Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 2663 – 2676
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doi:10.55730/1300-0098.3293

Approximation by sampling Kantorovich series in weighted spaces of functions

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Received: 15.02.2022 • **Accepted/Published Online:** 13.06.2022 • **Final Version:** 05.09.2022

Abstract: This paper studies the convergence of the so-called sampling Kantorovich operators for functions belonging to weighted spaces of continuous functions. This setting allows us to establish uniform convergence results for functions that are not necessarily uniformly continuous and bounded on $\mathbb{R}$. In that context we also prove quantitative estimates for the rate of convergence of the family of the above operators in terms of weighted modulus of continuity. Finally, pointwise convergence results in quantitative form by means of Voronovskaja type theorems have been also established.

Key words: Generalized sampling series, sampling Kantorovich operators, pointwise and uniform convergence, weighted approximation, asymptotic formulas

1. Introduction

The sampling type operators are very useful tools in order to provide an approximate version of the classical Whittaker–Kotel'nikov–Shannon (WKS) sampling theorem [23, 25], and it has been widely studied since 1980s. An approximate version of WKS sampling theorem was developed at RWTH Aachen by P. L. Butzer and his school in the late 1970s. This has been made by the introduction of the generalized sampling series defined by

$$\left(G_w^{\chi} f\right)(x)=\sum_{k\in\mathbb{Z}} f\left(\frac{k}{w}\right)\chi\left(wx-k\right), \quad x\in\mathbb{R}, w>0, \tag{1.1}$$

where $f:\mathbb{R}\to\mathbb{R}$ is any function for which the series (1.1) is convergent for every $x\in\mathbb{R}$, and $\chi:\mathbb{R}\to\mathbb{R}$ (called the *kernel* of the operator) denotes a continuous, discrete approximate identity which satisfies suitable assumptions (see [11] and see [12, 28] for their main properties). The main advantage of the operators G_w^{χ} is to reconstruct a given continuous signal f by its sample values in the form $f\left(k/w\right)$, $k\in\mathbb{Z}$, $w>0$. For a more comprehensive studies, we refer the readers to, among the others, [6, 8, 14–16, 26, 29, 30].

In order to approximate not necessarily continuous function, an L^1-version of the above operators was first introduced in [7], by replacing the sample values in G_w^{χ} with the mean values $w\int_{k/w}^{(k+1)/w} f\left(u\right)du$. The

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2010 *AMS Mathematics Subject Classification:* 41A25, 41A35.

so-called sampling Kantorovich operators are of the form

$$\left(K_w^\chi f\right)(x) := \sum_{k\in\mathbb{Z}} \left[w \int_{k/w}^{(k+1)/w} f(u)\, du \right] \chi(wx-k), \ x \in \mathbb{R}, \tag{1.2}$$

where f is a locally integrable function and χ is a kernel function.

The construction of the operators in (1.2) is the most natural way to obtain operators which can be well-defined also for general measurable, locally integrable functions, hence not necessarily continuous. One can face such a situation, especially, in all concrete applications in which a sample of a given signal f at the point k/w can not be matched exactly. This situation produces a kind of error in the reconstruction of a given signal which is called "time-jitter" error. It is clear that, the operator of type (1.2) allows us to reduce the time-jitter error by using the data taken in a neighborhood of the samples k/w in place of the exact pointwise values at the nodes k/w. Also in practice, in the process of acquiring an image, the sensor that measures the signal always averages the signal at the sensor (pixel) rather than measuring the signal at a point, and therefore the Kantorovich sampling model seems to be the one that appropriately describes what happens in reality. The sampling Kantorovich operators have been studied with respect to many aspects, see [4, 9, 10, 13, 17, 19, 27].

It is well-known that, there exist a natural relation between the generalized sampling series (1.1) and their Kantorovich forms (1.2). To show this relation, let us consider the functions

$$F(x) = \int_0^x f(y)dy$$

and

$$(\sigma_w F)(x) = F\left(x + \frac{1}{w}\right) - F(x). \tag{1.3}$$

In this case, it is easy to see the connection

$$\begin{aligned} \left(K_w^\chi f\right)(x) &= w \sum_{k\in\mathbb{Z}} \chi(wx-k) \left[F\left(\frac{k+1}{w}\right) - F\left(\frac{k}{w}\right) \right] \\ &= w \left(G_w^\chi(\sigma_w F)\right)(x). \end{aligned} \tag{1.4}$$

The studies for the above operators in the continuous setting are generally for functions f belonging to $C_B^U(\mathbb{R})$ (the space of uniformly continuous and bounded functions). Since the sampling series and their forms aim to reconstruct (in some sense) a given continuous signal f by a sequence of their sample values, assuming uniform continuity and boundedness of target function is very restrictive. In order to enlarge the class of functions for which we consider the approximation problems, generalized sampling operators G_w^χ have been studied in weighted spaces of functions in [1].

