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A Fredholm theory on Krein spaces and its application to Weyl-type theorems and homogeneous equations

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Abstract: In this paper, we review the approach presented by An and Heo on the study of Weyl-type theorems for self-adjoint operators on Krein spaces and show that this approach is not appropriate due to a fallacy. Motivated by this fact, we define a new modification of the kernel of a bounded linear operator on a Krein space, namely J-kernel, which allows us to successfully introduce a Fredholm theory in this context and study some variations of Weyl-type theorems for bounded linear operators defined on these spaces. In addition, we will describe the J-index in terms of solution sets of homogeneous equations.

Key words: Fredholm theory, Krein space, Weyl's theorem, homogeneous equation

1. Introduction

During the first decade of the 20th century, Weyl [26] studied the spectra of all compact perturbations of self-adjoint operators on Hilbert spaces and found that their intersection consists precisely of those points of the spectrum that are not isolated eigenvalues with finite multiplicity. This property is now known as Weyl's theorem and versions of it have been extended to hyponormal operators [9, 10], seminormal operators [6], Toeplitz operators [10] and to operators on Banach spaces [10, 18]; as well as some variants have been discussed by Harte and Lee [16], and Rakocevic [23].

In 2018, An and Heo [3] introduced a modification of the kernel of bounded linear operators on Krein spaces, namely J-kernel, to study Fredholm theory and some versions of Weyl-type theorems on these spaces. However, this idea has a flaw, because in general, the J-kernel they defined is not linear, so the notion of dimension of the J-kernel does not make sense and this prevents the successful development of the Fredholm theory and hence establish Weyl-type theorems in this context.

In order to satisfactorily study some Weyl-type theorems for bounded linear operators on Krein spaces, we will define a new modification of the kernel of a linear operator, called J-kernel, which is linear (because we build it from the classical kernel of a given operator) and that also allows recovering the classical theory when the fundamental symmetry is the identity. From this new definition, we introduce Fredholm, Weyl and Browder

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operators on Krein spaces, and we also give some conditions to guarantee that Weyl's and Browder's theorems hold for certain classes of operators. It is important to point out that the J-kernel of an operator T is built from all the vectors that T transforms into negative vectors, thus in a certain way its dimension measures how much the operator is "affected" by the indefinite inner product. This paper is organized as follows. In Section 1, we briefly review some notions of bounded operators on a Krein space and review local spectral theory of bounded operators. In Section 2, we introduce the definition of J-kernel of a operator and the notions of J-Fredholm, J-Weyl and J-Browder operators. We finish Section 2 with the study of J-Weyl's theorem and J-Browder's theorem for J-self-adjoint operators and J-unitary operators. In Section 3, we present an application that relates the J-kernel of an operator with the solutions of a certain homogeneous equation.

2. Fredhom theory on Krein spaces

2.1. Krein spaces

Spaces with indefinite inner product arise naturally in physics, for instance in special relativistic [20], quantum field theory [12, 13, 21] and quantum mechanics [17]. Krein spaces, i.e. complete indefinite inner product spaces, were formally defined by L. Pontrjagin [22] and Ju. Ginzburg [15] and their properties have been investigated by several mathematicians [4, 5, 7, 8, 25]. We briefly review some notions of Krein spaces, which are used in this paper and refer [5, 7, 8] for more detailed information. We write $(\mathcal{K}, [\cdot,\cdot], J)$ for a Krein space with fundamental decomposition $\mathcal{K}^+ \oplus \mathcal{K}^-$ and fundamental symmetry J that fulfills

$$J(x^+ + x^-) = x^+ - x^-, \qquad x^+ + x^- \in \mathcal{K}^+ \oplus \mathcal{K}^-. \tag{2.1}$$

Therefore, the J-inner product

$$\langle x^+ + x^-, y^+ + y^- \rangle_J := [x^+ + x^-, J(y^+ + y^-)] = [x^+, y^+] - [x^-, y^-], \quad x^\pm, y^\pm \in \mathcal{K}^\pm, \tag{2.2}$$

makes $(\mathcal{K}, \langle\cdot,\cdot\rangle_J)$ turn out to be a Hilbert space. Topological notions in $\mathcal{K}$ are determined from the topology induced by the J-norm, which is defined as

$$\|x\|_J := \sqrt{\langle x, x\rangle_J} = \sqrt{[x, Jx]}, \qquad x \in \mathcal{K}.$$

