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On a certain type of warped-twisted product submanifolds

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Abstract: We introduce a certain type of warped-twisted product submanifolds which is called warped-twisted product hemislant submanifolds of the form ${}_{f_2}M^{\perp} \times_{f_1} M^{\theta}$ with warping function f_2 on M^{θ} and twisting function f_1, where $M^{\perp}$ is a totally real and M^{θ} is a slant submanifold of a globally conformal Kaehler manifold. We prove that a warped-twisted product hemislant submanifold of a globally conformal Kaehler manifold is a locally doubly warped product. Then we establish a general inequality for doubly warped product mixed geodesic hemislant submanifolds and get some results for such submanifolds by using the equality sign of the general inequality.

Key words: Twisted product, warped product, totally real distribution, slant distribution, hemislant submanifold, globally conformal Kaehler manifold

1. Introduction

One of the ways to get a new Riemannian manifold (or, more generally, pseudo-Riemannian manifold) is to product two pseudo-Riemannian manifolds in the usual sense. The most general manifolds obtained in this way are the doubly twisted product manifolds [16]. In fact, this notion appeared in the literature a long time ago, under the name of conformally seperable spaces [23]. It is well known that the notion of doubly twisted product is a natural generalization of the notion of doubly warped product [11], twisted product [8], warped product [3] and direct product.

In [18], we defined two classes of doubly twisted products under the names of *nearly doubly twisted products of type 1* and *type 2*. In this article, we rename the nearly doubly twisted products of type 1 as *warped-twisted products*.

One of the popular research areas in differential geometry is the theory of submanifolds. Actually, there are well known classes of submanifolds such as holomorphic (invariant), totally real (antiinvariant) [22], CR-(semiinvariant)[1, 2], slant [5], semislant [15], hemislant or antislant [4, 17]. All classes are determined by the behavior of the almost complex or almost product structure of the ambient manifold. On the other hand, the theory of warped product submanifolds has become a popular research area since Chen [6] studied the warped product CR-submanifolds in Kaehler structures. Afterwards, several classes of warped product submanifolds appeared in the literature. Also, the warped product submanifolds have been studied in different kind of structures. Most of the studies related to this theory can be found in the book [7] and its list of references.

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In this paper, we consider and study a certain type of warped-twisted products which is warped-twisted product hemislant submanifolds in globally conformal Kaehler manifolds. More precisely, we study warped-twisted product submanifolds which have the form ${}_{f_2}M^{\perp} \times_{f_1} M^{\theta}$ with warping function f_2 on M^{θ} and twisting function f_1, where $M^{\perp}$ is a totally real and M^{θ} is a slant submanifold of the globally conformal Kaehler manifold. We obtain necessary and sufficient conditions for such submanifolds to be twisted product, base conformal warped product and direct product submanifolds. In the main theorem, we prove that a warped-twisted product hemislant submanifold of a globally conformal Kaehler manifold is locally a doubly warped product. After the main theorem mentioned, we focus on the study of doubly warped product hemislant submanifolds and we get some results for doubly warped product mixed geodesic hemislant submanifolds when the Lee vector field of the globally conformal Kaehler manifold is tangent to them. We say that a warped-twisted product is nontrivial if it is neither twisted nor base conformal warped or direct product.

2. Preliminiaries

In this section, we recall the fundamental definitions and notions needed for the further study. Actually, in subsection 2.1, we give the definition of doubly twisted and warped-twisted products and in subsection 2.2, we will recall the definitions of locally and globally conformal Kaehler manifolds. The basic background for submanifolds of Riemannian manifolds will be presented in subsection 2.3.

2.1. Warped-twisted products

Let M_1 and M_2 be Riemannian manifolds endowed with metric tensors g_1 and g_2, respectively and let f_1 and f_2 be positive smooth functions defined on $M_1 \times M_2$. Then the *doubly twisted product manifold* [16] ${}_{f_2}M_1 \times_{f_1} M_2$ is the product manifold $\bar{M} = M_1 \times M_2$ equipped with metric g given by

$$g = {f_2}^2 \pi_1^* g_1 + {f_1}^2 \pi_2^* g_2,$$

where $\pi_i : M_1 \times M_2 \to M_i$ is the canonical projection, for $i = 1, 2$. Each function f_i is called a *twisting function* of the doubly twisted product ${}_{f_2}M_1 \times_{f_1} M_2$. If the twisting functions f_1 and f_2 depend only on the points of M_1 and M_2 respectively, then ${}_{f_2}M_1 \times_{f_1} M_2$ becomes a *doubly warped product manifold* [11] and each function f_i is called a *warping function* of the doubly warped product manifold. In this case, if $f_1 \equiv 1$ or $f_2 \equiv 1$, then we get a *warped product* [3].

Let ${}_{f_2}M_1 \times_{f_1} M_2$ be doubly twisted product manifold. If $f_1 \equiv 1$ or $f_2 \equiv 1$, then we get a *twisted product* [8] with the twisting function f_1 or a twisted product with the twisting function f_2. In a warped or twisted product case, the notation ${}_{f_2}M_1 \times_{f_1} M_2$ is simplified to ${}_{f_2}M_1 \times M_2$ or $M_1 \times_{f_1} M_2$. In addition, if both f_1 and f_2 are constant, then we get a *usual* or *direct product manifold* [7].

Let us recall the definition of a warped-twisted product manifold. Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds and let $f_2 : M_2 \to (0, \infty)$ and $f_1 : M_1 \times M_2 \to (0, \infty)$ be smooth functions. The *warped-twisted product* ${}_{f_2}M_1 \times_{f_1} M_2$ [18] is the product manifold $M_1 \times M_2$ equipped with the metric tensor g defined by

$$g = (f_2 \circ \pi_2)^2 \pi_1^*(g_1) + f_1^2 \pi_2^*(g_2). \tag{2.1}$$

The function $f_2 \in C^\infty(M_2)$ is called a *warping function* and the function $f_1 \in C^\infty(M_1 \times M_2)$ is called a *twisting function* of ${}_{f_2}M_1 \times_{f_1} M_2$. In this case, if the function f_1 depends only on the points of M_2, then the warped-twisted product ${}_{f_2}M_1 \times_{f_1} M_2$ becomes a *base conformal warped product* [9]. We say that a warped-twisted product is *nontrivial* if it is neither doubly warped product nor warped product or base conformal warped product.

Let ${}_{f_2}M_1 \times_{f_1} M_2$ be a warped-twisted product manifold with the Levi-Civita connection $\bar{\nabla}$ of g, given in (2.1). Also, ∇^i denote by the Levi-Civita connection of g_i, for $i \in \{1,2\}$, respectively. By usual convenience, we denote the set of lifts of vector fields on M_i by $\mathcal{L}(M_i)$ and we use the same notation for a vector field and for its lift. On the other hand, each π_i is a positive homothety, so it preserves the Levi-Civita connection. Thus, there is no confusion using the same notation for a connection on M_i and for its pullback via π_i. Then, the covariant derivative formulas of the warped-twisted product manifold ${}_{f_2}M_1 \times_{f_1} M_2$ with warping function $f_2 \in C^\infty(M_2)$ and twisting function f_1 are given by

$$\bar{\nabla}_X Y = \nabla^1_X Y - g(X,Y)\bar{\nabla}\ln(f_2 \circ \pi_2), \tag{2.2}$$

$$\bar{\nabla}_V X = \bar{\nabla}_X V = V(\ln(f_2 \circ \pi_2))X + X(\ln f_1)V, \tag{2.3}$$

$$\bar{\nabla}_U V = \nabla^2_U V + U(\ln f_1)V + V(\ln f_1)U - g(U,V)\bar{\nabla}\ln f_1, \tag{2.4}$$

for any $X, Y \in \mathcal{L}(M_1)$ and $U, V \in \mathcal{L}(M_2)$. These formulas immediately come from Lemma 2.1 of [12] with $X(\ln(f_2 \circ \pi_2)) = Y(\ln(f_2 \circ \pi_2)) = 0$.