In the present paper, we investigate the approximation behaviors of generalized sampling Kantorovich operators K_w^χ for functions belonging to weighted space of functions. More precisely, for the above operators the pointwise and uniform convergence are established together with the rate of convergence by involving the weighted modulus of continuity. Finally, we prove quantitative Voronovskaja type theorem for the operators (1.2).

The paper is organized as follows. The first and second sections are devoted to fundamentals of generalized sampling and sampling Kantorovich series, as well as to some auxiliary results. The third section is devoted to well-definiteness of the operators between weighted spaces of functions, together with pointwise and uniform convergence. In Section 4, we present rate of convergence of the family of operators in terms of weighted modulus of continuity. The last section contains a pointwise convergence result in quantitative form by means of a Voronovskaja type theorem.

2. Preliminaries and auxiliary results

A function $\chi : \mathbb{R} \to \mathbb{R}$ is called a kernel function if it satisfies the following assumptions:

(χ1) χ is continuous on $\mathbb{R}$.

(χ2) the discrete algebraic moment of order 0 :

$$m_0(\chi, u) = \sum_{k \in \mathbb{Z}} \chi(u-k) = 1,$$

for every $u \in \mathbb{R}$.

(χ3) there exists $\beta > 0$, such that the discrete absolute moment of order β is finite, i.e.:

$$M_\beta(\chi) = \sup_{u \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |\chi(u-k)| \, |u-k|^\beta < +\infty.$$

The above assumptions are quite natural when sampling type series are studied. Several examples of kernels known in the literature can be found, e.g., in [20].

The following lemma holds (see [7]).

Lemma 2.1 *Let χ be a function satisfying (χ1) and (χ3). For every $\delta > 0$ there holds:*

$$\lim_{w \to +\infty} \sum_{|k-wx| > w\delta} |\chi(wx-k)| = 0,$$

uniformly with respect to $x \in \mathbb{R}$.

From [18, Lemma 2.1. (i)], if χ satisfies the assumptions (χ1) and (χ3), it follows that

$$M_\gamma(\chi) = \sup_{u \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |\chi(u-k)| \, |u-k|^\gamma < +\infty,$$

for every $0 \leq \gamma \leq \beta$.

Now, we introduce the weighted function spaces.

A function $\widetilde{w}$ is said to be a weight function if it is a positive continuous function on the whole real axis $\mathbb{R}$. Here, we consider the weight function

$$\widetilde{w}(x) = \frac{1}{1+x^2}, \ x \in \mathbb{R}.$$

By $B_{\widetilde{w}}(\mathbb{R})$, we denote the space

$$B_{\widetilde{w}}(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} : \sup_{x\in\mathbb{R}} \widetilde{w}(x) |f(x)| \in \mathbb{R} \right\}.$$

Denoting by $C^0(\mathbb{R})$ the space of continuous functions on the whole $\mathbb{R}$, the following natural subspaces of $B_{\widetilde{w}}(\mathbb{R})$ will be used in the rest of the paper:

$$\begin{aligned} C_{\widetilde{w}}(\mathbb{R}) &:= C^0(\mathbb{R}) \cap B_{\widetilde{w}}(\mathbb{R}), \\ C^*_{\widetilde{w}}(\mathbb{R}) &:= \left\{ f \in C_{\widetilde{w}}(\mathbb{R}) : \lim_{x\to\mp+\infty} \widetilde{w}(x) f(x) \in \mathbb{R} \right\}, \\ U_{\widetilde{w}}(\mathbb{R}) &:= \{ f \in C_{\widetilde{w}}(\mathbb{R}) : \widetilde{w}f \text{ is uniformly continuous} \}. \end{aligned}$$

The linear space of functions $B_{\widetilde{w}}(\mathbb{R})$, and its above subspaces are normed spaces with the norm

$$\|f\|_{\widetilde{w}} := \sup_{x\in\mathbb{R}} \widetilde{w}(x) |f(x)|$$

see [3, 5, 21, 22].

The weighted modulus of continuity, considered in [24] and denoted by $\Omega(f;\cdot)$, is defined for $f \in C_{\widetilde{w}}(\mathbb{R})$ by

$$\Omega(f;\delta) = \sup_{|h|<\delta,\ x\in\mathbb{R}} \frac{|f(x+h) - f(x)|}{(1+h^2)(1+x^2)}, \quad \text{for } \delta > 0. \tag{2.1}$$

Some elementary properties of $\Omega(f;\delta)$ are collected in the following lemma.

