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# Some recent results in plastic structure on Riemannian manifold

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**Abstract:** The plastic ratio is a fascinating topic that continually generates new ideas. The purpose of this paper is to point out and find some applications of the plastic ratio in the differential manifold. Precisely, we say that an $(1,1)$-tensor field $P$ on a $m$-dimensional Riemannian manifold $(M,g)$ is a plastic structure if it satisfies the equation $P^3 = P + I$, where $I$ is the identity. We establish several properties of the plastic structure. Then we show that a plastic structure induces on every invariant submanifold a plastic structure, too.

**Key words:** Plastic ratio, plastic structure, polynomial structure, Riemannian manifold

## 1. Introduction and preliminaries

In 1928, shortly after abandoning his architectural studies and becoming a novice monk, Hans van der Laan discovered a new, unique system of architectural proportions. Its construction is completely based on a single irrational value which he called the plastic number (also known as the plastic constant)

$$\rho \approx 1.324718... \approx \frac{4}{3}. \tag{1.1}$$

This number was originally studied by G. Cordonnier in 1924. Gérard Cordonnier $(1907-1977)$, was a French engineer. He studied the plastic number (which he called the radiant number) when he was just 17 years old. The word plastic was not intended to refer to a specific substance, but rather in its adjectival sense, meaning something that can be given a three-dimensional shape (see [10]). However, Hans van der Laan was the first who explained how it relates to the human perception of differences in size between three-dimensional objects and demonstrated his discovery in (architectural) design. His main premise was that the plastic number ratio is truly aesthetic in the original Greek sense, i.e. that its concern is not beauty but the clarity of perception [10]. Special number sequences have played an important role in mathematics and applied sciences. Moreover, some special number sequences such as Fibonacci, Lucas, Pell, Jacobsthal, Padovan, and Perrin sequences have many applications in art, music, photography, architecture, painting, engineering, geometry, and others. It is well-known that the term golden ratio is defined as a ratio of two consecutive Fibonacci numbers converging to

$$\frac{1+\sqrt{5}}{2} \approx 1.618034. \tag{1.2}$$

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In a similar way, the ratio of two consecutive Padovan or Perrin numbers converges to

$$\sqrt[3]{\frac{1}{2}+\frac{1}{6}\sqrt{\frac{23}{3}}}+\sqrt[3]{\frac{1}{2}-\frac{1}{6}\sqrt{\frac{23}{3}}}\approx 1.324718 \tag{1.3}$$

that is called as plastic ratio. Gerard Cordonnier described applications to architecture and illustrated the use of the plastic number in many buildings. The plastic number $\rho$ (also known as the plastic constant, the plastic ratio, the minimal Pisot number, the Platinum number, Siegel's number (or, in French, le nombre radiant) is a mathematical constant which is the unique real solution of the cubic equation

$$x^3 = x + 1 \tag{1.4}$$

it has the exact value

$$\rho=\sqrt[3]{\frac{9+\sqrt{69}}{18}}+\sqrt[3]{\frac{9-\sqrt{69}}{18}}. \tag{1.5}$$

Its decimal expansion begins with $1.324717957244746025960908854\ldots$

The idea of constructing a structure on a Riemannian manifold, called by us a plastic structure, is based on several results from geometrical structures constructed on Riemannian manifolds. Kentaro Yano introduced the notion of an $f$-structure [11]. Extending this structure, Goldberg and Yano [6] introduced the notion of the polynomial structure on a manifold, as a $C^\infty$ tensor field $F$ of type $(1,1)$ defined on a differentiable manifold $M$, such that the algebraic equation is satisfied:

$$Q(F) = F^d + a_1F^{d-1} + ... + a_{d-1}F + a_dI = 0 \tag{1.6}$$

where $a_1, a_2, ..., a_d$ are real numbers and $I$ is the identity tensor of type $(1,1)$.

**Definition 1.1** *A nylons structure is a polynomial structure with the structural polynomial*

$$Q(J) = J^3 - sJ - tI \tag{1.7}$$

*where $s$ and $t$ are positive integers and $I$ is the identity operator on the Lie algebra $\Gamma(M)$ of the vector fields on $M$. The positive solution of the equation $x^3 - sx - t = 0$ is named a member of the nylon family. All the members of the nylon family are positive cube root irrational numbers*

*i)* $\rho=\sqrt[3]{\frac{t}{2}+\sqrt{\frac{t^2}{4}-\frac{s^3}{27}}}+\sqrt[3]{\frac{t}{2}-\sqrt{\frac{t^2}{4}-\frac{s^3}{27}}}$ *if* $27t^2-4s^3>0$,