Remark 2.1 *Until Section 5, we will use the same symbol for the warping function f_2 and its pullback $f_2 \circ \pi_2$, i.e. we will put $f_2 = f_2 \circ \pi_2$.*

2.2. Locally and globally conformal Kaehler manifolds

Let $(\bar{M}, J, g)$ be a Hermitian manifold of dimension $2m$. Then it is called a *locally conformal Kaehler manifold* (briefly *l.c.K. manifold*) [10], if each point of $p \in \bar{M}$ has an open neighborhood $\mathcal{U}$ with smooth function $\sigma : \mathcal{U} \to R$ such that $\tilde{g} = e^{-\sigma} g\,|_{\mathcal{U}}$ is a Kaehler metric on $\mathcal{U}$. If one choose $\mathcal{U} = \bar{M}$, then $(\bar{M}, J, g)$ is called a *globally conformal Kaehler manifold* (briefly *g.c.K. manifold*).

Theorem 2.2 *([10]) Let $(\bar{M}, J, g)$ be a Hermitian manifold and let Ω be a $2-$ form defined by $\Omega(\bar{X}, \bar{Y}) = g(\bar{X}, J\bar{Y})$ for all vector fields $\bar{X}$ and $\bar{Y}$ in $\bar{M}$. Then $(\bar{M}, J, g)$ is a l.c.K. manifold if and only if there exists a globally defined $1-$ form ω such that*

$$d\Omega = \omega \wedge \Omega \qquad \textit{and} \qquad d\omega = 0. \tag{2.5}$$

The closed $1-$ form ω is called the *Lee form* of the l.c.K. manifold $(\bar{M}, J, g)$. In addition, the manifold $(\bar{M}, J, g)$ is g.c.K., if its Lee form ω is also exact. In this case, we have $\omega = d\sigma$ [21]. The *Lee vector field* B is defined by

$$\omega(\bar{X}) = g(B, \bar{X}), \tag{2.6}$$

for any vector fields $\bar{X}$ on $\bar{M}$. One can see that, the globally conformal Kaehler case is a special case of the locally conformal Kaehler case. We denote by $\tilde{\nabla}$ (resp. $\bar{\nabla}$) the Levi-Civita connection on a g.c.K. manifold $\bar{M}$

with respect to $\tilde{g} = e^{-\sigma} g$ (resp. g). Then we have [10]

$$\tilde{\nabla}_{\bar{X}} \bar{Y} = \bar{\nabla}_{\bar{X}} \bar{Y} - \tfrac{1}{2}\Big\{\omega(\bar{X})\bar{Y} + \omega(\bar{Y})\bar{X} - g(\bar{X},\bar{Y})B\Big\}, \tag{2.7}$$

for any vector fields $\bar{X}$ and $\bar{Y}$ on $\bar{M}$. The connection $\tilde{\nabla}$ is a torsionless linear connection on $\bar{M}$ which is called the *Weyl connection* of g. It is easy to see that the Weyl connection $\tilde{\nabla}$ satisfies the condition

$$\tilde{\nabla} J = 0. \tag{2.8}$$

Remark 2.3 *Throughout this paper, we denote by $(\bar{M}, J, \omega, g)$ the g.c.K. manifold with the Lee form ω.*

2.3. Submanifolds of Riemannian manifolds

Let M be an isometrically immersed submanifold in a Riemannian manifold $(\bar{M}, g)$. Let $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$ with respect to the metric g and let ∇ and $\nabla^{\perp}$ be the induced, and induced normal connection on M, respectively. Then, for all $X, Y \in TM$ and $Z \in T^{\perp}M$, the Gauss and Weingarten formulas are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.9}$$

$$\bar{\nabla}_X Z = -A_Z X + \nabla^{\perp}_X Z, \tag{2.10}$$

where TM is the tangent bundle and $T^{\perp}M$ is the normal bundle of M in $\bar{M}$. Additionally, h is the *second fundamental form* of M and A_Z is the Weingarten endomorphism associated with Z. The second fundamental form h and the *shape operator* A are related by

$$g(h(X,Y), Z) = g(A_Z X, Y). \tag{2.11}$$

The *mean curvature vector field* H of M is given by $H = \frac{1}{m}(trace\, h)$, where $dim(M) = m$. We say that the submanifold M is *totally geodesic* in $\bar{M}$ if $h = 0$, and *minimal* if $H = 0$. The submanifold M is called *totally umbilical* if $h(X,Y) = g(X,Y)H$ for all $X, Y \in TM$.

Let M be any submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then the Gauss and Weingarten formulas with respect to $\tilde{\nabla}$ are given by

$$\tilde{\nabla}_X Y = \hat{\nabla}_X Y + \tilde{h}(X,Y), \tag{2.12}$$

$$\tilde{\nabla}_X Z = -\tilde{A}_Z X + \tilde{\nabla}^{\perp}_X Z, \tag{2.13}$$

for $X, Y \in TM$ and $Z \in T^{\perp}M$. Thus, using (2.7), (2.9)–(2.13), we have the following relations

$$\hat{\nabla}_X Y = \nabla_X Y - \frac{1}{2}\Big\{\omega(X)Y + \omega(Y)X - g(X,Y)B^T\Big\}, \tag{2.14}$$

$$\tilde{A}_Z X = A_Z X + \frac{1}{2}\omega(Z)X, \tag{2.15}$$

$$\tilde{h}(X,Y) = h(X,Y) + \frac{1}{2}g(X,Y)B^N, \tag{2.16}$$

where $X, Y \in TM$ and $Z \in T^{\perp}M$, where B^T and B^N are the tangential and the normal part to M of B, respectively.

3. Hemislant submanifolds of a g.c.K. manifold

In this section, we recall some fundamental properties of hemislant submanifolds of a g.c.K. manifold given in [19] and we give some auxiliary results to prove our main theorem.

Let $(\bar{M}, J, g)$ be an almost Hermitian manifold and let M be a Riemannian manifold isometrically immersed in $\bar{M}$. A distribution $\mathcal{D}$ on M is called a *slant distribution* if for $V \in \mathcal{D}_p$, the angle θ between JV and $\mathcal{D}_p$ is constant, i.e. independent of $p \in M$ and $V \in \mathcal{D}_p$. The constant angle θ is called the *slant angle* of the slant distribution $\mathcal{D}$. We know that holomorphic and totally real distributions on M are slant distributions with $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant distribution is called *proper* if it is neither holomorphic nor totally real. A submanifold M of $\bar{M}$ is said to be a *slant submanifold* [5] if the tangent bundle TM of M is slant. For examples and more details, (see [5]).