Lemma 2.2 **([24])** *Let $\delta > 0, x \in \mathbb{R}$. Then,*

(i) $\Omega(f;\delta)$ is an increasing function of δ,

(ii) $\lim_{\delta\to 0^+} \Omega(f;\delta) = 0$ when $f \in C^*_{\widetilde{w}}(\mathbb{R})$,

(iii) for each $\lambda > 0$ and $f \in C_{\widetilde{w}}(\mathbb{R})$,

$$\Omega(f;\lambda\delta) \le 2(1+\lambda)\left(1+\delta^2\right)\Omega(f;\delta). \tag{2.2}$$

Remark 2.3 **([1])** *Using the inequality (2.2) with $\lambda = \frac{|y-x|}{\delta}$, x, $y \in \mathbb{R}$, $\delta > 0$, we have*

$$\begin{aligned} |f(y) - f(x)| &\le 2\left(1 + \frac{|y-x|}{\delta}\right)(1+\delta^2)(1+x^2)(1+(y-x)^2)\Omega(f;\delta) \\ &\le \begin{cases} 4(1+\delta^2)^2(1+x^2)\Omega(f;\delta), & |y-x| \le \delta, \\ 4(1+\delta^2)^2(1+x^2)\Omega(f;\delta)\frac{|y-x|^3}{\delta^3}, & |y-x| > \delta. \end{cases} \end{aligned}$$

Therefore, combining both the above cases, we obtain

$$|f(y) - f(x)| \le 4(1+\delta^2)^2(1+x^2)\Omega(f;\delta)\left(1 + \frac{|y-x|^3}{\delta^3}\right).$$

Moreover, choosing $0 < \delta \leq 1$ *we get*

$$|f(y) - f(x)| \leq 16(1 + x^2)\Omega(f;\delta)\left(1 + \frac{|y-x|^3}{\delta^3}\right), \tag{2.3}$$

for every $f \in C_{\widetilde{w}}(\mathbb{R})$, $x, y \in \mathbb{R}$.

3. Pointwise and uniform convergence of K_w^χ in weighted spaces

This section is devoted to pointwise and uniform convergence of the series $K_w^\chi f$ for functions f belonging to $C_{\widetilde{w}}(\mathbb{R})$ and $U_{\widetilde{w}}(\mathbb{R})$, respectively.

From now on, we will consider kernels χ according with the definition given in Section 2.

Let us first prove well-definiteness of sampling Kantorovich operators acting on weighted spaces of functions.

Theorem 3.1 *Let* χ *be a kernel satisfying* $(\chi 3)$ *for* $\beta = 2$. *Then, for a fixed* $w > 0$, *the operator* K_w^χ *is a linear operator from* $B_{\widetilde{w}}(\mathbb{R})$ *to* $B_{\widetilde{w}}(\mathbb{R})$ *and its operator norm turns out to be:*

$$\|K_w^\chi\|_{B_{\widetilde{w}}(\mathbb{R})\to B_{\widetilde{w}}(\mathbb{R})} \leq M_0(\chi)\left(1 + \frac{1}{3w^2} + \frac{1}{w}\right) + M_1(\chi)\left(\frac{1}{w} + \frac{1}{w^2}\right) + \frac{M_2(\chi)}{w^2}. \tag{3.1}$$

Proof For a fixed $w > 0$ and $x \in \mathbb{R}$, using the definition of the operators K_w^χ given in (1.2), we can write what follows

$$\begin{aligned}
|(K_w^\chi f)(x)| &\leq \sum_{k\in\mathbb{Z}} |\chi(wx-k)|\, w \int_{k/w}^{(k+1)/w} |f(y)| \frac{(1+y^2)}{1+y^2} dy \\
&\leq \|f\|_{\tilde{w}} \sum_{k\in\mathbb{Z}} |\chi(wx-k)|\, w \int_{k/w}^{(k+1)/w} \left(1+y^2\right) dy \\
&= \|f\|_{\tilde{w}} \sum_{k\in\mathbb{Z}} |\chi(wx-k)|\, w \left[\frac{1}{w} + \frac{1}{3}\left(\frac{k+1}{w}\right)^3 - \frac{1}{3}\left(\frac{k}{w}\right)^3\right] \\
&= \|f\|_{\tilde{w}} \sum_{k\in\mathbb{Z}} |\chi(wx-k)| \left[1 + \frac{1}{3w^2} + \frac{k^2}{w^2} + \frac{k}{w^2}\right] \\
&= \|f\|_{\tilde{w}} \sum_{k\in\mathbb{Z}} |\chi(wx-k)| \left[1 + \frac{1}{3w^2} + \left(\frac{k}{w} - x\right)^2 + 2x\left(\frac{k}{w} - x\right) + x^2 + \frac{1}{w}\left(\frac{k}{w} - x\right) + \frac{x}{w}\right] \\
&\leq \|f\|_{\tilde{w}} \sum_{k\in\mathbb{Z}} |\chi(wx-k)| \left(1+x^2\right) \left[1 + \frac{1}{3w^2} + \frac{|k-wx|^2}{w^2} + \frac{|k-wx|}{w} + \frac{|k-wx|}{w^2} + \frac{1}{w}\right] \\
&\leq \|f\|_{\tilde{w}} \left(1+x^2\right) \left[M_0(\chi)\left(1 + \frac{1}{3w^2} + \frac{1}{w}\right) + M_1(\chi)\left(\frac{1}{w} + \frac{1}{w^2}\right) + \frac{M_2(\chi)}{w^2}\right]
\end{aligned}$$