*ii)* $\rho_1=-2\sqrt[3]{\frac{-t}{2}}$ *and* $\rho_2=\sqrt[3]{\frac{-t}{2}}$, *if* $27t^2-4s^3=0$,

*iii)* $\rho_1=\frac{2}{\sqrt{3}}\sqrt{s}\sin(\frac{1}{3}\sin^{-1}(\frac{-3\sqrt{3}t}{2\sqrt[3]{s}}))$, $\rho_2=\frac{-2}{\sqrt{3}}\sqrt{s}\sin(\frac{1}{3}\sin^{-1}(\frac{-3\sqrt{3}t}{2\sqrt[3]{s}})+\frac{\pi}{3})$ *and*

$\rho_3=\frac{2}{\sqrt{3}}\sqrt{s}\cos(\frac{1}{3}\sin^{-1}(\frac{-3\sqrt{3}t}{2\sqrt[3]{s}})+\frac{\pi}{6})$ *if* $27t^2-4s^3<0$.

**Remark 1.2** *Let $s=t=1$ in the definition (1.1). Then we obtain plastic polynomial $Q(J)=J^3-J-1$ .*

The aim of the present paper is to investigate the geometry of the plastic structure on a Riemannian manifold and we give some examples of plastic structure on quaternion structure and on Euclidean manifold. In section 3, we give some properties of the induced structure on a submanifold in a plastic Riemannian manifold and we find a necessary and sufficient condition for this kind of submanifold to be a plastic Riemannian manifold.

## 2. Main results

Construction of the plastic number. First, we construct the number $\varphi$, called the golden ratio or divine proportion, an ancient aesthetic axiom, then we explain about plastic ratio. The latter works well in-plane (its simplest representation being the golden rectangle) but fails to generate harmonious relations within and between three-dimensional objects. Van der Laan, therefore, elevates the definition of the golden rectangle in terms of spatial dimension. The golden ratio can be calculated by sectioning the segment $AB$ into two parts $AC$ and $BC$ such that

$$\varphi = \frac{AB}{BC} = \frac{BC}{AC}. \tag{2.1}$$

Segments $AB$ and $BC$ are sides of the golden rectangle. Letting $AB = 1$ it follows

$$\varphi = \frac{1}{BC} = \frac{BC}{1 - BC} \Longrightarrow BC^2 = 1 - BC \Longrightarrow \varphi^2 = \varphi + 1 \tag{2.2}$$

Golden ratio is obtained by solving last equation in (2.2):

$$\varphi = 1.618034... \tag{2.3}$$

Van der Laan breaks segment $AB$ in the similar manner, but in three parts. If $C$ and $D$ are points of subdivision, plastic number $\rho$ is defined with

$$\rho = \frac{AB}{AD} = \frac{AD}{BC} = \frac{BC}{AC} = \frac{AC}{CD} = \frac{CD}{BD} \tag{2.4}$$

as illustrated on the Figure (2). The plastic constant P = 1.32471795..., is also called the plastic number, the plastic ratio, the minimal Pisot number, le nombre radiant (in French).
Letting $AB = 1$, from $AC = 1 - BC, BD = 1 - AD$ and (2.4) follows

$$\rho^3 = \rho + 1. \tag{2.5}$$

Using Cardano's formula, the number (1.3) is obtained from (2.5) as the only real solution. Segments $AC$, $CD$, and $BD$ can be interpreted as sides of a cuboid analogous to the golden rectangle.

**Theorem 2.1** *Let $P_n(x) = x^n - x - 1$, for given natural number $n > 1$. Then there exists real number $x_n \in (1,2)$ such that $P_n(x_n) = 0$ and $x_{n+1} < x_n$.*

**Proof** For $n > 1$, we have $p_n(1) < 0$ and $P_n(2) > 0$. Therefore existence of number $x_n \in (1,2)$ follows from the intermediate value theorem. For part two, assume the contrary, i.e. that there exists natural number $n > 1$ such that $x_{n+1} \geq x_n$. Then $P_n(x) = x^n - x - 1$, implies $x_{n+1}^n \geq x_n^n$. Thus $\dfrac{1}{x_{n+1}} + 1 \geq x_n + 1$. Therefore $x_n \cdot x_{n+1} \leq 1$ which is impossible. Hence $x_{n+1} < x_n$ for all $n > 1$. □