Thus, $\mathcal{K}^+ \oplus \mathcal{K}^-$ becomes the orthogonal sum of Hilbert spaces. For sake of notational simplicity, in some situations we write shortly $\mathcal{K}$ for a Krein space $(\mathcal{K}, [\cdot,\cdot], J)$, and $\mathcal{H}$ for a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. The set of all bounded linear operators on a Krein space $\mathcal{K}$ is denoted by $L(\mathcal{K})$. The unique J-adjoint operator $T^{[*]} : \mathcal{K} \to \mathcal{K}$ of a bounded linear operator $T : \mathcal{K} \to \mathcal{K}$ is given by

$$[Tx, y] = [x, T^{[*]}y] \quad \text{for all} \quad x, y \in \mathcal{K}.$$

In this way, the notions of J-self-adjoint, J-unitary and J-normal operators are taken with respect to the indefinite inner product, and the definition of self-adjoint, unitary and normal operators are taken with respect to the J-inner product associated. Furthermore, we have the following relation

$$T^{[*]} = JT^*J, \qquad T \in L(\mathcal{K}),$$

where T^* denotes the adjoint of T with respect to the J-inner product. The fundamental projections

$$P_+ := \tfrac{1}{2}(\mathrm{I} + J), \qquad P_- := \tfrac{1}{2}(\mathrm{I} - J) \tag{2.3}$$

acting on $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$ are defined by $P_+(x^+ + x^-) = x^+$ and $P_-(x^+ + x^-) = x^-$. In light of the equations (2.3), we immediately have that $P_\pm$ and J commute. Moreover, P_+ and P_- are orthogonal projections, i.e. $P_\pm^2 = P_\pm = P_\pm^*$, regardless of whether we consider $[\cdot,\cdot]$ or $\langle \cdot,\cdot \rangle_J$ on $\mathcal{K}$. We will denote by $End_J(\mathcal{K}) \subset L(\mathcal{K})$ the set of all bounded linear operators on $\mathcal{K}$ which commute with J. Fuglede's theorem [14] implies that $End_J(\mathcal{K})$ is a $*$-algebra, that is, an algebra closed under the involution $*$ given by the adjoint operator.

2.2. Local spectral theory

In this section, we recall some concepts of local spectral theory based in [1, 2, 9, 10, 16, 18, 23, 24]. Let $\mathcal{H}$ be a complex Hilbert space. For an operator $T \in L(\mathcal{H})$ we will denote by $\alpha(T)$ the dimension of the kernel $Ker(T)$ of T, and by $\beta(T)$ the codimension of the range $R(T)$ of T. An operator $T \in L(\mathcal{H})$ is *lower semi-Fredholm* if it has range with finite codimension and *upper semi-Fredholm* if it has closed range and finite-dimensional kernel. If T is either upper or lower semi-Fredholm, then T is called a *semi-Fredholm operator*, and the *index* of T is defined by

$$Ind(T) := \alpha(T) - \alpha(T^*).$$

If the quantities $\alpha(T)$ and $\alpha(T^*)$ are finite, then T is called a *Fredholm operator*. The *essential spectrum* $\sigma_e(T)$ is the set of all complex numbers λ such that $T-\lambda$ is not Fredholm. We say that T is a *Weyl operator* if it is Fredholm of index zero. The *Weyl spectrum* of T is the intersection of the spectra of its compact perturbations:

$$\sigma_w(T) := \{\lambda \in \mathbf{C} : T-\lambda \text{ is not Weyl}\} = \bigcap \{\sigma(T+S) : S \in K(\mathcal{H})\},$$

where $K(\mathcal{H})$ is the set of all compact operators acting on $\mathcal{H}$.

Recall that the family $\{Ker(T^k)\}$ forms an ascending sequence of subspaces of $\mathcal{H}$. The *ascent* of T, denoted by $p(T)$, is the smallest nonnegative integer k for which $Ker(T^k) = Ker(T^{k+1})$ holds. Similarly, we have that the family $\{R(T^k)\}$ forms a descending sequence and the smallest nonnegative integer k for which $R(T^k) = R(T^{k+1})$ is called the *descent* of T and is denoted by $q(T)$. An operator T is *Browder* if it is Fredholm with finite ascent and finite descent. The *Browder spectrum* of T is the intersection of the spectra of its commuting compact perturbations:

$$\sigma_b(T) := \{\lambda \in \mathbf{C} : T-\lambda \text{ is not Browder}\} = \bigcap \{\sigma(T+S) : S \in K(\mathcal{H}) \cap comm(T)\},$$

where $comm(T)$ is the set of all operators in $L(\mathcal{H})$ that commute with T. Weyl's theorem holds for T if and only if

$$\sigma(T) \setminus \sigma_w(T) = E^0(T),$$

where $E^0(T) = \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(T-\lambda) < \infty\}$, that is, the set of all isolated points of the spectrum of T which are eigenvalues of finite multiplicity. Browder's theorem holds for T if and only if

$$\sigma_b(T) = \sigma_w(T).$$

For $T \in L(\mathcal{H})$, we write $\sigma_p(T)$, $\sigma_a(T)$, $\sigma_s(T)$ and $\sigma_{com}(T)$ for the point spectrum, the approximate point spectrum, the surjective spectrum and the compression spectrum of T, respectively.

2.3. A counterexample

In 2018, An and Heo [3] presented a modification of the kernel of a linear operator on a Krein space in order to study the Fredholm theory on this type of space. They defined the J-kernel of an operator $T \in L(\mathcal{K})$ as follows:

$$J\text{-}ker(T) := \{x \in \mathcal{K} : \ [Tx, Tx] = 0\}.$$

However, this set is not a linear subspace of $\mathcal{K}$ in general, and thus the definition of the dimension of $J\text{-}ker(T)$ is meaningless. Therefore, J-Fredholm operators introduced by An and Heo are not well defined. To see the previous statement we will present the following example.