which implies that

$$\frac{|(K_w^\chi f)(x)|}{1+x^2} \leq \|f\|_{\tilde{w}} \left[M_0(\chi)\left(1 + \frac{1}{3w^2} + \frac{1}{w}\right) + M_1(\chi)\left(\frac{1}{w} + \frac{1}{w^2}\right) + \frac{M_2(\chi)}{w^2}\right] \tag{3.2}$$

for every $x \in \mathbb{R}$. According to the assumption $(\chi 3)$, since the fact that $M_2(\chi) < +\infty$ implies $M_j(\chi) < +\infty$ for $j = 0, 1$ we have $\|K_w^\chi f\|_{\widetilde{w}} < +\infty$ which means that $K_w^\chi f \in B_{\widetilde{w}}(\mathbb{R})$. Further, (3.1) can be directly obtained by taking supremum over $x \in \mathbb{R}$ and the supremum with respect to $f \in B_{\widetilde{w}}(\mathbb{R})$ with $\|f\|_{\widetilde{w}} \leq 1$ in (3.2). □

The main theorem of this section can be stated as follows:

Theorem 3.2 *Let χ be a kernel satisfying $(\chi 3)$ for $\beta = 2$ and $f \in C_{\widetilde{w}}(\mathbb{R})$ be fixed. Then,*

$$\lim_{w \to +\infty} (K_w^\chi f)(x) = f(x), \tag{3.3}$$

holds for every $x \in \mathbb{R}$. Moreover, if $f \in U_{\widetilde{w}}(\mathbb{R})$, then

$$\lim_{w \to +\infty} \|K_w^\chi f - f\|_{\widetilde{w}} = 0. \tag{3.4}$$

Proof Firstly, for all $x \in \mathbb{R}$, and $w > 0$, it is easy to see that the inequality

$$|f(u) - f(x)| \leq \widetilde{w}(u) |f(u)| \left| \frac{1}{\widetilde{w}(u)} - \frac{1}{\widetilde{w}(x)} \right| + \frac{1}{\widetilde{w}(x)} |\widetilde{w}(u) f(u) - \widetilde{w}(x) f(x)|$$

holds. Using the above inequality and $(\chi 2)$, we can write what follows:

$$\begin{aligned}
|(K_w^\chi f)(x) - f(x)| &\leq \sum_{k \in \mathbb{Z}} |\chi(wx - k)| \, w \int_{k/w}^{(k+1)/w} |f(u) - f(x)| \, du \\
&\leq \sum_{k \in \mathbb{Z}} |\chi(wx - k)| \, w \int_{k/w}^{(k+1)/w} \left[\widetilde{w}(u) |f(u)| \left| \frac{1}{\widetilde{w}(u)} - \frac{1}{\widetilde{w}(x)} \right| \right. \\
&\left. + \frac{1}{\widetilde{w}(x)} |\widetilde{w}(u) f(u) - \widetilde{w}(x) f(x)| \right] du \\
&\leq \|f\|_{\widetilde{w}} \sum_{k \in \mathbb{Z}} |\chi(wx - k)| \, w \int_{k/w}^{(k+1)/w} \left| u^2 - x^2 \right| du \\
&+ \frac{1}{\widetilde{w}(x)} \sum_{k \in \mathbb{Z}} |\chi(wx - k)| \, w \int_{k/w}^{(k+1)/w} |\widetilde{w}(u) f(u) - \widetilde{w}(x) f(x)| \, du \\
&:= I_1 + I_2.
\end{aligned}$$