A *hemislant submanifold* M [15] of an almost Hermitian manifold $(\bar{M}, J, g)$ is a submanifold such that its tangent bundle TM admits two orthogonal complementary totally real distribution $\mathcal{D}^{\perp}$ $(\forall X \in \mathcal{D}^{\perp}, JX \in T^{\perp}M)$ and slant distribution $\mathcal{D}^{\theta}$, i.e. we have

$$TM = \mathcal{D}^{\perp} \oplus \mathcal{D}^{\theta}. \tag{3.1}$$

We say that a hemislant submanifold M is *proper* if $dim(\mathcal{D}^{\perp}) \neq 0$ and $\theta \neq 0, \frac{\pi}{2}$.
For any $X \in TM$ we write

$$JX = PX + FX, \tag{3.2}$$

where PX is the tangential part of JX, and FX is the normal part of JX. Similarly, for any $Z \in T^{\perp}M$, we put

$$JZ = tZ + nZ, \tag{3.3}$$

where tZ is the tangential part of JZ, and nZ is the normal part of JZ. Then the normal bundle $T^{\perp}M$ of M is decomposed as

$$T^{\perp}M = J\mathcal{D}^{\perp} \oplus F\mathcal{D}^{\theta} \oplus \overline{\mathcal{D}}, \tag{3.4}$$

where $\overline{\mathcal{D}}$ is the orthogonal complementary distribution of $J\mathcal{D}^{\perp} \oplus F\mathcal{D}^{\theta}$ in $T^{\perp}M$ and it is an invariant subbundle of $T^{\perp}M$ with respect to J. For a hemislant submanifold, we have

$$P^2V = -\cos^2\theta V, \tag{3.5}$$

$$g(PU, PV) = \cos^2\theta g(U, V) \qquad \text{and} \qquad g(FU, FV) = \sin^2\theta g(U, V), \tag{3.6}$$

for $U, V \in \Gamma(\mathcal{D}^{\theta})$.

Lemma 3.1 *Let M be a hemislant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then we have*

$$g(\nabla_X Y, V) = -\sec^2\theta g\Big(A_{JY}PV - A_{FPV}Y - \tfrac{1}{2}\omega(FPV)Y,\, X\Big) - \tfrac{1}{2}\omega(V)g(X,Y), \tag{3.7}$$

and

$$g(\nabla_U V, X) = \sec^2\theta g\Big(A_{JX}PV - A_{FPV}X + \tfrac{1}{2}\omega(JX)PV,\, U\Big) - \tfrac{1}{2}\omega(X)g(U,V), \tag{3.8}$$

for $X, Y \in \Gamma(\mathcal{D}^\perp)$ and $U, V \in \Gamma(\mathcal{D}^\theta)$.

Proof Let M be a hemislant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$ and $X, Y \in \Gamma(\mathcal{D}^\perp)$ and $V \in \Gamma(\mathcal{D}^\theta)$. Since $(\bar{M}, J, \tilde{g} = e^{-\sigma}g)$ is a Kaehler manifold, by using (2.8), (2.13), (3.2) and (3.3), we have

$$\begin{aligned} \tilde{g}(\hat{\nabla}_X Y, V) = & \quad \tilde{g}(\tilde{\nabla}_X Y, V) = \tilde{g}(\tilde{\nabla}_X JY, JV) \\ = & \quad \tilde{g}(\tilde{\nabla}_X JY, PV) + \tilde{g}(\tilde{\nabla}_X JY, FV) \\ = & \quad -\tilde{g}(\tilde{A}_{JY}PV, X) - \tilde{g}(\tilde{\nabla}_X Y, tFV) - \tilde{g}(\tilde{\nabla}_X Y, nFV). \end{aligned}$$

Here, using the fact that $tF = -I - P^2$ and $nF = -FP$ (see, the equation (2.16) of [19]), we obtain

$$\begin{aligned} \tilde{g}(\hat{\nabla}_X Y, V) = & \quad -\tilde{g}(\tilde{A}_{JY}PV, X) + \sin^2\theta \tilde{g}(\hat{\nabla}_X Y, V) + \tilde{g}(\tilde{\nabla}_X Y, FPV) \\ = & \quad -\tilde{g}(\tilde{A}_{JY}X, PV) + \tilde{g}(\tilde{A}_{FPV}Y, X) + \sin^2\theta \tilde{g}(\hat{\nabla}_X Y, V). \end{aligned}$$

Hence, we get

$$\tilde{g}(\hat{\nabla}_X Y, V) = -\sec^2\theta\{\tilde{g}(\tilde{A}_{JY}PV - \tilde{A}_{FPV}Y, X)\}.$$

Now, by using (2.6), (2.14) and (2.15), we derive the conclusion (3.7). The other assertion (3.8) can be obtained by a similar method. □

By using (3.8), we can prove the following result.

Theorem 3.2 *Let M be a hemislant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then the slant distribution $\mathcal{D}^\theta$ on M is integrable if and only if*

$$g(A_{JX}PV - A_{FPV}X, U) = g(A_{JX}PU - A_{FPU}X, V) - \omega(JX)g(PV, U), \tag{3.9}$$

for $X \in \Gamma(\mathcal{D}^\perp)$ and $U, V \in \Gamma(\mathcal{D}^\theta)$.

Remark 3.3 *The integrability condition of the distribution $\mathcal{D}^\theta$ was given in Proposition 3.2 of [19] in a different way.*

Theorem 3.4 *Let M be a proper hemislant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then the totally real distribution $\mathcal{D}^\perp$ is always integrable.*

Proof The proof is very similar to the proof of Theorem 3.1 of [19]. So, we omit it. □

Remark 3.5 *Throughout this paper, for a hemislant submanifold M of a g.c.K. manifold $(\bar{M}, J, \omega, g)$, we write $B^M = B^\perp + B^\theta$, where B^T (resp. B^θ) is tangential part of B^M to $\mathcal{D}^\perp$ (resp. $\mathcal{D}^\theta$).*

4. Warped-twisted product hemislant submanifolds of a g.c.K. manifold

In this section, we give a characterization for a warped-twisted product hemislant submanifold in the form ${}_{f_2}M^{\perp} \times_{f_1} M^{\theta}$ with warping function $f_2 \in C^{\infty}(M^{\theta})$ and twisting function f_1 of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. We first give an (nontrivial) example of such a submanifold.

Example 4.1 *Let $(z_1, ..., z_6)$ be natural coordinates of the six-dimensional Euclidean space R^6 and let $\bar{R}^6 = \{(z_1, ..., z_6) \in R^6 : z_1 \neq \mp z_2 \, and \, z_5 \neq 0\}$. Then $(\bar{R}^6, J, g_0)$ is a Kaehler manifold with usual Kaehler structure (J, g_0). Now, we consider the Riemannian metric $g = e^{\sigma} g_0$ conformal to Kaehler metric g_0 on $\bar{R}^6$, where $e^{\sigma} = \dfrac{(z_2^2 - z_1^2)^2}{16}(z_5)^2$. So, $(\bar{R}^6, J, g)$ is a g.c.K. manifold. Let M be a submanifold given by*

$$z_1 = u - v\,,\ z_2 = u + v\,,\ z_3 = v\,,\ z_4 = 0\,,\ z_5 = x\,,\ z_6 = 0\,,$$

where $x, u, v \neq 0$. Then, the local frame field of the tangent bundle TM of M is given by

$$X = \partial_5\,,\quad U = \frac{1}{\sqrt{2}}(\partial_1 + \partial_2)\,,\ V = \frac{1}{\sqrt{3}}(-\partial_1 + \partial_2 + \partial_3),$$

where $\partial_i = \frac{\partial}{\partial z_i}$ for $i \in \{1, 2, ..., 6\}$. Then $\mathcal{D}^{\perp} = span\{X\}$ is a totally real and $\mathcal{D}^{\theta} = span\{U, V\}$ is a (proper) slant distribution with the slant angle $\theta = \cos^{-1}(\frac{2}{\sqrt{6}})$. Thus, M is a proper hemislant submanifold with the slant angle $\theta = \cos^{-1}(\frac{2}{\sqrt{6}})$.

By direct calculations, we see the distributions $\mathcal{D}^{\perp}$ and $\mathcal{D}^{\theta}$ are integrable. Let us denote the integral submanifolds of $\mathcal{D}^{\perp}$ and $\mathcal{D}^{\theta}$ by $M^{\perp}$ and M^{θ}, respectively. Let $g_{\perp}$ and g_{θ} be the induced metrics from the Kaehler metric g_0 on $M^{\perp}$ and M^{θ}, respectively. We choose the conformal Riemann metric $\bar{g}_{\perp} = x^2 g_{\perp}$ and $\bar{g}_{\theta} = u^2 g_{\theta}$ on $M^{\perp}$ and M^{θ}, respectively.

