZ, \] (3.3)

where \( tZ \) is the tangential part of \( JZ \), and \( nZ \) is the normal part of \( JZ \). Then the normal bundle \( T^\perp M \) of \( M \) is decomposed as

\[ T^\perp M = JD^\perp \oplus FD^\theta \oplus \bar{D}, \] (3.4)

where \( \bar{D} \) is the orthogonal complementary distribution of \( JD^\perp \oplus FD^\theta \) in \( T^\perp M \) and it is an invariant subbundle of \( T^\perp M \) with respect to \( J \). For a hemislant submanifold, we have

\[ P^2 V = - \cos^2 \theta V, \] (3.5)

\[ g(PU, PV) = \cos^2 \theta g(U, V) \quad \text{and} \quad g(FU, FV) = \sin^2 \theta g(U, V), \] (3.6)

for \( U, V \in \Gamma(D^\theta) \).
**Lemma 3.1** Let $M$ be a hemislant submanifold of a g.c.K. manifold $(\tilde{M}, J, \omega, g)$. Then we have

$$g(\nabla_X Y, V) = -\sec^2\theta g(A_{JY} PV - A_{FPV} Y - \frac{1}{2}\omega(FPV)Y, X) - \frac{1}{2}\omega(V)g(X, Y),$$ (3.7)

and

$$g(\nabla_U V, X) = \sec^2\theta g(A_{JX} PV - A_{FPV} X + \frac{1}{2}\omega(JX)PV, U) - \frac{1}{2}\omega(X)g(U, V),$$ (3.8)

for $X, Y \in \Gamma(\mathcal{D}^\perp)$ and $U, V \in \Gamma(\mathcal{D}^\theta)$.

**Proof** Let $M$ be a hemislant submanifold of a g.c.K. manifold $(\tilde{M}, J, \omega, g)$ and $X, Y \in \Gamma(\mathcal{D}^\perp)$ and $V \in \Gamma(\mathcal{D}^\theta)$. Since $(\tilde{M}, J, \tilde{g} = e^{-\sigma}g)$ is a Kaehler manifold, by using (2.8), (2.13), (3.2) and (3.3), we have

$$\tilde{g}(\tilde{\nabla}_X Y, V) = \tilde{g}(\tilde{\nabla}_X Y, J V) = \tilde{g}((\tilde{\nabla}_X JY, PV) = \tilde{g}(\tilde{\nabla}_X JY, FV)$$

$$= -\tilde{g}(\tilde{A}_{JY} PV, X) - \tilde{g}(\tilde{\nabla}_X Y, tFV) - \tilde{g}(\tilde{\nabla}_X Y, nFV).$$

Here, using the fact that $tF = -I - P^2$ and $nF = -FP$ (see, the equation (2.16) of [19]), we obtain

$$\tilde{g}(\tilde{\nabla}_X Y, V) = -\tilde{g}(\tilde{A}_{JY} PV, X) + \sin^2\theta\tilde{g}(\tilde{\nabla}_X Y, V) + \tilde{g}(\tilde{\nabla}_X Y, FPV)$$

$$= -\tilde{g}(\tilde{A}_{JY} X, PV) + \tilde{g}(\tilde{A}_{FPV} Y, X) + \sin^2\theta\tilde{g}(\tilde{\nabla}_X Y, V).$$

Hence, we get

$$\tilde{g}(\tilde{\nabla}_X Y, V) = -\sec^2\theta(\tilde{g}(\tilde{A}_{JY} PV - \tilde{A}_{FPV} Y, X)).$$

Now, by using (2.6), (2.14) and (2.15), we derive the conclusion (3.7). The other assertion (3.8) can be obtained by a similar method. $\square$

By using (3.8), we can prove the following result.

**Theorem 3.2** Let $M$ be a hemislant submanifold of a g.c.K. manifold $(\tilde{M}, J, \omega, g)$. Then the slant distribution $\mathcal{D}^\theta$ on $M$ is integrable if and only if

$$g(A_{JX} PV - A_{FPV} X, U) = g(A_{JX} PU - A_{FPV} X, V) - \omega(JX)g(PV, U),$$ (3.9)

for $X \in \Gamma(\mathcal{D}^\perp)$ and $U, V \in \Gamma(\mathcal{D}^\theta)$.

**Remark 3.3** The integrability condition of the distribution $\mathcal{D}^\theta$ was given in Proposition 3.2 of [19] in a different way.

**Theorem 3.4** Let $M$ be a proper hemislant submanifold of a g.c.K. manifold $(\tilde{M}, J, \omega, g)$. Then the totally real distribution $\mathcal{D}^\perp$ is always integrable.

**Proof** The proof is very similar to the proof of Theorem 3.1 of [19]. So, we omit it. $\square$

**Remark 3.5** Throughout this paper, for a hemislant submanifold $M$ of a g.c.K. manifold $(\tilde{M}, J, \omega, g)$, we write $B^M = B^\perp + B^\theta$, where $B^T$ (resp. $B^\theta$) is tangential part of $B^M$ to $\mathcal{D}^\perp$ (resp. $\mathcal{D}^\theta$).
4. Warped-twisted product hemislant submanifolds of a g.c.K. manifold

In this section, we give a characterization for a warped-twisted product hemislant submanifold in the form $f_2 M^\perp \times f_1 M^\theta$ with warping function $f_2 \in C^\infty(M^\theta)$ and twisting function $f_1$ of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. We first give an (nontrivial) example of such a submanifold.