Example 2.1 *Let $\mathcal{K} = \mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$ and let $J : \mathcal{K} \to \mathcal{K}$ be given by $J(e_1) = e_2$ and $J(e_2) = e_1$, where e_1, e_2 are the elements of the canonical basis of $\mathbb{C}^2$. According to the above, J is unitary and $J^2 = I$ (where I is the identity on $\mathcal{K}$), which implies that J defines a Krein space on $\mathcal{K}$. Now, e_1 and e_2 belongs to $J\text{-}ker(I)$, because*

$$[e_1\ ,\ e_1] = \langle Je_1\ ,\ e_1\rangle = \langle e_2\ ,\ e_1\rangle = \langle (0,1)\ ,\ (1,0)\rangle = 0 + 0 = 0.$$

Thus, $e_1 \in J\text{-}ker(I)$. Similarly, $[e_2\ ,\ e_2] = \langle e_1\ ,\ e_2\rangle = 0$ and so $e_2 \in J\text{-}ker(I)$. On the other hand,

$$[e_1 + e_2\ ,\ e_1 + e_2] = \langle J(e_1 + e_2)\ ,\ e_1 + e_2\rangle = \langle (1,1)\ ,\ (1,1)\rangle = 1 + 1 = 2.$$

Therefore, $e_1 + e_2 \notin J\text{-}ker(I)$, thus we conclude that $J\text{-}ker(I)$ is not linear.

3. Fredholm theory on Krein spaces

3.1. J-Fredholm operators

Throughout this section, we denote by $\mathcal{K}$ a Krein space $(\mathcal{K}, [\cdot,\cdot], J)$. For the sake of introducing a consistent definition which describes the behavior of the indefinite inner product and also generalizes the notion of the classical kernel, for an operator $T \in L(\mathcal{K})$ we define its *J-kernel* by

$$Ker_J(T) := Ker(P_+T).$$

We can easily see that $Ker(T)$ is a closed subspace of $Ker_J(T)$ and that if $J = I_{\mathcal{K}}$, then we obtain the equality $Ker(T) = Ker_J(T)$. Now let us consider the difference $r_J(T) = \alpha_J(T) - \alpha(T)$, where $\alpha_J(T) := dim\, Ker_J(T)$. We claim that $r_J(T) = dim\,[R(T) \cap \mathcal{K}^-]$. Indeed, there is a closed subspace $\mathcal{W}$ of $\mathcal{K}$ such that $Ker_J(T) = Ker(T) \oplus \mathcal{W}$ and, since $R(T) \cap \mathcal{K}^- = T(Ker_J(T))$, then we obtain that T is bijective on $\mathcal{W}$. Hence,

$$dim\,[R(T) \cap \mathcal{K}^-] = dim\,\mathcal{W} = dim\, Ker_J(T) - dim\, Ker(T) = r_J(T).$$

From the above, it follows that the J-kernel of T is built from all the vectors that T transforms into negative vectors.
The following result shows how to characterize the kernel of an operator in terms of the indefinite inner product of a Krein space.

Proposition 3.1 *Let $(\mathcal{K}, [\cdot,\cdot], J)$ be a Krein space and $T \in End_J(\mathcal{K})$. If $r_J(T) = 0$, then $Ker(T) = \{x \in \mathcal{K} : [Tx, Tx] = 0\}$.*

Proof It is easy to check that $Ker(T) \subset \{x \in \mathcal{K} : [Tx, Tx] = 0\}$. Now, if $[Tx, Tx] = 0$ then we have $\|Tx^+\|_J = \|Tx^-\|_J$. Since $r_J(T) = 0$ and T commutes with J, we obtain that $Tx^- \in R(T) \cap \mathcal{K}^- = \{0\}$. Therefore, $x \in Ker(T)$. □

It is important to highlight that $Ker_J(T)$ is a closed subspace of $\mathcal{K}$. Thus, we can define the J-index of $T \in L(\mathcal{K})$ by

$$Ind_J(T) := \alpha_J(T) - \alpha_J(T^*),$$

where $\alpha_J(T) = dim\, Ker_J(T)$ and $\alpha_J(T^*) = dim\, Ker_J(T^*)$. It is worth recalling that the J-index restricted to the $*$-algebra $End_J(\mathcal{K})$ is a classical index, i.e. if $T \in End_J(\mathcal{K})$ then $Ind_J(T) = Ind(P_+T)$. In fact, a simple calculation yields that

$$Ind_J(T) = \alpha_J(T) - \alpha_J(T^*) = \alpha(P_+T) - \alpha(P_+T^*) = \alpha(P_+T) - \alpha(T^*P_+) = Ind(P_+T).$$

The above implies that the map $Ind_J(\cdot) : End_J(\mathcal{K}) \to \mathbb{Z}$ define a surjective group homomorphism, because it is the Fredholm index (see [1, 11]).