Since $x = z_5$ and $uv = \dfrac{(z_2^2 - z_1^2)}{4}$ on M, the induced metric of M from the conformal Kaehler metric g is

$$\begin{aligned} ds^2 = &\quad (uv)^2 x^2 dx^2 + (uv)^2 x^2 (du^2 + dv^2) \\ = &\quad (uv)^2 x^2 g_{\perp} + (uv)^2 x^2 g_{\theta} \\ = &\quad (uv)^2 \bar{g}_{\perp} + (xv)^2 \bar{g}_{\theta}. \end{aligned}$$

Thus, M is a warped-twisted product of $(M^{\perp}, \bar{g}_{\perp})$ and $(M^{\theta}, \bar{g}_{\theta})$. So, ${}_{f_2}M^{\perp} \times_{f_1} M^{\theta}$ is a (nontrivial) warped-twisted product proper hemislant submanifold of the g.c.K. manifold $(\bar{R}^6, J, g)$ with warping function $f_2 = uv$ and twisting function $f_1 = vx$. Moreover, the Lee form of $(\bar{R}^6, J, g)$ is

$$\omega = 2\left(\frac{1}{x}dx + \frac{1}{u}du + \frac{1}{v}dv\right).$$

Consequently, the Lee vector field is

$$B = \frac{2}{(uv)^2 x^2}\left(\frac{1}{x}\frac{\partial}{\partial x} + \frac{1}{u}\frac{\partial}{\partial u} + \frac{1}{v}\frac{\partial}{\partial u}\right)$$

which is tangent to M.

Lemma 4.2 *Let* $M = {}_{f_2}M^{\perp} \times_{f_1} M^{\theta}$ *be a warped-twisted product hemislant submanifold with warping function* $f_2 \in C^{\infty}(M^{\theta})$ *and twisting function* f_1 *of a g.c.K. manifold* $(\bar{M}, J, \omega, g)$. *Then, for all* $X \in \mathcal{L}(M^{\perp})$, *we have*

$$\omega(X) = \tfrac{2}{3}X(\ln f_1). \tag{4.1}$$

Proof Let $M = {}_{f_2}M^{\perp} \times_{f_1} M^{\theta}$ be a warped-twisted product hemislant submanifold with warping function $f_2 \in C^{\infty}(M^{\theta})$ and twisting function f_1 of a g.c.K. manifold $(\bar{M}, J, \omega, g)$ and $U, V \in \mathcal{L}(M^{\theta})$ and $X \in \mathcal{L}(M^{\perp})$. Then, we have

$$\begin{aligned} 3d\Omega(X,U,V) = {} & X\Omega(U,V) + U\Omega(V,X) + V\Omega(X,U) \\ & - \Omega([X,U],V) - \Omega([U,V],X) - \Omega([V,X],U) \\ = {} & Xg(U,PV), \end{aligned}$$

since $[X,V] = [X,U] = 0$ from (2.3) and $[U,V] = \nabla^{\theta}_U V - \nabla^{\theta}_V U \in \Gamma(TM^{\theta})$ from (2.4). After some calculation in view of (2.3), we obtain

$$3d\Omega(X,U,V) = 2X(\ln f_1)g(U,PV), \tag{4.2}$$

since $PV \in \mathcal{L}(M^{\theta})$. On the other hand, we have

$$\begin{aligned} d\Omega(X,U,V) = {} & \omega \wedge \Omega(X,U,V) \\ = {} & \omega(X)\Omega(U,V) + \omega(U)\Omega(V,X) + \omega(V)\Omega(X,U) \\ = {} & \omega(X)g(U,PV), \end{aligned}$$

from (2.5). Namely,

$$d\Omega(X,U,V) = \omega(X)g(U,PV). \tag{4.3}$$

Thus, the assertion follows from (4.2) and (4.3). □

By Lemma 4.2, we immediately have the following result.

Theorem 4.3 *Let* $M = {}_{f_2} M^{\perp} \times_{f_1} M^{\theta}$ *be a warped-twisted product hemislant submanifold with warping function* $f_2 \in C^{\infty}(M^{\theta})$ *and twisting function* f_1 *of a g.c.K. manifold* $(\bar{M}, J, \omega, g)$. *Then* M *is a base conformal warped product submanifold in the form* ${}_{f_2}M^{\perp} \times_{f_1} M^{\theta}$ *if and only if the Lee vector field* B *is normal to* $M^{\perp}$.

Proof Let $M = {}_{f_2} M^{\perp} \times_{f_1} M^{\theta}$ be a warped-twisted product hemislant submanifold with warping function $f_2 \in C^{\infty}(M^{\theta})$ and twisting function f_1 of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. If M is a base conformal warped product submanifold in the form ${}_{f_2}M^{\perp} \times_{f_1} M^{\theta}$, then for any $X \in \mathcal{L}(M^{\perp})$, $X(\ln f_1)$=0, since f_1 depends only on the points of M^{θ}. From (4.1), we find $g(B,X) = 0$. So, the Lee vector field B is normal to $M^{\perp}$.

Conversely, if the Lee vector field B is normal to $M^{\perp}$, we have $g(B,X) = 0$. Then, we get $X(\ln f_1) = 0$ for any $X \in \mathcal{L}(M^{\perp})$ from (4.1). So f_1 depends only on the points of M^{θ}. Then the induced metric tensor g_M of M has the form $g_M = {f_2}^2 g_{\perp} \oplus \tilde{g}_{\theta}$, where f_2 is warping function and $\tilde{g}_{\theta} = {f_1}^2 g_{\theta}$. Thus, $M = {}_{f_2} M^{\perp} \times_{f_1} M^{\theta}$ is a base conformal warped product. □

Lemma 4.4 *Let* $M = {}_{f_2}M^{\perp} \times_{f_1} M^{\theta}$ *be a warped-twisted product hemislant submanifold with warping function* $f_2 \in C^{\infty}(M^{\theta})$ *and twisting function* f_1 *of a g.c.K. manifold* $(\bar{M}, J, \omega, g)$. *Then, for all* $V \in \mathcal{L}(M^{\theta})$, *we have*

$$\omega(V) = \tfrac{2}{3}V(\ln f_2). \tag{4.4}$$

Proof Let $M =_{f_2} M^{\perp} \times_{f_1} M^{\theta}$ be a warped-twisted product hemislant submanifold with the warping function $f_2 \in C^{\infty}(M^{\theta})$ and the twisting function f_1 of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then we have $d\omega = 0$, since $\omega = d\sigma$. Hence, using the exterior differentiation formula, we obtain $0 = d\omega(V, X) = V\omega(X) - X\omega(V) - \omega([V, X])$ for $V \in \mathcal{L}(M^{\theta})$ and $X \in \mathcal{L}(M^{\perp})$. Hence, it follows that

$$V\omega(X) = X\omega(V), \tag{4.5}$$

since $[V, X] = 0$. Here, using (4.1), (2.3) and (2.6), we have

$$\begin{aligned} \tfrac{3}{2}V\omega(X) = &\quad V[X(\ln f_1)] = V[g(X, \nabla \ln f_1)] \\ = &\quad g(\nabla_V X, \nabla \ln f_1) + g(X, \nabla_V \nabla \ln f_1) \\ = &\quad g\Big(V(\ln f_2)X + X(\ln f_1)V, \nabla \ln f_1\Big) \\ &\quad + g\Big(X, V(\ln f_2)\nabla \ln f_1 + \nabla \ln f_1(\ln f_1)V\Big) \\ = &\quad 2V(\ln f_2)X(\ln f_1)\,. \end{aligned} \tag{4.6}$$