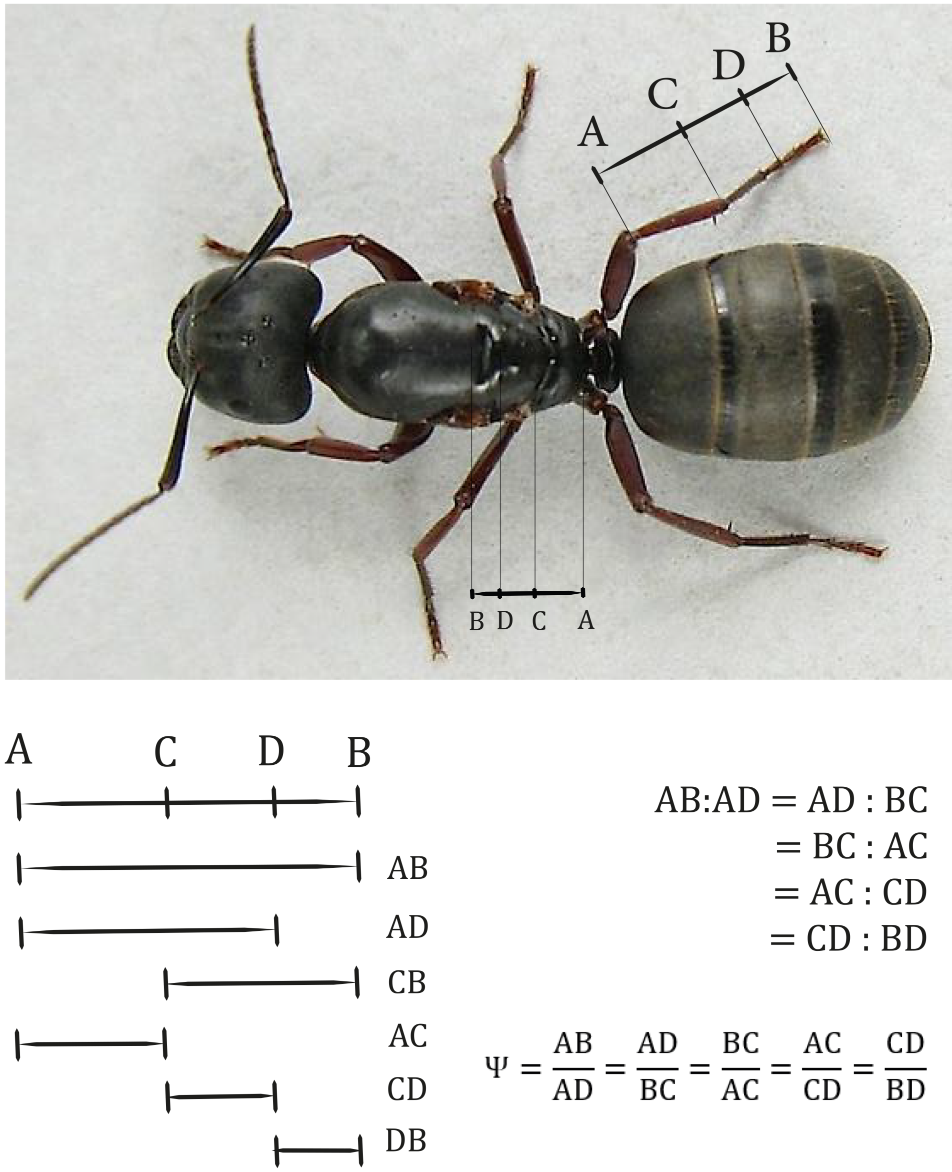


**Figure.** Breaking the segment $AB$ in 3 parts. The first picture shows an application of it for an ant.

### 2.1. Plastic Riemannian structure

In this section, we define a polynomial structure on an m-dimensional Riemannian manifold $(M, g)$, called by us a plastic structure, determined by a (1,1)-tensor field $P$ which satisfies the equation:

$$P^3 = P + I \tag{2.6}$$

where $I$ is the identity operator on the Lie algebra $\Gamma(M)$ of vector fields on $M$.

**Definition 2.2** *Let $(M, g)$ be a Riemannian manifold. A plastic structure on $(M, g)$ is a non-null tensor field*

*$P$ of type $(1,1)$ which satisfies the equation*

$$P^3 = P + I \tag{2.7}$$

*where $I$ is the identity transformation on the Lie algebra $\Gamma(M)$ of vector fields on $M$.*

We say that the matrix $g$ is $P$-compatible if the equality

$$g(P(X), Y) = g(X, P(Y)). \tag{2.8}$$

is satisfied for every tangent vector fields $X, Y \in \Gamma(M)$.

**Theorem 2.3** *If $\rho$ satisfies in $P^3 = P + 1$ ($\rho$ is the plastic ratio), then $\rho^{-1} = \rho^2 - 1$.*

**Proof** The plastic number is the real solution of the equation $P^3 - P - 1 = 0$

$$P^5 - P^4 - 1 = (P^3 - P - 1)(P^2 - P + 1). \tag{2.9}$$

So

$$P^4(P - 1) = 1, \tag{2.10}$$

it follows that the plastic number also satisfies $P - 1 = P^{-4}$.