In order to introduce J-Fredholm, J-Weyl and J-Browder operators, we need to define the notion of ascent and descent of an operator in the context of Krein spaces. For this, we will denote by $p_J(T)$ the *J-ascent* of an operator $T \in L(\mathcal{K})$, which is defined as

$$p_J(T) := p(P_+T),$$

and in a similar way, the *J-descent* of T, denoted by $q_J(T)$, is defined as

$$q_J(T) = q(P_+T).$$

Definition 3.2 *Let $(\mathcal{K}, J, [\cdot,\cdot])$ be a Krein space and $T \in L(\mathcal{K})$. We say that:*

1. *T is a J-*Fredholm operator*, if $\alpha_J(T)$, $\alpha_J(T^*)$ are finite and the range of T is closed.*
2. *T is a J-*Weyl operator*, if it is J-Fredholm and $Ind_J(T) = 0$.*
3. *T is a J-*Browder operator*, if it is J-Fredholm and both $p_J(T)$, $q_J(T)$ are finite.*

Example 3.3 *Every J-Fredholm self-adjoint operator is J-Weyl. In particular, let us consider $\mathcal{K} = \ell^2(\mathbb{N})$ with the inner product*

$$\left[\left(a_n\right)_{n\in\mathbb{N}}, \left(b_n\right)_{n\in\mathbb{N}}\right] := \sum_{n=1}^{5}(-1)^n a_n\overline{b_n} + \sum_{n=6}^{\infty} a_n\overline{b_n}.$$

If we take $J : \mathcal{K} \to \mathcal{K}$ as

$$Je_n := \begin{cases} -e_n, & for \quad n = 1, 3, 5; \\ e_n, & for \quad n \neq 1, 3, 5, \end{cases}$$

then extending J linearly on all $\mathcal{K}$, where the e_n are the canonical basis of $\ell^2(\mathbb{N})$, we can conclude that $(\mathcal{K}, [\cdot,\cdot], J)$ is a Krein space. Now, let us define the operator $T \in L(\mathcal{K})$ by

$$Ta = P_+a + \langle a, e_1\rangle_{\ell^2(\mathbb{N})}e_1, \qquad a := \left(a_n\right)_{n\in\mathbb{N}}.$$

Observe that T is self-adjoint and J-Fredholm, because $Ker(T) = span\{e_3, e_5\}$ and $r_J(T) = 1$ (the intersection of its range with $\mathcal{K}^-$ is the subspace spanned by e_1). Hence T is J-Weyl.

Example 3.4 *If $dim\,\mathcal{K}^- < \infty$, then every invertible operator is J-Fredholm, by Proposition 3.7. Additionally, since $Ker_J(P_+) = \mathcal{K}^-$, we conclude that P_+ is J-Fredholm under these conditions.*

Example 3.5 *If $(\mathcal{K}, [\cdot,\cdot], J)$ is a Krein space and $dim\,\mathcal{K}^- < \infty$, then every invertible operator T such that $T \in End_J(\mathcal{K})$, is J-Browder.*

We define the *J-essential spectrum*, *J-Browder spectrum* and *J-Weyl spectrum*, as follows:

$$J\text{-}\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } J\text{-Fredholm}\},$$

$$J\text{-}\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } J\text{-Browder}\},$$

$$J\text{-}\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } J\text{-Weyl}\}.$$

Observe that the following implications hold.

$$T \text{ is } J\text{-Browder} \implies T \text{ is } J\text{-Fredholm} \implies T \text{ is Fredholm}.$$

The converses of the above implications are not true in general, for example, if we consider the identity operator $\mathcal{K}$, then I is Fredholm because it is invertible, but if $dim\,\mathcal{K}^- = \infty$ we conclude that

$$\alpha_J(I) = \alpha(I) + r_J(I) = 0 + dim\,\mathcal{K}^- = \infty.$$

Therefore, under these conditions I is not J-Fredholm, in particular I is Browder but is not J-Browder.

Proposition 3.6 *Let T be a J-Browder operator. If $T \in End_J(\mathcal{K})$, then T is J-Weyl.*

Proof Since $p_J(T)$ and $q_J(T)$ are finite by definition and T is J-Fredholm, we only need to prove that the J-index of T is zero. From the definition of $p_J(T)$ and $q_J(T)$ together with [1, Theorem 3.4], we get that $\alpha(P_+T) = \alpha(T^*P_+)$. Furthermore, as $T \in End_J(\mathcal{K})$ we obtain that

$$Ind_J(T) = \alpha_J(T) - \alpha_J(T^*) = \alpha(P_+T) - \alpha(P_+T^*) = \alpha(P_+T) - \alpha(T^*P_+) = 0.$$

Thus, T is J-Weyl. □

Proposition 3.7 *Let $(\mathcal{K}, J, [\cdot,\cdot])$ be a Krein space and $T \in L(\mathcal{K})$. If $dim\,\mathcal{K}^- < \infty$, then J-$\sigma_e(T) \subset \sigma(T)$.*

Proof Let $\lambda \notin \sigma(T)$. Without loss of generality, let us assume that $\lambda = 0$. Then, T is invertible and hence, we have

$$\alpha_J(T) = r_J(T) + \alpha(T) = r_J(T) + 0 = dim\,\mathcal{K}^- < \infty,$$

$$\alpha_J(T^*) = r_J(T^*) + \alpha(T^*) = r_J(T^*) + 0 = dim\,\mathcal{K}^- < \infty.$$