Hence, we obtain

$$V\omega(X) = \tfrac{4}{3}V(\ln f_2)X(\ln f_1). \tag{4.7}$$

On the other hand, using (2.3) and (2.6), we have

$$\begin{aligned} X\omega(V) = &\quad Xg(B, V) = Xg(B^{\theta}, V) \\ = &\quad g(\nabla_X B^{\theta}, V) + g(B^{\theta}, \nabla_X V) \\ = &\quad g\Big(B^{\theta}(\ln f_2)X + X(\ln f_1)B^{\theta}, V\Big) + g\Big(B^{\theta}, V(\ln f_2)X + X(\ln f_1)V\Big) \\ = &\quad 2\omega(V)X(\ln f_1)\,. \end{aligned} \tag{4.8}$$

Namely, we have

$$X\omega(V) = 2\omega(V)X(\ln f_1). \tag{4.9}$$

Now, using (4.5)$\sim$(4.9), we get (4.4). ☐

By Lemma 4.8, we immediately have the following result.

Theorem 4.5 *Let $M =_{f_2} M^{\perp} \times_{f_1} M^{\theta}$ be a warped-twisted product hemislant submanifold with warping function $f_2 \in C^{\infty}(M^{\theta})$ and twisting function f_1 of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then M is a twisted product submanifold in the form $M^{\perp} \times_{f_1} M^{\theta}$ if and only if the Lee vector field B is normal to M^{θ}.*

Proof Let M is a twisted product submanifold in the form $M^{\perp} \times_{f_1} M^{\theta}$, where f_1 is a twisting function. Then, for any $V \in \mathcal{L}(M^{\theta})$, $V(\ln f_2)$=0, since f_2 is a constant. From (4.4), we find $g(B, V) = 0$, for any $V \in \mathcal{L}(M^{\theta})$. So, the Lee vector field B is normal to M^{θ}.

Conversely, if the Lee vector field B is normal to M^{θ}, we have $g(B, V) = 0$, for any $V \in \mathcal{L}(M^{\theta})$. Then, we get $V(\ln f_2) = 0$ from (4.4). So, f_2 is a constant, say $f_2 = c$. Then the induced metric tensor g_M of M has the form $g_M = c^2 g_{\perp} \oplus {f_1}^2 g_{\theta}$, where c is constant and f_1 is the twisting function. Thus, $M = M^{\perp} \times_{f_1} M^{\theta}$ is a twisted product. ☐

By Theorems 4.3 and 4.5, we get the following result.

Theorem 4.6 *Let $M = {}_{f_2}M^{\perp} \times_{f_1} M^{\theta}$ be a warped-twisted product hemislant submanifold with warping function $f_2 \in C^{\infty}(M^{\theta})$ and twisting function f_1 of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then M is a locally direct product manifold if and only if the Lee vector field B is normal to M.*

Proof Let $M = {}_{f_2}M^{\perp} \times_{f_1} M^{\theta}$ be a warped-twisted product hemislant submanifold with warping function $f_2 \in C^{\infty}(M^{\theta})$ and twisting function f_1 of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. If M is a locally direct product, then the functions f_1 and f_2 are constants. In that case, for any $X \in \mathcal{L}(M^{\perp})$ and $V \in \mathcal{L}(M^{\theta})$, we have $g(B, X) = g(B, V) = 0$ from (4.1) and (4.4), respectively. It follows that B is normal to M.

Conversely, let B is normal to M. Then, for any $X \in \mathcal{L}(M^{\perp})$ and $V \in \mathcal{L}(M^{\theta})$, we have $X(\ln f_1) = V(\ln f_2) = 0$. It follows that f_2 is a constant, say $f_2 = c$ and f_1 depends only on the points of M^{θ}. Then the induced metric tensor g_M of M has the form $g_M = c^2 g_{\perp} \oplus {f_1}^2 g_{\theta}$. Hence, we conclude that M is a locally direct product of $(M^{\perp}, \tilde{g_{\perp}})$ and $(M^{\theta}, \tilde{g_{\theta}})$, where $\tilde{g_{\perp}} = c^2 g_{\perp}$ and $\tilde{g_{\theta}} = {f_1}^2 g_{\theta}$. □

By using (3.7) and (4.4), we deduce the following result.

Lemma 4.7 *Let $M = {}_{f_2} M^{\perp} \times_{f_1} M^{\theta}$ be a warped-twisted product hemislant submanifold with warping function $f_2 \in C^{\infty}(M^{\theta})$ and twisting function f_1 of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then we have*

$$g(A_{JX}PV - A_{FPV}X, Y) = \left\{ \cos^2\theta\, \omega(V) + \tfrac{1}{2}\omega(FPV) \right\} g(X, Y) \tag{4.10}$$

for any $X, Y \in \mathcal{L}(M^{\perp})$ and $V \in \mathcal{L}(M^{\theta})$.

By using (3.8) and (4.1), we deduce the following result.

Lemma 4.8 *Let $M = {}_{f_2} M^{\perp} \times_{f_1} M^{\theta}$ be a warped-twisted product hemislant submanifold with warping function $f_2 \in C^{\infty}(M^{\theta})$ and twisting function f_1 of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then we have*

$$g(A_{JX}PV - A_{FPV}X, U) = -\cos^2\theta\, \omega(X) g(V, U) - \tfrac{1}{2}\omega(JX) g(PV, U), \tag{4.11}$$

for any $X \in \mathcal{L}(M^{\perp})$ and $U, V \in \mathcal{L}(M^{\theta})$.

Now, we recall the following two facts to prove the main theorem.

Lemma 4.9 *(Proposition 3-a [16]) Let g be a pseudo-Riemannian metric on the manifold $M = M_1 \times M_2$ and $(\mathcal{D}_1, \mathcal{D}_2)$ are called the canonical foliations. Suppose that $\mathcal{D}_1$ and $\mathcal{D}_2$ intersect perpendicularly everywhere. Then (M, g) is a doubly twisted product ${}_{f_2}M_1 \times_{f_1} M_2$ if and only if $\mathcal{D}_1$ and $\mathcal{D}_2$ are totally umbilic foliations.*

Lemma 4.10 *(Lemma 3.1.1 [13]) Let ${}_{f_2}M_1 \times_{f_1} M_2$ be a doubly twisted product. It is a doubly warped product if and only if the mean curvature vector fields of the canonical foliations are closed.*

Motivated by Lemma 4.9 and Lemma 4.10, we can obtain the following result.

Lemma 4.11 *Let ${}_{f_2}M_1 \times_{f_1} M_2$ be a doubly twisted product. It is a warped-twisted product with warping function $f_2 \in \mathcal{C}^{\infty}(M_2)$ and twisting function f_1 if and only if the mean curvature vector field of canonical foliation $\mathcal{D}_1$ is closed.*

Proof The proof is very similar to the proof of Lemma 2.3 [12], so we omit it. □

We now are ready to prove the main theorem.

Theorem 4.12 *Let M be a hemislant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then M is a locally warped-twisted product submanifold if and only if its shape operator A satisfies the following equation*

$$A_{JX}PV - A_{FPV}X = \left\{ \cos^2\theta\, \omega(V) + \frac{1}{2}\omega(FPV) \right\} X - \cos^2\theta\, \omega(X)V - \frac{1}{2}\omega(JX)PV, \tag{4.12}$$

for $X \in \Gamma(\mathcal{D}^\perp)$ and $V \in \Gamma(\mathcal{D}^\theta)$. Moreover, M is also a locally doubly warped product submanifold.