$\square$

**Definition 2.4** *A Riemannian manifold $(M, g)$, endowed with a plastic structure $P$ that the Riemannian metric $g$ is $P$-compatible (or, a plastic Riemannian manifold) and $(g, P)$ is named a plastic Riemannian structure on $M$.*

In a plastic Riemannian manifold $(M, g, P)$, the equality (2.8) satisfied the equation

$$g(P^2(X), P(Y)) = g(P(X), Y) + g(X, Y) \tag{2.11}$$

for every tangent vector fields $X, Y \in \Gamma(M)$.

For $n \geq 2$, let $X^n$ denotes the n-times Cartesian product $\underbrace{X \times \ldots \times X}_{n-times}$. To simplify, we let $(x_i)_{i=1}^n$ and $(x)_1^n$ stand for $(x_1, ..., x_n)$ and $(x)_{i=1}^n$, respectively.

**Example 2.5** *We consider that the ambient space $m$-dimension Euclidean space $\mathbb{E}^m$, $(m \in \mathbb{N})$. Let $P : \mathbb{E}^m \longrightarrow \mathbb{E}^m$ be an $(1,1)$ tensor field defined by*

$$P((x)_{i=1}^m) = ((\rho x)_{i=1}^m) \tag{2.12}$$

*for every point $((x)_{i=1}^m) \in \mathbb{E}^m$, where $\rho$ is the roots of the equation $x^3 = x + 1$.*
*On the other hand, for $((x)_{i=1}^m)$, $((z)_{i=1}^m) \in \mathbb{E}^m$, we have*

$$P^3((x)_{i=1}^m) = ((\rho^3 x)_{i=1}^m) = ((\rho^1 x)_{i=1}^m) + ((x)_{i=1}^m).$$

*Thus, we obtain $P^3 = P + I$ and we have*

$$< P((x)_{i=1}^m), ((z)_{i=1}^m) >=< ((x)_{i=1}^m), P((z)_{i=1}^m) >$$

*for every $((x)_{i=1}^m), ((z)_{i=1}^m) \in \mathbb{E}^m$. Hence, the scalar product $<>$ on $\mathbb{E}^m$ is $P$-compatible. Therefore, $P$ is a plastic structure defined on $(\mathbb{E}^m, <>)$. Consequently $(\mathbb{E}^m, <>, P)$ is a plastic Riemannian manifold.*

**Example 2.6** *(plastic matrices) Suppose $\mathbb{M}_n^n$ be the set of all real square. The matrix, $\varpi \in \mathbb{M}_n^n$ is called a plastic matrix if $\varpi$ satisfies in the equation*

$$\varpi^3 = \varpi + I, \tag{2.13}$$

*where $I$ is the identity matrix on $\mathbb{M}_n^n$. For the two-dimensional case, we obtain a three-parametric family of plastic structures by solving (2.13)*

$$\begin{bmatrix} a & \frac{a^3 - a - 1}{-c(2a+d)} \\ c & (a+d)^3 - a + 1 \end{bmatrix} \tag{2.14}$$

*where $a, d \in \mathbb{R}$ and $c \in \mathbb{R} - \{0\}$. For example a plastic matrix in $\mathbb{M}_2^2$ is given by*

$$\varpi = \begin{bmatrix} \rho & 0 \\ 0 & \rho \end{bmatrix} \tag{2.15}$$

*where $\rho$ is the plastic ratio.*

**Plastic quaternion.** We write any quaternion in the form $Q = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3$ or $Q = S_Q + \vec{V}_Q$, where the symbols $S_Q = q_0$ and $\vec{V}_Q = q_1 e_1 + q_2 e_2 + q_3 e_3$ denote the scalar and vector parts of $Q$.

**Definition 2.7** *Let $Q$ be a quaternion, if $Q$ satisfies the equation*

$$Q^3 = Q + 1. \tag{2.16}$$

*Then, we say that $Q$ is a plastic quaternion.*

**Example 2.8** *Let $Q = S_Q + \vec{V}_Q$. Then we obtain*

$$Q^3 = S_Q^3 + 3S_Q^2 V_Q - 3S_Q < \vec{V}_Q, \vec{V}_Q > -\vec{V}_Q < \vec{V}_Q, \vec{V}_Q > . \tag{2.17}$$