**Example 4.1** Let $(z_1, ..., z_6)$ be natural coordinates of the six-dimensional Euclidean space $\mathbb{R}^6$ and let $\bar{R}^6 = \{(z_1, ..., z_6) \in \mathbb{R}^6 : z_1 \neq 0, z_2 \neq 0, z_5 \neq 0\}$. Then $(\bar{R}^6, J, g_0)$ is a Kaehler manifold with usual Kaehler structure $(\bar{J}, g_0)$. Now, we consider the Riemannian metric $g = e^\sigma g_0$ conformal to Kaehler metric $g_0$ on $\bar{R}^6$, where $e^\sigma = \frac{(z_2^2 - z_3^2)^2}{16}(z_5)^2$. So, $(\bar{R}^6, J, g)$ is a g.c.K. manifold. Let $M$ be a submanifold given by

$$z_1 = u - v, \quad z_2 = u + v, \quad z_3 = v, \quad z_4 = 0, \quad z_5 = x, \quad z_6 = 0,$$

where $x, u, v \neq 0$. Then, the local frame field of the tangent bundle $TM$ of $M$ is given by

$$X = \partial_5, \quad U = \frac{1}{\sqrt{2}}(\partial_1 + \partial_2), \quad V = \frac{1}{\sqrt{3}}(-\partial_1 + \partial_2 + \partial_3),$$

where $\partial_i = \frac{\partial}{\partial z_i}$ for $i \in \{1, 2, ..., 6\}$. Then $D^\perp = \text{span}\{X\}$ is a totally real and $D^\theta = \text{span}\{U, V\}$ is a (proper) slant distribution with the slant angle $\theta = \cos^{-1}(\frac{2}{\sqrt{6}})$. Thus, $M$ is a proper hemislant submanifold with the slant angle $\theta = \cos^{-1}(\frac{2}{\sqrt{6}})$.

By direct calculations, we see the distributions $D^\perp$ and $D^\theta$ are integrable. Let us denote the integral submanifolds of $D^\perp$ and $D^\theta$ by $M^\perp$ and $M^\theta$, respectively. Let $g_\perp$ and $g_\theta$ be the induced metrics from the Kaehler metric $g_0$ on $M^\perp$ and $M^\theta$, respectively. We choose the conformal Riemann metric $\bar{g}_\perp = x^2 g_\perp$ and $\bar{g}_\theta = u^2 g_\theta$ on $M^\perp$ and $M^\theta$, respectively.

Since $x = z_5$ and $uv = \frac{(z_2^2 - z_3^2)}{4}$ on $M$, the induced metric of $M$ from the conformal Kaehler metric $g$ is

$$ds^2 = (uv)^2 x^2 dx^2 + (uv)^2 x^2 (du^2 + dv^2) = (uv)^2 x^2 g_\perp + (uv)^2 x^2 g_\theta = (uv)^2 \bar{g}_\perp + (uv)^2 \bar{g}_\theta.$$

Thus, $M$ is a warped-twisted product of $(M^\perp, \bar{g}_\perp)$ and $(M^\theta, \bar{g}_\theta)$. So, $f_2 M^\perp \times f_1 M^\theta$ is a (nontrivial) warped-twisted product proper hemislant submanifold of the g.c.K. manifold $(\bar{R}^6, J, g)$ with warping function $f_2 = uv$ and twisting function $f_1 = vx$. Moreover, the Lee form of $(\bar{R}^6, J, g)$ is

$$\omega = 2\left(\frac{1}{x} dx + \frac{1}{u} du + \frac{1}{v} dv\right).$$

Consequently, the Lee vector field is

$$B = \frac{2}{(uv)^2 x^2} \left(\frac{1}{x} \frac{\partial}{\partial x} + \frac{1}{u} \frac{\partial}{\partial u} + \frac{1}{v} \frac{\partial}{\partial u}\right)$$

which is tangent to $M$. 

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Lemma 4.2 Let $M = f_2 M^1 \times_{f_1} M^\theta$ be a warped-twisted product hemislant submanifold with warping function $f_2 \in C^\infty(M^\theta)$ and twisting function $f_1$ of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then, for all $X \in \mathcal{L}(M^\perp)$, we have
\[
\omega(X) = \frac{2}{3} X(\ln f_1).
\] (4.1)

Proof Let $M = f_2 M^1 \times_{f_1} M^\theta$ be a warped-twisted product hemislant submanifold with warping function $f_2 \in C^\infty(M^\theta)$ and twisting function $f_1$ of a g.c.K. manifold $(\bar{M}, J, \omega, g)$ and $U, V \in \mathcal{L}(M^\theta)$ and $X \in \mathcal{L}(M^\perp)$. Then, we have
\[
3d\Omega(X, U, V) = X\Omega(U, V) + U\Omega(V, X) + V\Omega(X, U) - \Omega([X, U], V) - \Omega([U, V], X) - \Omega([V, X], U) = Xg(U, PV),
\]
since $[X, V] = [X, U] = 0$ from (2.3) and $[U, V] = \nabla^G_\theta V - \nabla^G_\theta U \in \Gamma(TM^\theta)$ from (2.4). After some calculation in view of (2.3), we obtain
\[
3d\Omega(X, U, V) = 2X(\ln f_1)g(U, PV),
\] (4.2)
since $PV \in \mathcal{L}(M^\theta)$. On the other hand, we have
\[
d\Omega(X, U, V) = \omega \wedge \Omega(X, U, V) = \omega(X)\Omega(U, V) + \omega(U)\Omega(V, X) + \omega(V)\Omega(X, U) = \omega(X)g(U, PV),
\]
from (2.5). Namely,
\[
d\Omega(X, U, V) = \omega(X)g(U, PV).
\] (4.3)

Thus, the assertion follows from (4.2) and (4.3). \hfill \Box

By Lemma 4.2, we immediately have the following result.