Proof Let M be a warped-twisted product submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$ of type ${}_{f_2}M^\perp \times_{f_1} M^\theta$. For any $X \in \mathcal{L}(M^\perp)$ and $V \in \mathcal{L}(M^\theta)$, we write

$$A_{JX}PV - A_{FPV}X = \Big(A_{JX}PV - A_{FPV}X \Big)^\perp + \Big(A_{JX}PV - A_{FPV}X \Big)^\theta, \tag{4.13}$$

where $\Big(A_{JX}PV - A_{FPV}X \Big)^\perp$ is the tangent part of $A_{JX}PV - A_{FPV}X$ to $M^\perp$ and $\Big(A_{JX}PV - A_{FPV}X \Big)^\theta$ is the tangent part of $A_{JX}PV - A_{FPV}X$ to M^θ. Hence, for any $Y \in \mathcal{L}(M^\perp)$, using (4.10), we have

$$g(A_{JX}PV - A_{FPV}X,\, Y) = g\Big(\Big\{ \cos^2\theta\, \omega(V) + \frac{1}{2}\omega(FPV) \Big\} X,\, Y \Big).$$

Since $Y \in \mathcal{L}(M^\perp)$ is arbitrary and the metric g is Riemannian, it follows that

$$\Big(A_{JX}PV - A_{FPV}X \Big)^\perp = \Big\{ \cos^2\theta\, \omega(V) + \tfrac{1}{2}\omega(FPV) \Big\} X. \tag{4.14}$$

Similarly, for any $U \in \mathcal{L}(M^\theta)$, using (4.11), we have

$$g(A_{JX}PV - A_{FPV}X,\, U) = g\Big(-\cos^2\theta\, \omega(X)V - \frac{1}{2}\omega(JX)PV,\, U \Big).$$

Since $U \in \mathcal{L}(M^\theta)$ is arbitrary and the metric g is Riemannian, it follows that

$$\Big(A_{JX}PV - A_{FPV}X \Big)^\theta = -\cos^2\theta\, \omega(X)V - \tfrac{1}{2}\omega(JX)PV. \tag{4.15}$$

Thus, by (4.13)–(4.15), we get (4.12).

Conversely, suppose that M is a hemislant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$ such that (4.12) holds. Then, for any $X \in \Gamma(\mathcal{D}^\perp)$ and $U, V \in \Gamma(\mathcal{D}^\theta)$, using (4.12), we deduce that (3.9). Thus, by Theorem 3.2, the slant distribution $\mathcal{D}^\theta$ is integrable. On the other hand, from Theorem 3.4, we already know that the totally real distribution $\mathcal{D}^\perp$ is integrable. Let $M^\perp$ and M^θ be the integral manifolds of $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$, respectively and let denote by $h^{\perp}$ and h^{θ} the second fundamental forms of $M^{\perp}$ and M^{θ} in M, respectively. Then, for any $X, Y \in \Gamma(\mathcal{D}^{\perp})$ and $V \in \Gamma(\mathcal{D}^{\theta})$, using (2.9), we have

$$g(h^{\perp}(X,Y),V) = g(\nabla_X Y, V).$$

Here, if we use (3.7) and (4.12), we find

$$g(h^{\perp}(X,Y),V) = -\tfrac{3}{2}\omega(V)g(X,Y).$$

Using (2.6), we obtain

$$g(h^{\perp}(X,Y),V) = g(-g(X,Y)\tfrac{3}{2}B^{\theta}, V).$$

Hence, we conclude that

$$h^{\perp}(X,Y) = -g(X,Y)\tfrac{3}{2}B^{\theta}.$$

This equation says that $M^{\perp}$ is totally umbilic with the mean curvature vector field $-\frac{3}{2}B^{\theta}$. On the other hand, for any $X \in \Gamma(\mathcal{D}^{\perp})$ and $U, V \in \Gamma(\mathcal{D}^{\theta})$, using (2.9), we have

$$g(h^{\theta}(U,V),X) = g(\nabla_U V, X).$$

Here, if we use (3.8) and (4.12), we find

$$g(h^{\theta}(U,V),X) = -\tfrac{3}{2}\omega(X)g(U,V).$$

Using (2.6), we obtain

$$g(h^{\theta}(U,V),X) = g(-g(U,V)\tfrac{3}{2}B^{\perp}, X).$$

Hence, we conclude that

$$h^{\theta}(U,V) = -g(U,V)\tfrac{3}{2}B^{\perp}.$$

It means that M^{θ} is totally umbilic in M with the mean curvature vector field $-\frac{3}{2}B^{\perp}$.

Next, we prove $B^{\perp}$ and B^{θ} are closed. Let denote by $\omega^{\perp}$ (resp. ω^{θ}) the dual 1-form of $B^{\perp}$ (resp. B^{θ}). For any $X \in \Gamma(\mathcal{D}^{\perp})$, we have $\omega^{\perp}(X) = \omega(X)$. Thus, for $X, Y \in \Gamma(\mathcal{D}^{\perp})$, we obtain

$$d\omega^{\perp}(X,Y) = X\omega^{\perp}(Y) - Y\omega^{\perp}(X) - \omega^{\perp}([X,Y]) = X\omega(Y) - Y\omega(X) - \omega([X,Y]) = d\omega(X,Y).$$

It follows that $d\omega^{\perp} = 0$, since $d\omega = 0$. Namely, $\omega^{\perp}$ is closed. Hence, $B^{\perp}$ is closed, since its dual 1-form is closed. Thus, by Lemma 4.11, M is a locally warped-twisted product submanifold. Moreover, we can prove that B^{θ} is closed in a similar way. Thereby, by Lemma 4.10, M is also a locally doubly warped product submanifold. $\square$

Remark 4.13 *We have just proved that a warped-twisted product hemislant submanifold of a g.c.K. manifold* $(\bar{M}, J, \omega, g)$ *is also a doubly warped product submanifold in Theorem 4.12. Therefore, from now on we will focus on doubly warped product submanifolds of a g.c.K. manifold.*

5. An inequality for doubly warped product mixed geodesic hemislant submanifolds

In this section, we shall establish an inequality for the squared norm of the second fundamental form of a doubly warped product mixed geodesic hemislant submanifold in the form ${}_{f_2}M^{\perp} \times_{f_1} M^{\theta}$, where $M^{\perp}$ is a totally real and M^{θ} is a slant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Note that a general inequality for any doubly warped product submanifold in arbitrary Riemannian manifolds was established in Theorem 3 of [14].

Let ${}_{f_2}M_1 \times_{f_1} M_2$ be a doubly warped product manifold equipped with the metric g defined by

$$g = (f_2 \circ \pi_2)^2 \pi_1^*(g_1) + (f_1 \circ \pi_1)^2 \pi_2^*(g_2). \tag{5.1}$$

Then the covariant derivative formulas (2.2)–(2.4) become

$$\bar{\nabla}_X Y = \nabla^1_X Y - g(X,Y)\bar{\nabla}(\ln f_2 \circ \pi_2), \tag{5.2}$$

$$\bar{\nabla}_V X = \bar{\nabla}_X V = V(\ln f_2 \circ \pi_2)X + X(\ln f_1 \circ \pi_1)V, \tag{5.3}$$

$$\bar{\nabla}_U V = \nabla^2_U V - g(U,V)\bar{\nabla}(\ln f_1 \circ \pi_1), \tag{5.4}$$

for $X, Y \in \mathcal{L}(M_1)$ and $U, V \in \mathcal{L}(M_2)$. It follows that $M_1 \times \{p_2\}$ and $\{p_1\} \times M_2$ are totally umbilical submanifolds with closed mean curvature vector fields in ${}_{f_2}M_1 \times_{f_1} M_2$ [13], where $p_1 \in M_1$ and $p_2 \in M_2$. We say that a doubly warped product is nontrivial if it is neither warped nor a direct product.