*In this case, the equation $Q^3 - Q = 1$ is equivalent to*

$$S_Q^3 + 3S_Q^2 V_Q - 3S_Q < \vec{V}_Q, \vec{V}_Q > -\vec{V}_Q < \vec{V}_Q, \vec{V}_Q > -S_Q - \vec{V}_Q = 1 \tag{2.18}$$

*and we get*

$$\begin{cases} S_Q^3 - 3S_Q < \vec{V}_Q, \vec{V}_Q > -S_Q = 1 \\ 3S_Q^2 - < \vec{V}_Q, \vec{V}_Q > -1 = 0. \end{cases} \tag{2.19}$$

*Hence we obtain*

$$S_Q = \sqrt[3]{-\frac{1}{16} + \frac{1}{48}\sqrt{\frac{23}{3}}} + \sqrt[3]{-\frac{1}{16} - \frac{1}{48}\sqrt{\frac{23}{3}}} \qquad or \qquad \vec{V}_Q = 0. \tag{2.20}$$

*Then there are two cases for plastic quaternions* *i)* $\vec{V}_Q = 0 \Longrightarrow S_Q = \sqrt[3]{\frac{1}{2} + \frac{1}{6}\sqrt{\frac{23}{3}}} + \sqrt[3]{\frac{1}{2} - \frac{1}{6}\sqrt{\frac{23}{3}}}$ *(plastic ratio)*

*ii)* $\vec{V}_Q \neq 0 \Longrightarrow S_Q = \sqrt[3]{-\frac{1}{16} + \frac{1}{48}\sqrt{\frac{23}{3}}} + \sqrt[3]{-\frac{1}{16} - \frac{1}{48}\sqrt{\frac{23}{3}}}$ *and* $\mid \vec{V}_Q \mid^2 = 3S_Q^2 - 1$.

The Padovan sequence $\{\varphi_n\}_{n\geq 0}$ is defined by the initial values $P_0 = P_1 = P_2 = 1$ and the recurrence relation

$$P_{n+3} = P_{n+1} + P_n \qquad n \geq 0. \tag{2.21}$$

First few terms of this sequence are $1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28$ [1].

**Proposition 2.9** *A plastic Riemannian structure* $(M, g, P)$ *has the property*

$$P^n = \varphi_{n-4}P^2 + \varphi_{n-3}P + \varphi_{n-5}I \tag{2.22}$$

*for every integer number* $n > 4$*, where* $\{\varphi_n\}_{n\geq 0}$*, is the Padovan sequence.*

**Proof** As $P^3 = P + I$, we obtain $P^4 = P^2 + P$ , $P^5 = P^2 + P + I$ and $P^6 = P^2 + 2P + I$. Suppose that $P^n = \varphi_{n-4}P^2 + \varphi_{n-3}P + \varphi_{n-5}I$. Then we have

$$P^{n+1} = \varphi_{n-4}P^3 + \varphi_{n-3}P^2 + \varphi_{n-5}P. \tag{2.23}$$

So by ( 2.7) we obtain (2.22). □

**Theorem 2.10** *Let* $(1, 1)$ *tensor fields* $L$ *and* $K$ *defined by*

$$L = P^2 + (2\lambda^2 + \lambda)P \tag{2.24}$$

$$K = P^2 + \lambda P + 2\lambda I \tag{2.25}$$

*where*

$$\lambda = \sqrt[3]{\frac{25}{54} + \frac{1}{6}\sqrt{\frac{23}{3}}} + \sqrt[3]{\frac{25}{54} - \frac{1}{6}\sqrt{\frac{23}{3}}} - \frac{1}{3}. \tag{2.26}$$

*We have*

$$L^2 = K^2. \tag{2.27}$$

*Let* $(1, 1)$ *tensor fields* $T$ *and* $Z$ *defined by*

$$T = P^2 + 8P + 11I \tag{2.28}$$

$$Z = P^2 + 6P + 9I. \tag{2.29}$$

*Then we have*

$$\frac{1}{2}(T - Z) = P^3, \tag{2.30}$$

*where* $P$ *is a plastic structure.*

**Proof** The proof is straightforward by direct calculation. □

**Theorem 2.11** *Suppose $P$ is a plastic structure on a differentiable manifold $M$. We define two $(1,1)$ tensor fields $H$ and $Q$ by*

$$H = P^2 + (1 + \frac{2}{\eta})P + \eta I, \tag{2.31}$$

$$Q = P^2 - \frac{2}{\eta}P, \tag{2.32}$$

*where*

$$\eta = \sqrt[3]{-\frac{44}{27} + \sqrt{\frac{59}{27}}} + \sqrt[3]{-\frac{44}{27} - \sqrt{\frac{59}{27}}} - \frac{1}{3}. \tag{2.33}$$