Let us first estimate I_1. By a straightforward computation, we get

$$\begin{aligned} I_1 &\le \|f\|_{\widetilde{w}} \sum_{k\in\mathbb{Z}} |\chi(wx-k)|\, w \int_{k/w}^{(k+1)/w} \left[(u-x)^2 + 2|x|\,|u-x|\right] du \\ &\le \|f\|_{\widetilde{w}} \sum_{k\in\mathbb{Z}} |\chi(wx-k)|\, w \left\{ \frac{1}{3}\left[\left|\frac{k+1}{w}-x\right|^3 - \left|\frac{k}{w}-x\right|^3\right] + 2|x| \int_{k/w}^{(k+1)/w} |u-x|\, du \right\} \\ &\le \|f\|_{\widetilde{w}} \sum_{k\in\mathbb{Z}} |\chi(wx-k)|\, w \left\{ \frac{1}{3}\left[\frac{3}{w}\left|\frac{k}{w}-x\right|^2 + \frac{3}{w^2}\left|\frac{k}{w}-x\right| + \frac{1}{w^3}\right] \right. \\ &\quad \left. + 2|x| \left[\int_{k/w}^{(k+1)/w} \left(u-\frac{k}{w}\right) du + \int_{k/w}^{(k+1)/w} \left|\frac{k}{w}-x\right| du\right] \right\} \\ &= \|f\|_{\widetilde{w}} \sum_{k\in\mathbb{Z}} |\chi(wx-k)| \left\{ \frac{|k-wx|^2}{w^2} + \frac{|k-wx|}{w^2} + \frac{1}{3w^2} + 2|x|\left[\frac{1}{2w} + \frac{|k-wx|}{w}\right] \right\} \\ &\le \|f\|_{\widetilde{w}} \left[\frac{M_2(\chi)}{w^2} + \frac{M_1(\chi)}{w}\left(\frac{1}{w} + 2|x|\right) + \frac{M_0(\chi)}{w}\left(\frac{1}{3w} + |x|\right)\right]. \end{aligned} \tag{3.5}$$

Now let us consider I_2. Let $x \in \mathbb{R}$ and $\varepsilon > 0$ be fixed. Since f is continuous at x, $\widetilde{w}f$ is also continuous at x, hence there exists $\delta > 0$ such that $|\widetilde{w}(u) f(u) - \widetilde{w}(x) f(x)| < \varepsilon$ whenever $|u-x| < \delta$, $u \in \mathbb{R}$. Hence we can write

$$\begin{aligned} I_2 &= \frac{1}{\widetilde{w}(x)} \sum_{k\in\mathbb{Z}} |\chi(wx-k)|\, w \int_{k/w}^{(k+1)/w} |\widetilde{w}(u) f(u) - \widetilde{w}(x) f(x)|\, du \\ &= \frac{1}{\widetilde{w}(x)} \left\{ \sum_{|k-wx|\le \frac{w\delta}{2}} + \sum_{|k-wx|> \frac{w\delta}{2}} \right\} |\chi(wx-k)|\, w \int_{k/w}^{(k+1)/w} |\widetilde{w}(u) f(u) - \widetilde{w}(x) f(x)|\, du \\ &=: I_{2,1} + I_{2,2}. \end{aligned}$$

Let w' be fixed in a such way that $\frac{1}{w} < \frac{\delta}{2}$ for every $w > w'$. For $u \in \left[\frac{k}{w}, \frac{k+1}{w}\right]$, if $|k-wx| \le \frac{w\delta}{2}$, we have

$$|u-x| \le |u-k/w| + |k/w-x| \le \frac{1}{w} + \frac{\delta}{2} < \delta, \quad \text{for } w > w'.$$

Thus we obtain

$$I_{2,1} \le \frac{\varepsilon}{\widetilde{w}(x)} \sum_{|k-wx|\le \frac{w\delta}{2}} |\chi(wx-k)| < \frac{\varepsilon M_0(\chi)}{\widetilde{w}(x)}. \tag{3.6}$$

On the other hand, by using Lemma 2.1 we have for sufficiently large $w > 0$ that

$$I_{2,2} \le \frac{2\|f\|_{\widetilde{w}}}{\widetilde{w}(x)} \sum_{|k-wx|> \frac{w\delta}{2}} |\chi(wx-k)| \le \frac{2\|f\|_{\widetilde{w}}}{\widetilde{w}(x)} \varepsilon. \tag{3.7}$$

Combining the inequalities (3.5), (3.6), and (3.7) we have

$$|(K_w^\chi f)(x) - f(x)| \leq \|f\|_{\widetilde{w}} \left[\frac{M_2(\chi)}{w^2} + \frac{M_1(\chi)}{w} \left(\frac{1}{w} + 2|x| \right) + \frac{M_0(\chi)}{w} \left(\frac{1}{3w} + |x| \right) \right] + \frac{\varepsilon}{\widetilde{w}(x)} \left(M_0(\chi) + 2\|f\|_{\widetilde{w}} \right) \tag{3.8}$$

Finally, taking the limit of both sides as $w \to +\infty$, the assertion (3.3) follows.