Theorem 4.3 Let $M = f_2 M^1 \times_{f_1} M^\theta$ be a warped-twisted product hemislant submanifold with warping function $f_2 \in C^\infty(M^\theta)$ and twisting function $f_1$ of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then $M$ is a base conformal warped product submanifold in the form $f_2 M^1 \times_{f_1} M^\theta$ if and only if the Lee vector field $B$ is normal to $M^\perp$.

Proof Let $M = f_2 M^1 \times_{f_1} M^\theta$ be a warped-twisted product hemislant submanifold with warping function $f_2 \in C^\infty(M^\theta)$ and twisting function $f_1$ of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. If $M$ is a base conformal warped product submanifold in the form $f_2 M^1 \times_{f_1} M^\theta$, then for any $X \in \mathcal{L}(M^\perp)$, $X(\ln f_1) = 0$, since $f_1$ depends only on the points of $M^\theta$. From (4.1), we find $g(B, X) = 0$. So, the Lee vector field $B$ is normal to $M^\perp$.

Conversely, if the Lee vector field $B$ is normal to $M^\perp$, we have $g(B, X) = 0$. Then, we get $X(\ln f_1) = 0$ for any $X \in \mathcal{L}(M^\perp)$ from (4.1). So $f_1$ depends only on the points of $M^\theta$. Then the induced metric tensor $g_M$ of $M$ has the form $g_M = f_2^2 g_\perp \oplus g_\theta$, where $f_2$ is warping function and $g_\theta = f_1^2 g_\theta$. Thus, $M = f_2 M^1 \times_{f_1} M^\theta$ is a base conformal warped product. \hfill \Box

Lemma 4.4 Let $M = f_2 M^1 \times_{f_1} M^\theta$ be a warped-twisted product hemislant submanifold with warping function $f_2 \in C^\infty(M^\theta)$ and twisting function $f_1$ of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then, for all $V \in \mathcal{L}(M^\theta)$, we have
\[
\omega(V) = \frac{2}{3} V(\ln f_2).
\] (4.4)
Proof Let $M = f_2 \, M^\perp \times f_1, \, M^\theta$ be a warped-twisted product hemislant submanifold with the warping function $f_2 \in C^\infty(M^\theta)$ and the twisting function $f_1$ of a g.c.K. manifold $(M, J, \omega, g)$. Then we have $d\omega = 0$, since $\omega = d\sigma$. Hence, using the exterior differentiation formula, we obtain $0 = d\omega(V, X) = V\omega(X) - X\omega(V) - \omega([V, X])$ for $V \in \mathcal{L}(M^\theta)$ and $X \in \mathcal{L}(M^\perp)$. Hence, it follows that

$$V\omega(X) = X\omega(V), \quad (4.5)$$

since $[V, X] = 0$. Here, using (4.1), (2.3) and (2.6), we have

$$\frac{3}{2}V\omega(X) = V[X(\ln f_1)] = V[g(X, \nabla \ln f_1)] = g(\nabla V X, \nabla \ln f_1) + g(X, \nabla \nabla \ln f_1) = g \left( V(\ln f_2)X + X(\ln f_1)V, \nabla \ln f_1 \right) + g(\nabla V(\ln f_2)X, \nabla \ln f_1) = 2V(\ln f_2)X(\ln f_1). \quad (4.6)$$

Hence, we obtain

$$V\omega(X) = \frac{4}{3}V(\ln f_2)X(\ln f_1). \quad (4.7)$$

On the other hand, using (2.3) and (2.6), we have

$$X\omega(V) = Xg(B, V) = Xg(B^\theta, V) = g(\nabla X B^\theta, V) + g(B^\theta, \nabla X V) = g \left( B^\theta(\ln f_2)X + X(\ln f_1)B^\theta, V \right) + g(\nabla V(\ln f_2)X, \nabla \ln f_1) = 2\omega(V)X(\ln f_1). \quad (4.8)$$

Namely, we have

$$X\omega(V) = 2\omega(V)X(\ln f_1). \quad (4.9)$$

Now, using (4.5)~(4.9), we get (4.4). □

By Lemma 4.8, we immediately have the following result.

**Theorem 4.5** Let $M = f_2 \, M^\perp \times f_1, \, M^\theta$ be a warped-twisted product hemislant submanifold with warping function $f_2 \in C^\infty(M^\theta)$ and twisting function $f_1$ of a g.c.K. manifold $(M, J, \omega, g)$. Then $M$ is a twisted product submanifold in the form $M^\perp \times f_1, \, M^\theta$ if and only if the Lee vector field $B$ is normal to $M^\theta$.

**Proof** Let $M$ is a twisted product submanifold in the form $M^\perp \times f_1, \, M^\theta$, where $f_1$ is a twisting function. Then, for any $V \in \mathcal{L}(M^\theta)$, $V(\ln f_2) = 0$, since $f_2$ is a constant. From (4.4), we find $g(B, V) = 0$, for any $V \in \mathcal{L}(M^\theta)$. So, the Lee vector field $B$ is normal to $M^\theta$.