Since T has a closed range, T is J-Fredholm and hence, $\lambda \notin J$-$\sigma_e(T)$. □

Corollary 3.8 *Let $(\mathcal{K}, J, [\cdot,\cdot])$ be a Krein space and $T \in End_J(\mathcal{K})$. If $dim\,\mathcal{K}^- < \infty$, then J-$\sigma_w(T) \subset J$-$\sigma_b(T) \subset \sigma(T)$.*

Proof We know that T is Fredholm by Proposition 3.7. Now it only remains to prove that under these conditions, if T is invertible, then it has finite J-ascent and J-descent. Indeed, since T commutes with J and is invertible it follows that

$$Ker\big((P_+T)^n\big) = Ker(P_+T^n) = \mathcal{K}^-, \quad R\big((P_+T)^n\big) = R(P_+T^n) = \mathcal{K}^-, \quad \forall n \in \mathbb{N}.$$

Therefore, T has finite J-ascent and J-descent, this is, J-$\sigma_b(T) \subset \sigma(T)$. The inclusion J-$\sigma_w(T) \subset J$-$\sigma_b(T)$ is an immediate consequence of Proposition 3.6. □

Proposition 3.9 *Let $(\mathcal{K}, J, [\cdot,\cdot])$ be a Krein space and $T \in End_J(\mathcal{K})$. Then, T is J-Fredholm if and only if T is Fredholm and $dim\,\mathcal{K}^- < \infty$.*

Proof $\Rightarrow$) Notice that if T is J-Fredholm, then T is Fredholm. Now, as T commutes with J and $\mathcal{K}^- = Ker(P_+)$, we get that

$$\mathcal{K}^- \subset Ker(TP_+) = Ker(P_+T) = Ker_J(T).$$

Thus, $\alpha_J(T)$ is finite whenever $dim\,\mathcal{K}^- < \infty$.

$\Leftarrow$) If T is Fredholm, then $\alpha(T)$ and $\alpha(T^*)$ are finite. Also, if $dim\,\mathcal{K}^- < \infty$ then $r_J(T)$ and $r_J(T^*)$ are finite. In this way, we have

$$\alpha_J(T) = \alpha(T) + r_J(T) < \infty,$$

$$\alpha_J(T^*) = \alpha(T^*) + r_J(T^*) < \infty.$$

Thus, we obtain that T is J-Fredholm. □

Corollary 3.10 *Let $(\mathcal{K}, J, [\cdot,\cdot])$ be a Krein space and $T \in End_J(\mathcal{K})$. Then, T is J-Fredholm if and only if T restricted to $\mathcal{K}^+$ is Fredholm and $dim\,\mathcal{K}^- < \infty$.*

Next, we show the relationship between the spectrum and the J-Weyl spectrum for self-adjoint operators.

Proposition 3.11 *If T and S are J-Fredholm operators in $(\mathcal{K}, J, [\cdot,\cdot])$, then ST and TS are J-Fredholm operators.*

Proof By hypothesis, the following quantities are finite: $\alpha(T)$, $\alpha(S)$, $r_J(T)$ and $r_J(S)$. Therefore,

$$\alpha_J(ST) \le \alpha_J(S) + \alpha(T) = \alpha(S) + \alpha(T) + r_J(S) < \infty,$$

$$\alpha_J(TS) \le \alpha_J(T) + \alpha(S) = \alpha(S) + \alpha(T) + r_J(T) < \infty.$$

Using the same argument we show that $\alpha_J(T^*S^*)$ and $\alpha_J(S^*T^*)$ are finite and this completes the proof. □

3.2. J-Weyl's theorems

Analogous to the spectral subsets $E^0(T)$ and $\Pi^0(T)$ introduced in [24], we define the following sets:

$$J\text{-}E^0(T) = \{\lambda \in \mathrm{iso}\,\sigma(T) : 0 < \alpha_J(T-\lambda) < \infty\} \;\text{ and }\; J\text{-}\Pi^0(T) = \sigma(T) \setminus J\text{-}\sigma_b(T).$$

Definition 3.12 *Let $(\mathcal{K}, J, [\cdot,\cdot])$ be a Krein space and $T \in L(\mathcal{K})$. We say that:*

1. *J-Weyl's theorems holds for T, if $\sigma(T) \setminus J\text{-}\sigma_w(T) = J\text{-}E^0(T)$.*

2. *J-Browder's theorems holds for T if $\sigma(T) \setminus J\text{-}\sigma_w(T) = J\text{-}\Pi^0(T)$.*

Proposition 3.13 *Let T and S be two operators in $L(\mathcal{K})$. If $T = JSJ$, then $\alpha_J(T-\lambda) = \alpha_J(S-\lambda)$ for each $\lambda \in \mathbb{C}$.*

Proof We affirm that $Ker_J(T-\lambda) = J(Ker_J(S-\lambda))$. Indeed, if $x \in Ker_J(T-\lambda)$, then

$$(S-\lambda)P_+Jx = (S-\lambda)JP_+x = J(T-\lambda)P_+x = J(0) = 0,$$

which implies that $Jx \in Ker_J(S-\lambda)$ and equality holds for each $\lambda \in \mathbb{C}$ because J is unitary, which guarantees what we affirm. Then, as J is unitary, we conclude that $\alpha_J(T-\lambda) = \alpha_J(S-\lambda)$. □