Remark 5.1 *[11] For a doubly warped product manifold ${}_{f_2}M_1 \times_{f_1} M_2$, we have*

$$\begin{aligned} \bar{\nabla}(\ln f_1 \circ \pi_1) &= \frac{1}{(f_2 \circ \pi_2)^2}\nabla^1(\ln f_1 \circ \pi_1), \\ \bar{\nabla}(\ln f_2 \circ \pi_2) &= \frac{1}{(f_1 \circ \pi_1)^2}\nabla^2(\ln f_2 \circ \pi_2). \end{aligned} \tag{5.5}$$

In view of the above convenience together with (5.1) and (5.5), the covariant derivative formulas (5.2) and (5.4) become

$$\bar{\nabla}_X Y = \nabla^1_X Y - \frac{(f_2 \circ \pi_2)^2}{(f_1 \circ \pi_1)^2} g_1(X,Y)\nabla^2(\ln f_2 \circ \pi_2), \tag{5.6}$$

$$\bar{\nabla}_U V = \nabla^2_U V - \frac{(f_1 \circ \pi_1)^2}{(f_2 \circ \pi_2)^2} g_2(U,V)\nabla^1(\ln f_1 \circ \pi_1), \tag{5.7}$$

for $X, Y \in \mathcal{L}(M_1)$ and $U, V \in \mathcal{L}(M_2)$.

For more details on doubly warped products, we refer to the papers [11], [12], [13] and [20].

Remark 5.2 *From now on, we will use the same symbol for a warping function f_i and its pullback $f_i \circ \pi_i$ for $i = 1, 2$, i.e. we will put $f_i = f_i \circ \pi_i$.*

Lemma 5.3 *Let M be a doubly warped product hemislant submanifold in the form ${}_{f_2}M^{\perp} \times_{f_1} M^{\theta}$, where $M^{\perp}$ is a totally real and M^{θ} is a slant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. Then, we have*

$$g(h(X,Y),FV) = g(h(V,Y),JX) + \left\{\tfrac{2}{3}PV(\ln f_2) - \tfrac{1}{2}\omega(FV)\right\}g(X,Y), \tag{5.8}$$

$$g(h(U,V),JX) = g(h(U,X),FV) + \tfrac{2}{3}X(\ln f_1)g(PV,U) - \tfrac{1}{2}\omega(JX)g(U,V), \tag{5.9}$$

for $X,Y \in \mathcal{L}(M^{\perp})$ and $U,V \in \mathcal{L}(M^{\theta})$.

Proof Let $X,Y \in \mathcal{L}(M^{\perp})$ and $U,V \in \mathcal{L}(M^{\theta})$, by interchanging V with PV in (4.10) and (4.11), respectively, and using (2.11) and (3.5), we get (5.8) and (5.9) respectively. □

Remark 5.4 *We say that a hemislant submanifold M is* mixed geodesic*, if $h(X,V) = 0$ for $X \in \Gamma(\mathcal{D}^{\perp})$ and $V \in \Gamma(\mathcal{D}^{\theta})$.*

By Remark 5.4 together with (2.6), we have that:

Corollary 5.5 *Let $M =_{f_2} M^{\perp} \times_{f_1} M^{\theta}$ be a doubly warped product mixed geodesic hemislant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. If the Lee vector field B is tangent to M, then Equations (5.8) and (5.9) become*

$$g(h(X,Y),FV) = \tfrac{2}{3}PV(\ln f_2)g(X,Y), \tag{5.10}$$

$$g(h(U,V),JX) = \tfrac{2}{3}X(\ln f_1)g(PV,U), \tag{5.11}$$

respectively, where $X,Y \in \mathcal{L}(M^{\perp})$ and $U,V \in \mathcal{L}(M^{\theta})$.

Let $M =_{f_2} M^{\perp} \times_{f_1} M^{\theta}$ be a (m_1+m_2)-dimensional doubly warped product hemislant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$. We choose a canonical orthonormal basis $\{e_1,...,e_{m_1},\bar{e}_1,...,\bar{e}_{m_2},\ Je_1,...,Je_{m_1},e_1^*,...,e_{m_2}^*,\hat{e_1},...,\hat{e_l}\}$ of $\bar{M}$ such that $\{e_1,...,e_{m_1}\}$ is an orthonormal basis of $\mathcal{D}^{\perp}$, $\{\bar{e}_1,...,\bar{e}_{m_2}\}$ is an orthonormal basis of $\mathcal{D}^{\theta}$, $\{Je_1,...,Je_{m_1}\}$ is an orthonormal basis of $J\mathcal{D}^{\perp}$, $\{e_1^*,...,e_{m_2}^*\}$ is an orthonormal basis of $F\mathcal{D}^{\theta}$ and $\{\hat{e_1},...,\hat{e_l}\}$ is an orthonormal basis of $\overline{\mathcal{D}}$. Here, $m_1 = dim(\mathcal{D}^{\perp})$, $m_2 = dim(\mathcal{D}^{\theta})$ and $l = dim(\overline{\mathcal{D}})$.

Remark 5.6 *In view of (3.6), we can observe that $\{\sec\theta P\bar{e}_1,...,\sec\theta P\bar{e}_{m_2}\}$ is also an orthonormal basis of $\mathcal{D}^{\theta}$ and $\{\csc\theta F\bar{e}_1,...,\csc\theta F\bar{e}_{m_2}\}$ is also an orthonormal basis of $F\mathcal{D}^{\theta}$, where θ is the slant angle of $\mathcal{D}^{\theta}$.*

Theorem 5.7 *Let $M =_{f_2} M^{\perp} \times_{f_1} M^{\theta}$ be a doubly warped product mixed geodesic hemislant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$ such that the Lee vector field B is tangent to M. Then the squared norm of the second fundamental form h of M satisfies*

$$\|h\|^2 \geq m_1 \cot^2\theta \|B^{\theta}\|_{\theta}^2 + m_2(m_2-1)\cos^2\theta\|B^{\perp}\|_{\perp}^2, \tag{5.12}$$

where $m_1 = dim(M^{\perp})$ and $m_2 = dim(M^{\theta})$ and $\|.\|_{\perp}$ (resp. $\|.\|_{\theta}$) is calculated with respect to the metric $g_{\perp}$ (resp. g_{θ}).

Proof By the hypothesis, the squared norm of the second fundamental form h can be written as

$$\|h\|^2 = \|h(\mathcal{D}^\perp,\mathcal{D}^\perp)\|^2 + \|h(\mathcal{D}^\theta,\mathcal{D}^\theta)\|^2.$$

In view of the decomposition (3.4), which can be explicitly written as follows:

$$\begin{aligned}\|h\|^2 = & \sum_{i,j,k=1}^{m_1} g(h(e_i,e_j),Je_k)^2 + \sum_{i,j=1}^{m_1}\sum_{a=1}^{m_2} g(h(e_i,e_j),e_a^*)^2 \\ & + \sum_{a,b=1}^{m_2}\sum_{i=1}^{m_1} g(h(\bar{e}_a,\bar{e}_b),Je_i)^2 + \sum_{a,b,c=1}^{m_1} g(h(\bar{e}_a,\bar{e}_b),e_c^*)^2 \\ & + \sum_{r,s=1}^{m_1+m_2}\sum_{t=1}^{l} g(h(\tilde{e}_r,\tilde{e}_s),\hat{e}_t)^2,\end{aligned} \tag{5.13}$$

where the set $\{\tilde{e}_r\}_{1\le r\le(m_1+m_2)}$ is an orthonormal basis of M. Hence, we get

$$\|h\|^2 \ge \sum_{i,j=1}^{m_1}\sum_{a=1}^{m_2} g(h(e_i,e_j),e_a^*)^2 + \sum_{a,b=1}^{m_2}\sum_{i=1}^{m_1} g(h(\bar{e}_a,\bar{e}_b),Je_i)^2.$$