Conversely, if the Lee vector field $B$ is normal to $M^\theta$, we have $g(B, V) = 0$, for any $V \in \mathcal{L}(M^\theta)$. Then, we get $V(\ln f_2) = 0$ from (4.4). So, $f_2$ is a constant, say $f_2 = c$. Then the induced metric tensor $g_M$ of $M$ has the form $g_M = c^2 g_\perp \oplus f_1^2 g_\theta$, where $c$ is constant and $f_1$ is the twisting function. Thus, $M = M^\perp \times f_1, \, M^\theta$ is a twisted product. □

By Theorems 4.3 and 4.5, we get the following result.
Theorem 4.6 Let $M = f_2 M^\perp \times f_1 M^\theta$ be a warped-twisted product hemislant submanifold with warping function $f_2 \in C^\infty(M^\theta)$ and twisting function $f_1$ of a g.c.K. manifold $(M, J, \omega, g)$. Then $M$ is a locally direct product manifold if and only if the Lee vector field $B$ is normal to $M$.

Proof Let $M = f_2 M^\perp \times f_1 M^\theta$ be a warped-twisted product hemislant submanifold with warping function $f_2 \in C^\infty(M^\theta)$ and twisting function $f_1$ of a g.c.K. manifold $(M, J, \omega, g)$. If $M$ is a locally direct product, then the functions $f_1$ and $f_2$ are constants. In that case, for any $X \in \mathcal{L}(M^\perp)$ and $V \in \mathcal{L}(M^\theta)$, we have $g(B, X) = g(B, V) = 0$ from (4.1) and (4.4), respectively. It follows that $B$ is normal to $M$.

Conversely, let $B$ be normal to $M$. Then, for any $X \in \mathcal{L}(M^\perp)$ and $V \in \mathcal{L}(M^\theta)$, we have $X(\ln f_1) = V(\ln f_2) = 0$. It follows that $f_2$ is a constant, say $f_2 = c$ and $f_1$ depends only on the points of $M^\theta$. Then the induced metric tensor $g_M$ of $M$ has the form $g_M = c^2 g_\perp \oplus f_1^2 g_\theta$. Hence, we conclude that $M$ is a locally direct product of $(M^\perp, g_\perp)$ and $(M^\theta, g_\theta)$, where $g_\perp = c^2 g_\perp$ and $g_\theta = f_1^2 g_\theta$. □

By using (3.7) and (4.4), we deduce the following result.

Lemma 4.7 Let $M = f_2 M^\perp \times f_1 M^\theta$ be a warped-twisted product hemislant submanifold with warping function $f_2 \in C^\infty(M^\theta)$ and twisting function $f_1$ of a g.c.K. manifold $(M, J, \omega, g)$. Then we have

$$g(A_{JXPV} - A_{FPV}X, Y) = \left\{\cos^2 \theta \omega(V) + \frac{1}{2} \omega(FPV)\right\} g(X, Y) \quad (4.10)$$

for any $X, Y \in \mathcal{L}(M^\perp)$ and $V \in \mathcal{L}(M^\theta)$.

By using (3.8) and (4.1), we deduce the following result.

Lemma 4.8 Let $M = f_2 M^\perp \times f_1 M^\theta$ be a warped-twisted product hemislant submanifold with warping function $f_2 \in C^\infty(M^\theta)$ and twisting function $f_1$ of a g.c.K. manifold $(M, J, \omega, g)$. Then we have

$$g(A_{JXPV} - A_{FPV}X, U) = - \cos^2 \theta \omega(X) g(V, U) - \frac{1}{2} \omega(JX) g(PV, U), \quad (4.11)$$

for any $X \in \mathcal{L}(M^\perp)$ and $U, V \in \mathcal{L}(M^\theta)$.

Now, we recall the following two facts to prove the main theorem.

Lemma 4.9 (Proposition 3-a [16]) Let $g$ be a pseudo-Riemannian metric on the manifold $M = M_1 \times M_2$ and $(\mathcal{D}_1, \mathcal{D}_2)$ are called the canonical foliations. Suppose that $\mathcal{D}_1$ and $\mathcal{D}_2$ intersect perpendicularly everywhere. Then $(M, g)$ is a doubly twisted product $f_2 M_1 \times f_1 M_2$ if and only if $\mathcal{D}_1$ and $\mathcal{D}_2$ are totally umbilic foliations.

Lemma 4.10 (Lemma 3.1.1 [13]) Let $f_2 M_1 \times f_1 M_2$ be a doubly twisted product. It is a doubly warped product if and only if the mean curvature vector fields of the canonical foliations are closed.

Motivated by Lemma 4.9 and Lemma 4.10, we can obtain the following result.

Lemma 4.11 Let $f_2 M_1 \times f_1 M_2$ be a doubly twisted product. It is a warped-twisted product with warping function $f_2 \in C^\infty(M_2)$ and twisting function $f_1$ if and only if the mean curvature vector field of canonical foliation $\mathcal{D}_1$ is closed.
Proof. The proof is very similar to the proof of Lemma 2.3 [12], so we omit it. □

We now are ready to prove the main theorem.

**Theorem 4.12** Let $M$ be a hemislant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then $M$ is a locally warped-twisted product submanifold if and only if its shape operator $A$ satisfies the following equation

$$A_{JX}PV - A_{FPV}X = \left\{\cos^2\theta \omega(V) + \frac{1}{2}\omega(FPV)\right\}X - \cos^2\theta \omega(X)V - \frac{1}{2}\omega(JX)PV,$$

for $X \in \Gamma(D^\perp)$ and $V \in \Gamma(D^\theta)$. Moreover, $M$ is also a locally doubly warped product submanifold.