Lemma 3.14 *Let T and S be two operators in $L(\mathcal{K})$ such that $T = JSJ$ and $\lambda \in \mathbb{C}$.*

1. *$T-\lambda$ has finite ascent if and only if $S-\lambda$ has finite ascent.*

2. *$T-\lambda$ has finite descent if and only if $S-\lambda$ has finite descent.*

3. *$T-\lambda$ has finite J-ascent if and only if $S-\lambda$ has finite J-ascent.*

4. *$T-\lambda$ has finite J-descent if and only if $S-\lambda$ has finite J-descent.*

Proof 1. Assume that $T-\lambda$ has finite ascent, i.e. $Ker(T-\lambda)^p = Ker(T-\lambda)^{p+1}$ for some positive integer p. We will prove that $Ker(S-\lambda)^{p+1} \subset Ker(S-\lambda)^p$. If $x \in Ker(S-\lambda)^{p+1}$, then

$$(T-\lambda)^{p+1}Jx = J(S-\lambda)^{p+1}x = 0.$$

Hence, $Jx \in Ker(T-\lambda)^{p+1} = Ker(T-\lambda)^p$, so we have

$$(S-\lambda)^p x = (S-\lambda)^p J^2 x = J(T-\lambda)^p Jx = 0.$$

Therefore, $x \in Ker(S-\lambda)^p$, which implies that $Ker(S-\lambda)^{p+1} \subset Ker(S-\lambda)^p$. Since in general, the reverse inclusion is true, we have $S-\lambda$ has finite ascent. The reverse implication is also valid.
2. Suppose that $T-\lambda$ has finite descent, i.e. $R(T-\lambda)^q = R(T-\lambda)^{q+1}$ for some positive integer q. For each $y \in R(S-\lambda)^q$, there is $x \in \mathcal{K}$ such that $(S-\lambda)^q x = y$. Then,

$$Jy = J(S-\lambda)^q x = (T-\lambda)^q Jx \in R(T-\lambda)^q = R(T-\lambda)^{q+1}.$$

Hence, there exists a vector $z \in \mathcal{K}$ such that $Jy = (T-\lambda)^{q+1}z$, which implies that

$$y = J(T-\lambda)^{q+1}z = (S-\lambda)^{q+1}Jz \in R(S-\lambda)^{q+1}.$$

Thus, we obtain the inclusion $R(T^*-\lambda)^q \subset R(T^*-\lambda)^{q+1}$. On the other hand, the reverse inclusion is trivial. Therefore, $T^*-\lambda$ has finite descent. A similar argument shows that the reverse implication is also valid.
3. Let us note that

$$JP_+(T-\lambda)J = P_+(JTJ - \lambda J^2) = P_+(S-\lambda).$$

Thus, from part 1, we get

$$q_J(T-\lambda) = q\big(P_+(T-\lambda)\big) = q\big(P_+(S-\lambda)\big) = q_J(S-\lambda).$$

4. It is obtained using the same argument of part 3 together with the result of part 2. □

Theorem 3.15 *Let T and S be two operators in $L(\mathcal{K})$ such that $T = JSJ$. Then:*

1. *J-Weyl's theorem holds for T if and only if J-Weyl's theorem holds for S.*
2. *J-Browder's theorem holds for T if and only if J-Browder's theorem holds for S.*

Proof 1. By virtue of Proposition 3.13, we see that

$$\begin{aligned} Ind_J(S-\lambda) &= \alpha_J(S-\lambda) - \alpha_J(S^*-\overline{\lambda}) = \alpha_J(T-\lambda) - \alpha_J(T^*-\overline{\lambda}) \\ &= Ind_J(T-\lambda). \end{aligned}$$

This shows that J-$\sigma_w(T) = J$-$\sigma_w(S)$. The fact that $\sigma(T) = \sigma(JTJ) = \sigma(S)$, followed by equality J-$E^0(T) = J$-$E^0(S)$ obtained of Proposition 3.13, completes the proof.
2. By Lemma 3.14, J-$\sigma_b(T) = J$-$\sigma_b(S)$. Thus, from part 1, J-$\sigma_b(T) = J$-$\sigma_w(T)$ if and only if J-$\sigma_b(S) = J$-$\sigma_w(S)$. □

Since a J-self-adjoint operator T satisfies that $T = JT^*J$ and a J-unitary operator T satisfies that $T^{-1} = JT^*J$, then as an immediate application of Theorem 3.15, we obtain the following corollaries.