*Then we have*

$$HQ = QH = 1. \tag{2.34}$$

**Proposition 2.12** *The plastic structure $P$, defined on a $m$-dimensional Riemannian manifold $(M,g)$, is an isomorphism on the tangent space of the manifold $M$, $T_pM$, for every $p \in M$.*

**Proof** Since $kerP = \{X \in T_pM | PX = 0, \forall p \in M\}$ and $P^3 = PX + X$. Thus $kerP = \{0\}$. We remark that $P$ is an isomorphism on $T_pM$ for all $p \in M$. □

**Proposition 2.13** *The trace of the plastic structure $P$ defined on a $m$-dimensional Riemannian manifold $(M,g)$ has the property*

$$trace(P^3) = trace(P) + m. \tag{2.35}$$

**Proof** Denoting a local orthonormal basis of the tangent space $T_pM$ in a point $p \in M$ by $\{e_1, e_2, ..., e_m\}$. From (2.7), we have

$$g(P^3 e_i, e_i) = g(Pe_i, e_i) + g(e_i, e_i).$$

The proof is straightforward (the summing by $i$). □

**Proposition 2.14** *If $P$ is a plastic structure of rank $d$ on $m$-dimensional Riemannian manifold $(M,g)$ and $d < m$. Then transpose of $P$ is a plastic structure on $(M,g)$ with the same rank.*

**Proof** $(P^3)^t = (P^t)^3$. □

### 2.2. Submanifolds of plastic Riemannian manifold

Let $S$ be an $n$-dimensional submanifold of codimension $r$, isometrically immersed in an $m$-dimensional plastic Riemannian manifold $(M, g, P)$ (where $m, n, r \in \mathbb{N}$, $n + r = m \geq 2$). We denote the tangent space of $S$ in a point $p \in S$ by $T_pS$. The normal space of $S$ in $p$, for all $p \in S$ by $T_p(S)^\perp$. Let $i_*$ be the differential of the immersion $i : S \longrightarrow M$. The induced Riemannian metric $\tilde{g}$ of $S$ is given by

$$\tilde{g}(X, Y) = g(i_*X, i_*Y) \tag{2.36}$$

for every $X, Y \in \Gamma(M)$. We consider a local orthonormal basis $\{N_1, ..., N_r\}$ of the normal space $T_p(S)^{\perp}$ in a point $p \in S$. We suppose that the indices verify that $\alpha, \beta, \gamma, ... \in \{1, ..., r\}$ and $i, j, k, ... \in \{1, ..., n\}$.
For every $X \in T_p(S)$, the tensor fields $P(i_*X)$ and $P(N_\alpha)$ can be decomposed in tangential and normal components at $S$. Therefore

$$P(i_*X) = i_*(\tilde{P}(X)) + \sum_{\alpha=1}^{r} u_\alpha(X)N_\alpha, \qquad \forall X \in \Gamma(S), \tag{2.37}$$

$$P(N_\alpha) = \epsilon i_*(\xi_\alpha) + \sum_{\beta=1}^{r} a_{\alpha\beta}N_\beta, \qquad (\epsilon = \pm 1) \tag{2.38}$$

where $\tilde{P}$ is an $(1,1)-$tensor field on $S$. The $\xi_\alpha$ are tangent vector fields on $S$. The vectors $u_\alpha$ are 1-forms on $S$ and the matrix $(a_{\alpha\beta})_r$ is a matrix of real functions of order $r \times r$ on $S$[2].

**Theorem 2.15** *If $S$ is an $n$-dimensional submanifold of codimension $r$ and isometrically immersed in a plastic Riemannian manifold $(M, g, P)$. Then the structure $(\tilde{P}, \tilde{g}, u_\alpha, \epsilon\xi_\alpha)$, induced on $S$ by the plastic structure $P$, verifies these equalities.*