**Proof.** Let $M$ be a warped-twisted product submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$ of type $f_2M^\perp \times f_1M^\theta$. For any $X \in \mathcal{L}(M^\perp)$ and $V \in \mathcal{L}(M^\theta)$, we write

$$A_{JX}PV - A_{FPV}X = \left(A_{JX}PV - A_{FPV}X\right)^\perp + \left(A_{JX}PV - A_{FPV}X\right)^\theta,$$

where $\left(A_{JX}PV - A_{FPV}X\right)^\perp$ is the tangent part of $A_{JX}PV - A_{FPV}X$ to $M^\perp$ and $\left(A_{JX}PV - A_{FPV}X\right)^\theta$ is the tangent part of $A_{JX}PV - A_{FPV}X$ to $M^\theta$. Hence, for any $Y \in \mathcal{L}(M^\perp)$, using (4.10), we have

$$g(A_{JX}PV - A_{FPV}X, Y) = \left\{\cos^2\theta \omega(V) + \frac{1}{2}\omega(FPV)\right\}X, Y.$$  

Since $Y \in \mathcal{L}(M^\perp)$ is arbitrary and the metric $g$ is Riemannian, it follows that

$$\left(A_{JX}PV - A_{FPV}X\right)^\perp = \left\{\cos^2\theta \omega(V) + \frac{1}{2}\omega(FPV)\right\}X.$$  

(4.14)

Similarly, for any $U \in \mathcal{L}(M^\theta)$, using (4.11), we have

$$g(A_{JX}PV - A_{FPV}X, U) = g\left(-\cos^2\theta \omega(X)V - \frac{1}{2}\omega(JX)PV, U\right).$$

Since $U \in \mathcal{L}(M^\theta)$ is arbitrary and the metric $g$ is Riemannian, it follows that

$$\left(A_{JX}PV - A_{FPV}X\right)^\theta = -\cos^2\theta \omega(X)V - \frac{1}{2}\omega(JX)PV.$$  

(4.15)

Thus, by (4.13)–(4.15), we get (4.12).

Conversely, suppose that $M$ is a hemislant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$ such that (4.12) holds. Then, for any $X \in \Gamma(D^\perp)$ and $U, V \in \Gamma(D^\theta)$, using (4.12), we deduce that (3.9). Thus, by Theorem 3.2, the slant distribution $D^\theta$ is integrable. On the other hand, from Theorem 3.4, we already know that the totally real distribution $D^\perp$ is integrable. Let $M^\perp$ and $M^\theta$ be the integral manifolds of $D^\perp$ and $D^\theta$, respectively and
let denote by \( h^\perp \) and \( h^\theta \) the second fundamental forms of \( M^\perp \) and \( M^\theta \) in \( M \), respectively. Then, for any \( X, Y \in \Gamma(D^\perp) \) and \( V \in \Gamma(D^\theta) \), using (2.9), we have

\[
g(h^\perp(X, Y), V) = g(\nabla_X Y, V).
\]

Here, if we use (3.7) and (4.12), we find

\[
g(h^\perp(X, Y), V) = -\frac{3}{2} \omega(V) g(X, Y).
\]

Using (2.6), we obtain

\[
g(h^\perp(X, Y), V) = g(-g(X, Y) \frac{3}{2} B^\theta, V).
\]

Hence, we conclude that

\[
h^\perp(X, Y) = -g(X, Y) \frac{3}{2} B^\theta.
\]

This equation says that \( M^\perp \) is totally umbilic with the mean curvature vector field \( -\frac{3}{2} B^\theta \). On the other hand, for any \( X \in \Gamma(D^\perp) \) and \( U, V \in \Gamma(D^\theta) \), using (2.9), we have

\[
g(h^\theta(U, V), X) = g(\nabla_U V, X).
\]

Here, if we use (3.8) and (4.12), we find

\[
g(h^\theta(U, V), X) = -\frac{3}{2} \omega(X) g(U, V).
\]

Using (2.6), we obtain

\[
g(h^\theta(U, V), X) = g(-g(U, V) \frac{3}{2} B^\perp, X).
\]

Hence, we conclude that

\[
h^\theta(U, V) = -g(U, V) \frac{3}{2} B^\perp.
\]

It means that \( M^\theta \) is totally umbilic in \( M \) with the mean curvature vector field \( -\frac{3}{2} B^\perp \).

Next, we prove \( B^\perp \) and \( B^\theta \) are closed. Let denote by \( \omega^\perp \) (resp. \( \omega^\theta \)) the dual 1-form of \( B^\perp \) (resp. \( B^\theta \)). For any \( X \in \Gamma(D^\perp) \), we have \( \omega^\perp(X) = \omega(X) \). Thus, for \( X, Y \in \Gamma(D^\perp) \), we obtain

\[
d\omega^\perp(X, Y) = X \omega^\perp(Y) - Y \omega^\perp(X) - \omega^\perp([X, Y]) = X \omega(Y) - Y \omega(X) - \omega([X, Y]) = d\omega(X, Y).
\]

It follows that \( d\omega^\perp = 0 \), since \( d\omega = 0 \). Namely, \( \omega^\perp \) is closed. Hence, \( B^\perp \) is closed, since its dual 1-form is closed. Thus, by Lemma 4.11, \( M \) is a locally warped-twisted product submanifold. Moreover, we can prove that \( B^\theta \) is closed in a similar way. Thereby, by Lemma 4.10, \( M \) is also a locally doubly warped product submanifold. 

**Remark 4.13** We have just proved that a warped-twisted product hemislant submanifold of a g.c.K. manifold \((\bar{M}, J, \omega, g)\) is also a doubly warped product submanifold in Theorem 4.12. Therefore, from now on we will focus on doubly warped product submanifolds of a g.c.K. manifold.