Corollary 3.16 *Let $T \in L(\mathcal{K})$. If T is J-self-adjoint, then the following statements hold:*

1. *J-Weyl's theorem holds for T if and only if J-Weyl's theorem holds for T^*.*
2. *J-Browder's theorem holds for T if and only if J-Browder's theorem holds for T^*.*

Corollary 3.17 *Let $T \in L(\mathcal{K})$. If T is J-unitary, then the following statements hold:*

1. *J-Weyl's theorem holds for T^{-1} if and only if J-Weyl's theorem holds for T^*.*
2. *J-Browder's theorem holds for T^{-1} if and only if J-Browder's theorem holds for T^*.*

Theorem 3.18 *Suppose that $dim\,\mathcal{K}^- < \infty$. If J-Weyl's theorem holds for $T \in End_J(\mathcal{K})$, then J-$E^0(T) = J$-$\Pi^0(T)$.*

Proof By Corollary 3.8, the condition $dim\,\mathcal{K}^- < \infty$ guarantees that J-$\sigma_w(T) \subset J$-$\sigma_b(T) \subset \sigma(T)$. If $\lambda \in J$-$\Pi^0(T)$, then $T-\lambda$ is J-Browder. The latter, together with the fact that $T \in End_J(\mathcal{K})$ imply that $T-\lambda$ is J-Weyl, by Proposition 3.6. Since J-Weyl's theorem holds for T, it follows that $\lambda \in J$-$E^0(T)$.
Conversely, let λ any point of J-$E^0(T)$. To simplify the notation, let us call $T_\lambda := T-\lambda$ and we claim that $Ker\big((P_+T_\lambda)^{n+1}\big) \subset Ker\big((P_+T_\lambda)^{n}\big)$ for all $n \in \mathbb{N}$. Indeed, if $x \in Ker\big((P_+T_\lambda)^{n+1}\big)$, then $P_+T_\lambda^{n+1}x = (P_+T_\lambda)^{n+1}x = 0$, because T commutes with P_+. Therefore, $T(T^nx) = T^{n+1}x \in \mathcal{K}^-$ and as $T \in End_J(\mathcal{K})$, we get that $T^nx \in \mathcal{K}^-$. From the above, we conclude that $(P_+T_\lambda)^nx = P_+T^nx = 0$ and hence $x \in Ker\big((P_+T_\lambda)^n\big)$.

Thus, the claim follows and by induction, $p(P_+T_\lambda) = p_J(T_\lambda) = 1$. Since the J-Weyl's theorem holds for T, we have that T_λ is J-Weyl and hence

$$\alpha(P_+T_\lambda) = \alpha_J(T_\lambda) = \alpha_J(T_\lambda^*) = \alpha(P_+T_\lambda^*) = \alpha(T_\lambda^* P_+) = \beta(P_+T_\lambda) < \infty.$$

Finally, by [1, Theorem 3.4], $q_J(T_\lambda) = q(P_+T_\lambda) = 1$ and we obtain that T_λ is J-Browder, which shows that $\lambda \in J\text{-}\Pi^0(T)$. □

The following result is an immediate consequence of the previous theorem.

Corollary 3.19 *Let $T \in End_J(\mathcal{K})$. The following statements are equivalent:*

1. *J-Weyl's theorem holds for T.*
2. *J-Browder's theorem holds for T and $J\text{-}E^0(T) = J\text{-}\Pi^0(T)$.*

4. Application of the J-Kernel to homogeneous equations

We know that $\alpha(T-\lambda)$ is exactly the number of linearly independent solutions x of the homogeneous equation:

$$Tx - \lambda x = 0, \qquad \lambda \in \mathbb{C}. \tag{4.1}$$

Also, x is a solution of Equation (4.1) if and only if for each y that is a solution of the equation

$$T^*z - \overline{\lambda}z = y, \qquad \lambda \in \mathbb{C},$$

we study similar arguments, i.e. the relationship of $\alpha_J(T-\lambda)$ with respect to an equation of the form (4.1), as well as characterize the J-index in terms of solutions of equations.

We start with the following proposition that characterizes the points of the J-kernel of an operator, whose proof is an immediate consequence of the definition of the J-kernel and the fact that T commutes with J.

Proposition 4.1 *Let T be a bounded linear operator on a Krein space $(\mathcal{K}, [\cdot,\cdot], J)$. The following statements hold:*

1. *$x \in Ker_J(T-\lambda)$ for every $\lambda \in \mathbb{C}$ if and only if $P_+Tx - \lambda P_+x = 0$.*
2. *$x \in Ker_J(T-\lambda)$ for every $\lambda \in \mathbb{C}$ if and only if P_+x is a solution of the homogeneous equation $Tx-\lambda x = 0$, provided that $T \in End_J(T)$.*

The following theorem guarantees the conditions for the equation $P_+Tx - \lambda P_+x = 0$ to have a solution.

Theorem 4.2 *Let T be a bounded linear operator on a Krein space $(\mathcal{K}, [\cdot,\cdot], J)$. The equation*

$$P_+Tx - \lambda P_+x = 0, \tag{4.2}$$

has a solution $x \in \mathcal{K}$ if and only if $\langle x, y\rangle_J = 0$ for each y such that the equation

$$T^*P_+z - \overline{\lambda}P_+z = y, \tag{4.3}$$

has a solution $z \in \mathcal{K}$.