By Remark 5.6, we arrive at

$$\|h\|^2 \ge \sum_{i,j=1}^{m_1}\sum_{a=1}^{m_2} g(h(e_i,e_j),\csc\theta F\bar{e}_a)^2 + \sum_{a,b=1}^{m_2}\sum_{i=1}^{m_1} g(h(\bar{e}_a,\bar{e}_b),Je_i)^2.$$

Now, using (5.10) and (5.11), we obtain

$$\begin{aligned}\|h\|^2 \ge & \tfrac{4}{9}\csc^2\theta \sum_{i,j=1}^{m_1}\sum_{a=1}^{m_2} (P\bar{e}_a(\ln f_2))^2 g^2(e_i,e_j) \\ & +\tfrac{4}{9}\sum_{a,b=1}^{m_2}\sum_{i=1}^{m_1}(e_i(\ln f_1))^2 g^2(P\bar{e}_a,\bar{e}_b).\end{aligned}$$

Again, by Remark 5.6, we can put $P\bar{e}_a = \cos\theta\acute{e}_c$, where $\{\acute{e}_c\}_{1\le c\le m_2}$ is an orthonormal basis of $\mathcal{D}^\theta$, so the last equation becomes

$$\begin{aligned}\|h\|^2 \ge & \tfrac{4}{9}\cot^2\theta \sum_{i,j=1}^{m_1}\sum_{c=1}^{m_2} (\acute{e}_c(\ln f_2))^2 g^2(e_i,e_j) \\ & +\tfrac{4}{9}\sum_{a,b=1}^{m_2}\sum_{i=1}^{m_1}(e_i(\ln f_1))^2 g^2(P\bar{e}_a,\bar{e}_b).\end{aligned}$$

Here, for $a,b\in\{1,2,...,m_2\}$, we have

$$g(P\bar{e}_a,\bar{e}_b) = \{ \begin{array}{ll} \cos\theta & if \ \ a\ne b, \\ 0 & if \ \ a=b, \end{array}$$

since $\mathcal{D}^\theta$ is a slant distribution with slant angle θ. Thus, by a direct calculation, we obtain the following inequality.

$$\|h\|^2 \ \ge \ \frac{4}{9}\Big\{m_1\cot^2\theta\|\nabla\ln f_2\|^2 + m_2(m_2-1)\cos^2\theta\|\nabla\ln f_1\|^2\Big\}. \tag{5.14}$$

On the other hand, by (2.6) and (5.5), we conclude that

$$B^{\perp} = \frac{2}{3{f_2}^2}\nabla^{\perp}(\ln f_1) \qquad \textit{and} \qquad B^{\theta} = \frac{2}{3{f_1}^2}\nabla^{\theta}(\ln f_2) \tag{5.15}$$

from (4.1) and (4.4), respectively. Hence, using (5.5) and (5.15) in (5.14), we get the inequality (5.12). □

Theorem 5.8 *Let $M =_{f_2} M^{\perp} \times_{f_1} M^{\theta}$ be a doubly warped product mixed geodesic hemislant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$ such that the Lee vector field B is tangent to M. If the invariant subnormal bundle $\overline{\mathcal{D}} = \{0\}$, then the equality sign of (5.12) holds if and only if $A_{JX}Y \in \mathcal{L}(M^{\theta})$ and $A_{FU}V \in \mathcal{L}(M^{\perp})$, where $X, Y \in \mathcal{L}(M^{\perp})$ and $U, V \in \mathcal{L}(M^{\theta})$.*

Proof Under the given hypothesis, we see that the equality sign of (5.12) holds if and only if

$$g(h(\mathcal{D}^{\perp}, \mathcal{D}^{\perp}), J\mathcal{D}^{\perp}) = 0 \qquad \text{and} \qquad g(h(\mathcal{D}^{\theta}, \mathcal{D}^{\theta}), F\mathcal{D}^{\theta}) = 0$$

from (5.13). These are equivalent to

$$g(h(Y, Z), JX) = 0 \qquad \text{and} \qquad g(h(V, W), FU) = 0$$

for $X, Y, Z \in \mathcal{L}(M^{\perp})$ and $U, V, W \in \mathcal{L}(M^{\theta})$. But, with the help of (2.11), we know these conditions hold if and only if

$$A_{JX}Y \in \mathcal{L}(M^{\theta}) \qquad \text{and} \qquad A_{FU}V \in \mathcal{L}(M^{\perp}).$$

□

Theorem 5.9 *Let $M =_{f_2} M^{\perp} \times_{f_1} M^{\theta}$ be a nontrivial doubly warped product mixed geodesic hemislant submanifold of a g.c.K. manifold $(\bar{M}, J, \omega, g)$ such that the Lee vector field B is tangent to M and the invariant subnormal bundle $\overline{\mathcal{D}} = \{0\}$. If the equality sign of (5.12) holds, then M^{θ} is also totally umbilic in the ambient manifold $\bar{M}$.*

Proof Let $\bar{h}^{\theta}$ denote the second fundamental form of M^{θ} in $\bar{M}$. Then, for $a \in \{1, ..., m_2\}$, we have

$$\bar{h}^{\theta}(\bar{e}_a, \bar{e}_a) = h^{\theta}(\bar{e}_a, \bar{e}_a) + h(\bar{e}_a, \bar{e}_a), \tag{5.16}$$

where $\{\bar{e}_1, ..., \bar{e}_{m_2}\}$ is an orthonormal basis of M^{θ} and h^{θ} is the second fundamental form of M^{θ} in M, and h is the second fundamental form of M in $\bar{M}$. Since $M =_{f_2} M^{\perp} \times_{f_1} M^{\theta}$ is a nontrivial doubly warped product, we see that

$$h^{\theta}(\bar{e}_a, \bar{e}_a) = -\tfrac{2}{{f_2}^2}\nabla^{\perp}(\ln f_1) \neq 0,$$

from (5.7). On the other hand, we know $h(\mathcal{D}^{\theta}, \mathcal{D}^{\theta}) \subseteq J\mathcal{D}^{\perp}$ from Theorem 5.8. Thus, we have

$$h(\bar{e}_a, \bar{e}_a) = \sum_{i=1}^{m_1} g(h(\bar{e}_a, \bar{e}_a), Je_i)Je_i,$$

where $\{e_1, ..., e_{m_1}\}$ is an orthonormal basis of $M^\perp$. Here, for each $a \in \{1, ..., m_2\}$ and $i \in \{1, ..., m_1\}$, using (5.11), we find

$$g(h(\bar{e}_a, \bar{e}_a), Je_i) = \frac{2}{3} e_i(\ln f_1) g(P\bar{e}_a, \bar{e}_a) = 0,$$

since $g(P\bar{e}_a, \bar{e}_a) = 0$. Which means that

$$h(\bar{e}_a, \bar{e}_a) = 0,$$

for each $a \in \{1, ..., m_2\}$. It follows that

$$\bar{h}^\theta(\bar{e}_a, \bar{e}_a) = h^\theta(\bar{e}_a, \bar{e}_a),$$

from (5.16). Thus, M^θ is totally umbilic in $\bar{M}$, since it is totally umbilic in M. □

Remark 5.10 *Whether the Lee form ω is exact or not does not change all the results in this paper. Thus, these results also hold for locally conformal Kaehler case.*

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