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5. An inequality for doubly warped product mixed geodesic hemislant submanifolds

In this section, we shall establish an inequality for the squared norm of the second fundamental form of a doubly warped product mixed geodesic hemislant submanifold in the form $f_2 M^+ \times_{f_1} M^\theta$, where $M^+$ is a totally real and $M^\theta$ is a slant submanifold of a g.c.K. manifold $(\overline{M}, J, \omega, g)$. Note that a general inequality for any doubly warped product submanifold in arbitrary Riemannian manifolds was established in Theorem 3 of [14].

Let $f_2 M_1 \times_{f_1} M_2$ be a doubly warped product manifold equipped with the metric $g$ defined by

$$g = (f_2 \circ \pi_2)^2 \pi_1^*(g_1) + (f_1 \circ \pi_1)^2 \pi_2^*(g_2).$$

(5.1)

Then the covariant derivative formulas (2.2)–(2.4) become

$$\bar{\nabla}_X Y = \nabla^1_X Y - g(X, Y) \nabla (\ln f_2 \circ \pi_2),$$

(5.2)

$$\bar{\nabla}_V X = \nabla_X V = V(\ln f_2 \circ \pi_2) X + X(\ln f_1 \circ \pi_1) V,$$

(5.3)

$$\bar{\nabla}_U V = \nabla^2_U V - g(U, V) \nabla (\ln f_1 \circ \pi_1),$$

(5.4)

for $X, Y \in \mathcal{L}(M_1)$ and $U, V \in \mathcal{L}(M_2)$. It follows that $M_1 \times \{p_2\}$ and $\{p_1\} \times M_2$ are totally umbilical submanifolds with closed mean curvature vector fields in $f_2 M_1 \times_{f_1} M_2$ [13], where $p_1 \in M_1$ and $p_2 \in M_2$. We say that a doubly warped product is nontrivial if it is neither warped nor a direct product.

**Remark 5.1** [11] For a doubly warped product manifold $f_2 M_1 \times_{f_1} M_2$, we have

$$\bar{\nabla}(\ln f_1 \circ \pi_1) = \frac{1}{(f_2 \circ \pi_2)^2} \nabla^1 (\ln f_1 \circ \pi_1),$$

$$\bar{\nabla}(\ln f_2 \circ \pi_2) = \frac{1}{(f_1 \circ \pi_1)^2} \nabla^2 (\ln f_2 \circ \pi_2).$$

(5.5)

In view of the above convenience together with (5.1) and (5.5), the covariant derivative formulas (5.2) and (5.4) become

$$\bar{\nabla}_X Y = \nabla^1_X Y - \frac{(f_2 \circ \pi_2)^2}{(f_1 \circ \pi_1)^2} g_1(X, Y) \nabla^2 (\ln f_2 \circ \pi_2),$$

(5.6)

$$\bar{\nabla}_U V = \nabla^2_U V - \frac{(f_1 \circ \pi_1)^2}{(f_2 \circ \pi_2)^2} g_2(U, V) \nabla^1 (\ln f_1 \circ \pi_1),$$

(5.7)

for $X, Y \in \mathcal{L}(M_1)$ and $U, V \in \mathcal{L}(M_2)$.

For more details on doubly warped products, we refer to the papers [11], [12], [13] and [20].

**Remark 5.2** From now on, we will use the same symbol for a warping function $f_i$ and its pullback $f_i \circ \pi_i$ for $i = 1, 2$, i.e. we will put $f_i = f_i \circ \pi_i$. 

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Lemma 5.3 Let $M$ be a doubly warped product hemislant submanifold in the form $f_2M^\perp \times f_1M^\theta$, where $M^\perp$ is a totally real and $M^\theta$ is a slant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then, we have

$$g(h(X,Y), FV) = g(h(V,Y), JX) + \left\{ \frac{\sqrt{2}}{2} PV(\ln f_2) - \frac{1}{2} \omega(FV) \right\} g(X,Y), \quad (5.8)$$

$$g(h(U,V), JX) = g(h(U, X), FV) + \frac{3}{2} X(\ln f_1) g(\overline{PV}, U) - \frac{1}{2} \omega(JX) g(U, V), \quad (5.9)$$

for $X, Y \in \mathcal{L}(M^\perp)$ and $U, V \in \mathcal{L}(M^\theta)$.

Proof Let $X, Y \in \mathcal{L}(M^\perp)$ and $U, V \in \mathcal{L}(M^\theta)$, by interchanging $V$ with $PV$ in (4.10) and (4.11), respectively, and using (2.11) and (3.5), we get (5.8) and (5.9) respectively. \hfill \Box

Remark 5.4 We say that a hemislant submanifold $M$ is mixed geodesic, if $h(X, V) = 0$ for $X \in \Gamma(D^\perp)$ and $V \in \Gamma(D^\theta)$.

By Remark 5.4 together with (2.6), we have that:

Corollary 5.5 Let $M = f_2M^\perp \times f_1M^\theta$ be a doubly warped product mixed geodesic hemislant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. If the Lee vector field $B$ is tangent to $M$, then Equations (5.8) and (5.9) become

$$g(h(X,Y), FV) = \frac{3}{2} PV(\ln f_2) g(X,Y), \quad (5.10)$$

$$g(h(U,V), JX) = \frac{3}{2} X(\ln f_1) g(\overline{PV}, U), \quad (5.11)$$

respectively, where $X, Y \in \mathcal{L}(M^\perp)$ and $U, V \in \mathcal{L}(M^\theta)$.