Now let us prove the assertion (3.4). For functions $f \in U_{\widetilde{w}}(\mathbb{R})$, let us follow the steps of above proof and replace δ with the corresponding parameter of the uniform continuity of $\widetilde{w}f$. Also considering inequality (3.8) we can write

$$\begin{aligned} \widetilde{w}(x)|(K_w^\chi f)(x) - f(x)| &\leq \frac{\widetilde{w}(x)\|f\|_{\widetilde{w}}}{w^2} M_2(\chi) + \frac{\|f\|_{\widetilde{w}}}{w} M_1(\chi)\widetilde{w}(x) \left(\frac{1}{w} + 2|x| \right) \\ &+ \frac{\|f\|_{\widetilde{w}} M_0(\chi)}{w} \widetilde{w}(x) \left(\frac{1}{3w} + |x| \right) + \varepsilon \left(M_0(\chi) + 2\|f\|_{\widetilde{w}} \right) \\ &\leq \frac{\|f\|_{\widetilde{w}}}{w^2} M_2(\chi) + \frac{\|f\|_{\widetilde{w}}}{w} M_1(\chi) \left(\frac{1+2w}{w} \right) + \frac{\|f\|_{\widetilde{w}}}{w} M_0(\chi) \left(\frac{1+3w}{3w} \right) \\ &+ \varepsilon \left(M_0(\chi) + 2\|f\|_{\widetilde{w}} \right), \end{aligned} \tag{3.9}$$

and passing to the supremum in (3.9) over $x \in \mathbb{R}$ we have (3.4) for $w \to +\infty$. This completes the proof. □

4. Rate of convergence for K_w^χ

In this section, we propose to obtain the rate of convergence for the family of generalized sampling Kantorovich operators via weighted modulus of continuity (2.1).

Theorem 4.1 *Let χ be a kernel satisfying $(\chi 3)$ with $\beta = 3$. For $f \in C_{\widetilde{w}}^*(\mathbb{R})$, we have*

$$\|K_w^\chi f - f\|_{\widetilde{w}} \leq 32\,\Omega(f; w^{-1}) \left[M_0(\chi) + 2M_3(\chi) \right],$$

for every $w \geq 1$.

Proof For every $x \in \mathbb{R}$, using $(\chi 2)$ and (2.3), we can write what follows:

$$|(K_w^\chi f)(x) - f(x)| \leq \sum_{k \in \mathbb{Z}} |\chi(wx - k)| w \int_{k/w}^{(k+1)/w} |f(u) - f(x)|\, du$$

$$\leq 16(1 + x^2)\Omega(f; \delta) \sum_{k \in \mathbb{Z}} |\chi(wx - k)| w \int_{k/w}^{(k+1)/w} \left(1 + \frac{|u - x|^3}{\delta^3} \right) du,$$

for every $\delta \leq 1$. Now, by standard computations we immediately get:

$$|(K_w^\chi f)(x) - f(x)| \leq 16(1 + x^2)\Omega(f; \delta) \sum_{k \in \mathbb{Z}} |\chi(wx - k)| \times$$

$$w\left[\frac{1}{w}+\frac{4}{\delta^3}\int_{k/w}^{(k+1)/w}\left(\left|u-\frac{k}{w}\right|^3+\left|\frac{k}{w}-x\right|^3\right)du\right]$$

$$\leq 16(1+x^2)\Omega(f;\delta)\sum_{k\in\mathbb{Z}}|\chi(wx-k)|\left[1+\frac{1}{(w\delta)^3}+\frac{4}{(w\delta)^3}|wx-k|^3\right]$$

$$\leq 16(1+x^2)\Omega(f;\delta)\left[M_0(\chi)\left(1+\frac{1}{(w\delta)^3}\right)+M_3(\chi)\frac{4}{(w\delta)^3}\right].$$

Now, setting $\delta = w^{-1}$, $w \geq 1$, and taking the supremum for $x \in \mathbb{R}$ we immediately obtain:

$$\|K_w^{\chi}f - f\|_{\widetilde{w}} \leq 32\,\Omega(f;w^{-1})\left[M_0(\chi)+2M_3(\chi)\right],$$

for every $w \geq 1$. □

5. Voronovskaja type formulae

Let $j \in \mathbb{N}$, then the algebraic moment of order j of a kernel χ is defined by

$$m_j\left(\chi,u\right)=\sum_{k\in\mathbb{Z}}\chi\left(u-k\right)\left(k-u\right)^j.$$

In order to present Voronovskaja theorem in quantitative form, we need of a further assumption on kernel function χ, i.e. there exists $r \in \mathbb{N}$ such that for every $j \in \mathbb{N}_0, j \leq r$, there holds:

(χ4) $m_j\left(\chi,u\right) =: m_j\left(\chi\right) \in \mathbb{R}$, is independent of u.

Now, we recall the estimation for the Peano's remainder in the Taylor expansion of the corresponding function.