$$\begin{cases} (i)\ u_\gamma(X) = u_\gamma(\tilde{P}^2(X)) + \sum_\alpha u_\alpha(\tilde{P}(X))a_{\alpha\gamma} + \epsilon\sum_\alpha u_\alpha u_\alpha(\xi_\alpha) + \sum_\alpha\sum_\beta u_\alpha(X)a_{\alpha\beta}a_{\beta\gamma} \\ (ii)\ \tilde{P}^3(X) = \tilde{P}(X) - X - \epsilon\sum_\alpha u_\alpha(\tilde{P}(X))(\xi_\alpha) - \epsilon\sum_\alpha u_\alpha(X)[\tilde{P}(\xi_\alpha) + \sum_\beta a_{\alpha\beta}(\xi_\beta)] \\ (iii)\ \tilde{P}^2(\xi_\alpha) = \xi_\alpha - \sum_\beta a_{\alpha\beta}\tilde{P}(\xi_\beta) - (\epsilon\sum_\gamma u_\gamma(X) + \sum_\beta\sum_\gamma a_{\alpha\beta}a_{\beta\gamma})\xi_\gamma \\ (iv)\ \sum_\gamma a_{\alpha\gamma}N_\gamma = \epsilon\sum_\gamma u_\gamma(\tilde{P}(\xi_\alpha))N_\gamma + \epsilon\sum_\beta\sum_\gamma a_{\alpha\beta}u_\gamma(\xi_\beta)N_r \\ +(\epsilon\sum_\gamma u_\gamma(X) + \sum_\beta\sum_\gamma a_{\alpha\beta}a_{\beta\gamma})\sum_\theta a_{\gamma\theta}N_\theta - N_\gamma \\ (v)\ a_{\alpha\beta} = a_{\beta\alpha} \\ (vi)\ u_\alpha(X) = \epsilon\tilde{g}(X, \xi_\alpha) \end{cases} \tag{2.39}$$

**Proof** Using two times the operator $P$ in (2.37), we obtain

$$\begin{aligned} &i_*(\tilde{P}^3(X)) + \sum_\gamma u_\gamma(\tilde{P}^2(X))N_\gamma + \epsilon\sum_\alpha u_\alpha(\tilde{P}(X))i_*(\xi_\alpha) + \sum_\alpha\sum_\gamma u_\alpha(\tilde{P}(X))a_{\alpha\gamma}N_\gamma \\ &+\epsilon\sum_\alpha u_\alpha(X)i_*\tilde{P}(\xi_\alpha) + \epsilon\sum_\alpha\sum_\gamma u_\alpha(X)u_\gamma(\xi_\alpha)N_\gamma + \epsilon\sum_\alpha\sum_\beta u_\alpha(X)a_{\alpha\beta}(i_*(\xi_\beta)) \\ &+\sum_\alpha\sum_\beta\sum_\gamma u_\alpha(X)a_{\alpha\beta}a_{\beta\gamma}N_\gamma = i_*(\tilde{P}(X)) + \sum_\gamma u_\gamma(X)N_\gamma + i_*(X) \end{aligned} \tag{2.40}$$

for $X \in \Gamma(S)$. Equalizing the tangential and normal parts, respectively, and from the last equality, we conclude the relations (i) and (ii) from (2.39). Applying the compatibility relation (2.8) for the normal vector fields $N_\alpha$

and $N_\beta$ gives $g(\epsilon i_*(\xi_\alpha)+\sum\limits_{\gamma=1} a_{\alpha\gamma}N_\gamma, N_\beta)=g(N_\alpha, \epsilon i_*(\xi_\beta)+\sum\limits_{\gamma=1} a_{\beta\gamma}N_\gamma)$. We obtain the relation (v) from (2.39). Applying (2.7) to the normal vector field $N_\alpha$, we obtain $P^3(N_\alpha)=P(N_\alpha)+N_\alpha$ and from (2.38). We conclude that

$$\epsilon i_*(\tilde{p}^2(\xi_\alpha))+\epsilon\sum_\gamma u_\gamma(\tilde{P}(\xi_\alpha))N_\gamma+\epsilon\sum_\beta a_{\alpha\beta}(i_*\tilde{P}(\xi_\beta)+\sum_\gamma u_\gamma(\xi_\beta)N_\gamma) \tag{2.41}$$

$$+(\epsilon\sum_\gamma u_\gamma(X)+\sum_\beta\sum_\gamma a_{\alpha\beta}a_{\beta\gamma})(\epsilon i_*(\xi_\gamma)+\sum_\theta a_{\gamma\theta}N_\theta)=\epsilon i_*(\xi_\alpha)+\sum_\gamma a_{\alpha\gamma}N_\gamma+N_\alpha$$

We obtain the relations (iii) and (iv) from (2.39) (equalizing the tangential and normal parts respectively from the last equality). From (2.8), we have $g(P(X),N_\alpha)=g(X,P(N_\alpha))$ which follows that

$$g(i_*\tilde{P}(X)+\sum_\beta u_\beta(X)N_\beta, N_\alpha)=g(X,\epsilon i_*\xi_\alpha+\sum_\beta a_{\alpha\beta}N_\beta) \tag{2.42}$$

for every $X\in\Gamma(S)$ and $N_\alpha\in T_p(S)^\perp$ (for all $p\in S$). It is possible to conclude that the relationship (vi) from (2.39) and the last equality. □