Let $M = f_2M^\perp \times f_1M^\theta$ be a $(m_1 + m_2)$-dimensional doubly warped product hemislant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. We choose a canonical orthonormal basis $\{e_1, ..., e_{m_1}, \bar{e}_1, ..., \bar{e}_{m_2}, J\bar{e}_1, ..., J\bar{e}_{m_1}, e_{m_1}^*, ..., e_{m_2}^*, \bar{e}_{m_1}^*, ..., \bar{e}_{m_2}^*\}$ of $\mathcal{M}$ such that $\{e_1, ..., e_{m_1}\}$ is an orthonormal basis of $D^\perp$, $\{\bar{e}_1, ..., \bar{e}_{m_2}\}$ is an orthonormal basis of $D^\theta$, $\{J\bar{e}_1, ..., J\bar{e}_{m_1}\}$ is an orthonormal basis of $J\mathcal{D}^\perp$, $\{e_{m_1}^*, ..., e_{m_2}^*\}$ is an orthonormal basis of $\mathcal{F}D^\theta$ and $\{\bar{e}_{m_1}^*, ..., \bar{e}_{m_2}^*\}$ is an orthonormal basis of $\overline{\mathcal{F}D}$. Here, $m_1 = \dim(D^\perp)$, $m_2 = \dim(D^\theta)$ and $l = \dim(\overline{\mathcal{F}D})$.

Remark 5.6 In view of (3.6), we can observe that $\{\sec \theta P\bar{e}_1, ..., \sec \theta P\bar{e}_{m_2}\}$ is also an orthonormal basis of $D^\theta$ and $\{\csc \theta \bar{F}e_1, ..., \csc \theta \bar{F}e_{m_2}\}$ is also an orthonormal basis of $\mathcal{F}D^\theta$, where $\theta$ is the slant angle of $D^\theta$.

Theorem 5.7 Let $M = f_2M^\perp \times f_1M^\theta$ be a doubly warped product mixed geodesic hemislant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$ such that the Lee vector field $B$ is tangent to $M$. Then the squared norm of the second fundamental form $h$ of $M$ satisfies

$$\|h\|^2 \geq m_1 \cot \theta \|B^\perp\|^2_\theta + m_2 (m_2 - 1) \cos^2 \theta \|B^\perp\|^2_\perp, \quad (5.12)$$

where $m_1 = \dim(M^\perp)$ and $m_2 = \dim(M^\theta)$ and $\|\cdot\|_\perp$ (resp. $\|\cdot\|_\theta$) is calculated with respect to the metric $g_\perp$ (resp. $g_\theta$).
Proof. By the hypothesis, the squared norm of the second fundamental form $h$ can be written as

$$\|h\|^2 = \|h(D^1, D^2)\|^2 + \|h(D^\theta, D^\theta)\|^2.$$  

In view of the decomposition (3.4), which can be explicitly written as follows:

$$\|h\|^2 = \sum_{i,j=1}^{m_1} \sum_{a=1}^{m_2} g(h(e_i, e_j), Je_i)^2 + \sum_{i=1}^{m_1} \sum_{a,b=1}^{m_2} g(h(e_i, e_j), e_a^*)^2$$

$$+ \sum_{a,b=1}^{m_2} \sum_{i=1}^{m_1} g(h(\bar{e}_a, \bar{e}_b), Je_i)^2 + \sum_{a=1}^{m_2} \sum_{i=1}^{m_1} g(h(\bar{e}_a, \bar{e}_b), e_i^*)^2$$

(5.13)

where the set $\{\bar{e}_r\}_{1 \leq r \leq (m_1+m_2)}$ is an orthonormal basis of $M$. Hence, we get

$$\|h\|^2 \geq \sum_{i,j=1}^{m_1} \sum_{a=1}^{m_2} g(h(e_i, e_j), e_a^*)^2 + \sum_{a,b=1}^{m_2} \sum_{i=1}^{m_1} g(h(\bar{e}_a, \bar{e}_b), Je_i)^2.$$  

By Remark 5.6, we arrive at

$$\|h\|^2 \geq \sum_{i,j=1}^{m_1} \sum_{a=1}^{m_2} g(h(e_i, e_j), \csc\theta \bar{e}_a)^2 + \sum_{a,b=1}^{m_2} \sum_{i=1}^{m_1} g(h(\bar{e}_a, \bar{e}_b), Je_i)^2.$$  

Now, using (5.10) and (5.11), we obtain

$$\|h\|^2 \geq \frac{4}{9} \csc^2\theta \sum_{i,j=1}^{m_1} \sum_{a=1}^{m_2} (P\bar{e}_a(\ln f_2))^2 g^2(e_i, e_j)$$

$$+ \frac{4}{9} \sum_{a,b=1}^{m_2} \sum_{i=1}^{m_1} (e_i(\ln f_1))^2 g^2(P\bar{e}_a, \bar{e}_b).$$  

Again, by Remark 5.6, we can put $P\bar{e}_a = \cos\theta \bar{e}_c$, where $\{\bar{e}_c\}_{1 \leq c \leq m_2}$ is an orthonormal basis of $D^\theta$, so the last equation becomes

$$\|h\|^2 \geq \frac{4}{9} \cot^2\theta \sum_{i,j=1}^{m_1} \sum_{c=1}^{m_2} (\bar{e}_c(\ln f_2))^2 g^2(e_i, e_j)$$

$$+ \frac{4}{9} \sum_{a,b=1}^{m_2} \sum_{i=1}^{m_1} (e_i(\ln f_1))^2 g^2(P\bar{e}_a, \bar{e}_b).$$  