Remark 5.1 (**[1]**) *Let $f \in C^r\left(\mathbb{R}\right)$, $r \in \mathbb{N}$, the space of r-times continuously differentiable functions. The remainder in Taylor's formula at the point $x \in \mathbb{R}$ is given by*

$$R_r\left(f;t,x\right)=f\left(t\right)-\sum_{k=0}^{r}\frac{f^{(k)}\left(x\right)}{k!}\left(t-x\right)^k$$

which can be written in the form

$$R_r\left(f;t,x\right)=\frac{\left(t-x\right)^r}{r!}\left(f^{(r)}\left(\xi\right)-f^{(r)}\left(x\right)\right), \tag{5.1}$$

where ξ is a number lying between t and x.

According to inequality (2.3), with similar method presented in [2], we can easily have the estimate

$$\left|R_r\left(f;t,x\right)\right| \leq \frac{16}{r!}\left(1+x^2\right)\Omega(f^{(r)};\delta)\left(\left|t-x\right|^r+\frac{\left|t-x\right|^{r+3}}{\delta^3}\right). \tag{5.2}$$

Theorem 5.2 *Let χ be a kernel satisfying the assumptions $(\chi 1)$, $(\chi 2)$ and $(\chi 3)$ for $\beta = r+3, r \in \mathbb{N}$, where r is the parameter of condition $(\chi 4)$. Then, for $f \in C^r(\mathbb{R})$ such that $f^{(r)} \in C^*_{\widetilde{w}}(\mathbb{R})$, there holds*

$$\left| w\left[\left(K_w^\chi f\right)(x) - f(x)\right] - \sum_{j=1}^{r} \frac{f^{(j)}(x)}{j! w^{j-1}} \sum_{l=0}^{j} \binom{j}{l} \frac{m_l(\chi)}{(j-l+1)} \right|$$
$$\leq \frac{2^{r+3}}{w^r r!}\left(1+x^2\right)\Omega\left(f^{(r)}, w^{-1}\right)\left[M_r(\chi) + \left(\frac{M_0(\chi)}{r+1}\right) + 8M_{r+3}(\chi) + \frac{8M_0(\chi)}{r+4}\right].$$

Proof Since $f^{(r)} \in C^*_{\widetilde{w}}(\mathbb{R})$, using Taylor expansion of f at a point $x \in \mathbb{R}$ and by the definition of $K_w^\chi f$, we immediately write

$$\left(K_w^\chi f\right)(x) = \sum_{k\in\mathbb{Z}} \chi(wx-k)\, w \int_{k/w}^{(k+1)/w} \sum_{j=0}^{r} \frac{f^{(j)}(x)}{j!}(t-x)^j\, dt$$
$$+ \sum_{k\in\mathbb{Z}} \chi(wx-k)\, w \int_{k/w}^{(k+1)/w} R_r(f;t,x)\, dt$$
$$:= I_1 + I_2.$$

Let us first consider I_1. In view of Binomial expansion, we obtain

$$I_1 = \sum_{j=0}^{r} \frac{f^{(j)}(x)}{j!} \sum_{k\in\mathbb{Z}} \chi(wx-k)\, w \int_{k/w}^{(k+1)/w} \sum_{l=0}^{j} \binom{j}{l}\left(\frac{k}{w}-x\right)^l \left(t-\frac{k}{w}\right)^{j-l} dt$$
$$= \sum_{j=0}^{r} \frac{f^{(j)}(x)}{j!} \sum_{l=0}^{j} \binom{j}{l} \frac{1}{w^{l-1}} \sum_{k\in\mathbb{Z}} \chi(wx-k)(k-wx)^l \int_{k/w}^{(k+1)/w} \left(t-\frac{k}{w}\right)^{j-l} dt$$
$$= \sum_{j=0}^{r} \frac{f^{(j)}(x)}{j! w^j} \sum_{l=0}^{j} \binom{j}{l} \frac{1}{(j-l+1)} m_l(\chi, wx)$$
$$= f(x) + \frac{1}{w} \sum_{j=1}^{r} \frac{f^{(j)}(x)}{j! w^{j-1}} \sum_{l=0}^{j} \binom{j}{l} \frac{1}{(j-l+1)} m_l(\chi, wx). \tag{5.3}$$

On the other hand, concerning I_2 by the inequality (5.2), we have

$$|I_2| \leq \frac{16w}{r!}\left(1+x^2\right)\Omega\left(f^{(r)},\delta\right)\sum_{k\in\mathbb{Z}}|\chi(wx-k)|\int_{k/w}^{(k+1)/w}\left[|t-x|^r+\frac{|t-x|^{r+3}}{\delta^3}\right]dt$$

$$\leq \frac{16w}{r!}\left(1+x^2\right)\Omega\left(f^{(r)},\delta\right)\left\{\sum_{k\in\mathbb{Z}}|\chi(wx-k)|\int_{k/w}^{(k+1)/w}\left[2^{r-1}\left(\left|t-\frac{k}{w}\right|^r+\left|\frac{k}{w}-x\right|^r\right)\right]dt\right.$$