**Proposition 2.16** *If $S$ is an $n$-dimensional submanifold of codimension $r$ and isometrically immersed in a plastic Riemannian manifold $(M,g,P)$, then*

$$\tilde{g}(\tilde{P}(X),Y)=\tilde{g}(X,\tilde{P}(Y)). \tag{2.43}$$

**Proof** According to (2.37) and (2.38), we have

$$\tilde{g}(\tilde{P}(X),Y)-\tilde{g}(X,\tilde{P}(Y))=g(i_*\tilde{P}(X),i_*Y)-g(i_*X,i_*\tilde{P}(Y)) \tag{2.44}$$

$$=g(P(i_*X)-\sum_\alpha u_\alpha(X)N_\alpha, i_*Y)-g(i_*X,P(i_*Y)-\sum_\beta u_\beta(Y)N_\beta)$$

$$=g(P(i_*X),i_*Y)-g(i_*X,P(i_*Y))=0$$

and this completes the proof. □

**Proposition 2.17** *If we suppose that $(\xi_i)_{i=1}^r$ are linearly independent tangent vector fields on $M$, then the 1-forms $(u_i)_{i=1}^r$ are also linearly independent.*

**Proof** The equality $\sum\limits_{\alpha=1}^r \mu^\alpha u_\alpha(X)=0$ is equivalent with

$$0=\sum_\alpha \mu^\alpha g(X,\xi_\alpha)=g(X,\sum_\alpha\mu^\alpha\xi_\alpha) \qquad \forall X\in\Gamma(M). \tag{2.45}$$

Thus, we have

$$\sum_{\alpha=1}^r \mu^\alpha\xi_\alpha=0\Rightarrow\mu^\alpha=0. \tag{2.46}$$

Therefore $(u_i)_{i=1}^r$ are linearly independent on $M$. □

If $S$ is a $n$-dimensional invariant submanifold of codimension $r$ (i.e. $P(T_p(S)) \subseteq T_p(S)$), isometrically immersed in a plastic Riemannian manifold $(M, g, P)$, then $\xi_\alpha(\alpha \in \{1, 2, ..., r\})$ are zero vector fields and the 1-forms $u_\alpha$ vanishes identically on $S$, ( or $u_\alpha(X) = \tilde{g}(X, \xi_\alpha) = 0$). Consequently, (2.37) and (2.38) are respectively written as

$$P(i_* X) = i_*(\tilde{P}(X)), \qquad P(N_\alpha) = \sum_\beta a_{\alpha\beta} N_\beta \tag{2.47}$$

for every $X \in \Gamma(S)$ and $\alpha \in \{1, 2, ..., r\}$. In this case the properties of the structure elements $\tilde{P}, \tilde{g}, u_\alpha, \epsilon\xi_\alpha, (a_{\alpha\beta})_r$, verify that

$$\begin{cases} (i)\ \tilde{P}^3(X) = \tilde{P}(X) - X \\ (ii)\ \sum\limits_\gamma a_{\alpha\gamma} N_\gamma = (\sum\limits_\beta \sum\limits_\gamma a_{\alpha\beta} a_{\beta\gamma}) \sum\limits_\theta a_{\gamma\theta} N_\theta - N_\gamma \\ (iii)\ a_{\alpha\beta} = a_{\beta\alpha} \end{cases} \tag{2.48}$$

for every $X, Y \in \Gamma(S)$ and $\alpha, \beta \in \{1, 2, ..., r\}$.

**Theorem 2.18** *Let $S$ be an $n$-dimensional submanifold of codimension $r$, isometrically immersed in a plastic Riemannian manifold $(M, g, P)$. Let $(\tilde{P}, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})r)$ be the induced structure on $S$ by structure $(g, P)$. A necessary and sufficient condition for $S$ to be invariant is that the induced structure $(\tilde{P}, \tilde{g})$ on $S$ is a plastic Riemannian structure, whenever $\tilde{P}$ is nontrivial.*

**Proof** If $S$ is an invariant submanifold in a plastic Riemannian manifold $(M, g, P)$, then $(\tilde{P}, \tilde{g})$ is a plastic Riemannian structure by (2.48) (i) and (2.16). Conversely, if we suppose that $(S, \tilde{g}, \tilde{P})$ is a plastic Riemannian structure, then $\sum\limits_\alpha u_\alpha(X)\xi_\alpha = 0$. Consequently, we obtain

$$\sum_\alpha u_\alpha(X)\tilde{g}(X, \xi_\alpha) = \sum_\alpha (u_\alpha(X))^2 = 0$$

from which $u_\alpha(X) = 0$ for $\alpha \in \{1, 2, ..., r\}$. Therefore submanifold $S$ is invariant. □

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**Competing interests.**

The authors declare that they have no competing interests.

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