Here, for $a, b \in \{1, 2, ..., m_2\}$, we have

$$g(P\bar{e}_a, \bar{e}_b) = \begin{cases} \cos\theta & \text{if } a \neq b, \\ 0 & \text{if } a = b, \end{cases}$$

since $D^\theta$ is a slant distribution with slant angle $\theta$. Thus, by a direct calculation, we obtain the following inequality.

$$\|h\|^2 \geq \frac{4}{9} \left\{ m_1 \cot^2\theta \|\ln f_2\|^2 + m_2(m_2 - 1) \cos^2\theta \|\ln f_1\|^2 \right\}.$$  

(5.14)
On the other hand, by (2.6) and (5.5), we conclude that
\[
B^\perp = \frac{2}{3f_2^2} \nabla^\perp (\ln f_1) \quad \text{and} \quad B^\theta = \frac{2}{3f_1^2} \nabla^\theta (\ln f_2)
\] (5.15)
from (4.1) and (4.4), respectively. Hence, using (5.5) and (5.15) in (5.14), we get the inequality (5.12).

**Theorem 5.8** Let \( M = f_2 \ M^\perp \times f_1 \ M^\theta \) be a doubly warped product mixed geodesic hemislant submanifold of a g.c.K. manifold \( (\bar{M}, J, \omega, g) \) such that the Lee vector field \( B \) is tangent to \( M \). If the invariant subnormal bundle \( \overline{D} = \{0\} \), then the equality sign of (5.12) holds if and only if \( A_{JX}Y \in \mathcal{L}(M^\theta) \) and \( A_{FU}V \in \mathcal{L}(M^\perp) \), where \( X, Y \in \mathcal{L}(M^\perp) \) and \( U, V \in \mathcal{L}(M^\theta) \).

**Proof** Under the given hypothesis, we see that the equality sign of (5.12) holds if and only if
\[
g(h(D^\perp, D^\perp), J D^\perp) = 0 \quad \text{and} \quad g(h(D^\theta, D^\theta), F D^\theta) = 0
\] from (5.13). These are equivalent to
\[
g(h(Y, Z), JX) = 0 \quad \text{and} \quad g(h(V, W), FU) = 0
\]
for \( X, Y, Z \in \mathcal{L}(M^\perp) \) and \( U, V, W \in \mathcal{L}(M^\theta) \). But, with the help of (2.11), we know these conditions hold if and only if
\[
A_{JX}Y \in \mathcal{L}(M^\theta) \quad \text{and} \quad A_{FU}V \in \mathcal{L}(M^\perp).
\]

**Theorem 5.9** Let \( M = f_2 \ M^\perp \times f_1 \ M^\theta \) be a nontrivial doubly warped product mixed geodesic hemislant submanifold of a g.c.K. manifold \( (\bar{M}, J, \omega, g) \) such that the Lee vector field \( B \) is tangent to \( M \) and the invariant subnormal bundle \( \overline{D} = \{0\} \). If the equality sign of (5.12) holds, then \( M^\theta \) is also totally umbilic in the ambient manifold \( \bar{M} \).

**Proof** Let \( \bar{h}^\theta \) denote the second fundamental form of \( M^\theta \) in \( \bar{M} \). Then, for \( a \in \{1, ..., m_2\} \), we have
\[
\bar{h}^\theta(\bar{e}_a, \bar{e}_a) = h^\theta(\bar{e}_a, \bar{e}_a) + h(\bar{e}_a, \bar{e}_a),
\] (5.16)
where \( \{\bar{e}_1, ..., \bar{e}_{m_2}\} \) is an orthonormal basis of \( M^\theta \) and \( h^\theta \) is the second fundamental form of \( M^\theta \) in \( \bar{M} \), and \( h \) is the second fundamental form of \( M \) in \( \bar{M} \). Since \( M = f_2 \ M^\perp \times f_1 \ M^\theta \) is a nontrivial doubly warped product, we see that
\[
h^\theta(\bar{e}_a, \bar{e}_a) = -\frac{2}{f_2^2} \nabla^\perp (\ln f_1) \neq 0,
\]
from (5.7). On the other hand, we know \( h(D^\theta, D^\theta) \subseteq JD^\perp \) from Theorem 5.8. Thus, we have
\[
h(\bar{e}_a, \bar{e}_a) = \sum_{i=1}^{m_1} g(h(\bar{e}_a, \bar{e}_a), J e_i) J e_i,
\]
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where \( \{e_1, \ldots, e_{m_1}\} \) is an orthonormal basis of \( M^\perp \). Here, for each \( a \in \{1, \ldots, m_2\} \) and \( i \in \{1, \ldots, m_1\} \), using (5.11), we find
\[
g(h(\bar{e}_a, \bar{e}_a), J e_i) = \frac{2}{f} e_i (\ln f_1) g(P \bar{e}_a, \bar{e}_a) = 0,
\]
since \( g(P \bar{e}_a, \bar{e}_a) = 0 \). Which means that
\[
h(\bar{e}_a, \bar{e}_a) = 0,
\]
for each \( a \in \{1, \ldots, m_2\} \). It follows that
\[
\bar{h}_\theta(\bar{e}_a, \bar{e}_a) = h_\theta(\bar{e}_a, \bar{e}_a),
\]
from (5.16). Thus, \( M^\theta \) is totally umbilic in \( \bar{M} \), since it is totally umbilic in \( M \).

\[\square\]

**Remark 5.10** Whether the Lee form \( \omega \) is exact or not does not change all the results in this paper. Thus, these results also hold for locally conformal Kaehler case.

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**References**