Proof $\Rightarrow$) If x is a solution of Equation (4.2) and y is as in the statement, then

$$\begin{aligned}\left\langle x,y\right\rangle_J &= \left\langle\, x,\, T^*P_+z-\overline{\lambda}P_+z\,\right\rangle_J = \left\langle\, x,\, (T^*P_+-\overline{\lambda}P_+)z\,\right\rangle_J\\ &= \left\langle\, (P_+T-\lambda P_+)x,\, z\,\right\rangle_J = \left\langle\, P_+Tx-\lambda P_+x,\, z\right\rangle_J = \langle 0,z\rangle_J = 0.\end{aligned}$$

$\Leftarrow$) If $x\in\mathcal{K}$ is such that $\langle x,y\rangle_J=0$ for each solution y of Equation (4.3), then for each $z\in\mathcal{K}$ it is satisfied that

$$0=\left\langle\, x,\, T^*P_+z-\overline{\lambda}P_+z\,\right\rangle_J = \left\langle\, x,\, (T^*P_+-\overline{\lambda}P_+)z\,\right\rangle_J = \left\langle\, (P_+T-\lambda P_+)x,\, z\,\right\rangle_J.$$

Since the above is true for each z, we conclude that $P_+Tx-\lambda P_+x=0$; that is, x is a solution of Equation (4.2). □

Proposition 4.3 *Assuming the same conditions of Theorem 4.2, we have that if $\lambda\notin J\text{-}\sigma_e(T)$ then Equation (4.2) has a finite number of linearly independent solutions.*

Proof By hypothesis, $T-\lambda$ is J-Fredholm, so in particular $\alpha_J(T-\lambda)$ is finite. The above together with the proposition 4.1 imply that $\alpha_J(T-\lambda)$ is the number of solutions of Equation (4.2), which completes the proof. □

For each $\lambda\in\mathbb{C}$, let us consider the following sets:

$$\begin{aligned} S(\lambda) &= \{\,x\in\mathcal{K}\,:\,Tx-\lambda x=0\}, \quad S(\lambda)_\pm = \{\,x\in\mathcal{K}^\pm\,:\,Tx-\lambda x=0\},\\ S(\lambda)^* &= \{\,x\in\mathcal{K}\,:\,T^*x-\overline{\lambda}x=0\}, \quad S(\lambda)^*_\pm = \{\,x\in\mathcal{K}^\pm\,:\,T^*x-\overline{\lambda}x=0\}.\end{aligned}\tag{4.4}$$

Each of the previous sets are vector subspaces of $\mathcal{K}$ and $\mathcal{K}^\pm$ in the respective cases, because they are sets of solutions of homogeneous equations. In particular, if T is a compact operator on the Hilbert space associated with $\mathcal{K}$, then by the Fredholm alternative (see [19, Chapter 8]) all the above sets have finite dimension; that is, the number of linearly independent solutions of each of the equations involved is finite, provided that $\lambda\neq 0$.

Theorem 4.4 *Let T be a bounded linear operator on a Krein space $(\mathcal{K},[\cdot,\cdot],J)$. If $T\in End_J(\mathcal{K})$ and $dim\,\mathcal{K}^-<\infty$, then for each $\lambda\in\mathbb{C}$ it is satisfied that*

$$Ind_J(T-\lambda)=dim\,S(\lambda)_+-dim\,S(\lambda)^*_+.$$

Proof As a consequence of Proposition 4.1, we have that $x\in Ker_J(T-\lambda)$ if and only if $P_+x\in S(\lambda)_+$. In this way, the function $Ker_J(T-\lambda)\ni x\mapsto P_+x\in S(\lambda)_+$ is linear and surjective, whose kernel is $\mathcal{K}^-$ by the definition of P_+. The above together with the isomorphism theorem for vector spaces imply that $Ker_J(T-\lambda)/\mathcal{K}^-$ and $S(\lambda)_+$ are isomorphic as vector spaces, which implies that

$$\alpha_J(T-\lambda):=dim\,Ker_J(T-\lambda)=dim\,S(\lambda)_++dim\,\mathcal{K}^-.$$

Therefore,

$$\begin{aligned}Ind_J(T-\lambda) &= \alpha_J(T-\lambda)-\alpha_J(T^*-\overline{\lambda})\\ &= dim\,S(\lambda)_++dim\,\mathcal{K}^- -(dim\,S(\lambda)^*_++dim\,\mathcal{K}^-)\\ &= dim\,S(\lambda)_+-dim\,S(\lambda)^*_+.\end{aligned}$$

□

Remark 4.5 *Under the same conditions of Theorem 4.4, we have that if T is J-Weyl, then the spaces $S(\lambda)_+$ and $S(\lambda)_+^*$ are isomorphic as vector spaces, because they have the same dimension.*

5. Conclusion

In this article, the theory of Fredholm operators on Krein spaces has been presented, starting with a review of the theory presented by An and Heo. Subsequently, the reason why the modification of the kernel of an operator introduced by these authors is not linear has been exhibited; which made it impossible to study Fredholm operators in Krein spaces under the approach presented by them. Then, a new modification of the kernel of an operator has been built, which has allowed us to introduce and study the Fredholm operators in this type of space. Finally, as an application of the approach that has been proposed, some relationships between the J-Fredholm theory (developed in Section 2) and certain types of homogeneous equations were exhibited.

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