$$\left.+\frac{1}{\delta^3}\sum_{k\in\mathbb{Z}}|\chi(wx-k)|\int_{k/w}^{(k+1)/w}\left[2^{r+2}\left(\left|t-\frac{k}{w}\right|^{r+3}+\left|\frac{k}{w}-x\right|^{r+3}\right)\right]dt\right\}$$

$$=\frac{2^{r+3}}{w^r r!}\left(1+x^2\right)\Omega\left(f^{(r)},\delta\right)\left\{\frac{1}{(r+1)}\sum_{k\in\mathbb{Z}}|\chi(wx-k)|+\sum_{k\in\mathbb{Z}}|\chi(wx-k)|\,|k-wx|^r\right.$$

$$\left.+\frac{8}{\delta^3w^3(r+4)}\sum_{k\in\mathbb{Z}}|\chi(wx-k)|+\frac{8}{w^3\delta^3}\sum_{k\in\mathbb{Z}}|\chi(wx-k)|\,|k-wx|^{r+3}\right\}$$

Now, choosing $\delta=w^{-1}$, we have

$$|I_2| \leq \frac{2^{r+3}}{w^r r!}\left(1+x^2\right)\Omega\left(f^{(r)},w^{-1}\right)\left[\frac{M_0(\chi)}{r+1}+M_r(\chi)+\frac{8M_0(\chi)}{r+4}+8M_{r+3}(\chi)\right].$$

Hence, by substituting I_1 and I_2 in (5.3) and by using $(\chi 4)$, we conclude

$$\left|w\left[\left(K_w^\chi f\right)(x)-f(x)\right]-\sum_{j=1}^{r}\frac{f^{(j)}(x)}{j!w^{j-1}}\sum_{l=0}^{j}\binom{j}{l}\frac{m_l(\chi)}{(j-l+1)}\right|$$

$$\leq\frac{2^{r+3}}{w^{r-1}r!}\left(1+x^2\right)\Omega\left(f^{(r)},w^{-1}\right)\left[\frac{M_0(\chi)}{r+1}+M_r(\chi)+\frac{8M_0(\chi)}{r+4}+8M_{r+3}(\chi)\right]$$

which is the desired. □

Corollary 5.3 *Under the assumption of Theorem 5.2, in view of Lemma 2.2 (ii), we have a qualitative form of the asymptotic formula for K_w^χ by f', i.e. for $r=1$*

i)

$$\lim_{w\to+\infty} w\left[\left(K_w^\chi f\right)(x)-f(x)\right]=f'(x)\,m_1(\chi)+\frac{f'(x)}{2}.$$

ii) In addition to the assumptions of Theorem 5.2, if we also assume

$$\sum_{l=0}^{j}\binom{j}{l}\frac{m_l(\chi)}{(j-l+1)}=0, \qquad j=1,...,r-1, \tag{5.4}$$

then we have

$$\left| w^r \left[\left(K_w^\chi f \right)(x) - f(x) \right] - \frac{f^{(r)}(x)}{r!} \sum_{l=0}^{r} \binom{r}{l} \frac{m_l(\chi)}{(r-l+1)} \right|$$
$$\leq \frac{2^{r+3}}{r!} \left(1 + x^2\right) \Omega \left(f^{(r)}, w^{-1} \right) \left[M_r(\chi) + \left(\frac{M_0(\chi)}{r+1} \right) + 8M_{r+3}(\chi) + \frac{8M_0(\chi)}{r+4} \right]$$

and

$$\lim_{w \to +\infty} w^r \left[\left(K_w^\chi f \right)(x) - f(x) \right] = \frac{f^{(r)}(x)}{r!} \sum_{l=0}^{r} \binom{r}{l} \frac{m_l(\chi)}{(r-l+1)}.$$

Examples of kernels satisfying (5.4) can be easily constructed following a strategy similar to that considered in [12].

Funding

D. Costarelli and G. Vinti have been partially supported within the project ”Metodiche di Imaging non invasivo mediante angiografia OCT sequenziale per lo studio delle Retinopatie degenerative dell'Anziano (M.I.R.A.)”, funded by Fondazione Cassa di Risparmio di Perugia (FCRP) 2019. Further, G. Vinti has been also partially supported by the projects: (1) Ricerca di Base 2019 dell'Università degli Studi di Perugia - ”Integrazione, Approssimazione, Analisi Nonlineare e loro Applicazioni”, (2) ”Metodi e processi innovativi per lo sviluppo di una banca di immagini mediche per fini diagnostici” funded by the FCRP, 2018.

T. Acar, O. Alagöz, M. Turgay, and G. Vinti have been supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK) (3501-Project 119F263).

Acknowledgments

The authors D. Costarelli and G. Vinti are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM), of the network RITA (Research ITalian network on Approximation), and of the UMI group “Teoria dell'Approssimazione e Applicazioni (T.A.A.)”.

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