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Explicit motion planning in digital projective product spaces

Seher FİŞEKCİ*, **İsmet KARACA**
Department of Mathematics, Faculty of Science, Ege University, Izmir, Turkey

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Abstract: We introduce digital projective product spaces based on Davis' projective product spaces. We determine an upper bound for the digital LS-category of digital projective product spaces. In addition, we obtain an upper bound for the digital topological complexity of these spaces through an explicit motion planning construction, which shows digital perspective validity of results given by S. Fişekci and L. Vandembroucq. We apply our outcomes on specific spaces in order to be more clear.

Key words: LS-category, topological complexity, motion planning, digital projective product spaces, digital topology

1. Introduction

Topological robotics has emerged as a new mathematical discipline, having been inspired by robotics and engineering. The discipline is devoted to study the concept of configuration spaces, motion planning, and the topological complexity with diverse algebraic material and methods. A configuration space is a given mechanical location which describes the configurations as desired. The motion planning algorithm determines the rule of a continuous motion in the system of given initial and final positions. It should have instability, which arises from topological reasons. The notion of the topological complexity is introduced by M. Farber in 2003 in order to inform topological measures of the complexity of the motion planning problem in robotics [14]. In other words, there may exist discontinuity in the motion planner on the configuration space X. The tool that measures this discontinuity is the topological complexity of the space X, denoted by $TC(X)$. This is a numerical homotopy invariant that can be difficult to determine. Particularly, computing the topological complexity of the n-dimensional real projective space is shown to be linked to the classical problem of determining the Euclidian space of minimal dimension in which this projective space can be immersed [16].

In recent years, digital topology has played an active role in the field of topological robotics. Karaca and Is defined the concept of digital topological complexity in 2018 [20]. The notion of the digital higher topological complexity is added to the literature in [21]. Ege and Karaca define cohomological operations in digital setting and prove the deficiency of the Künneth formula in [12]. Based on this fact, the cohomological lower bound, particularly zero-divisor cup-length property, does not hold for the digital topological complexity as shown in [21]. Studies on digital topology in the finite digital images and given counterexamples underline the differences between digital topological complexity and Farber's topological complexity [22, 23]. The study in [24] shows that there exists another way to state the digital topological complexity by using digital functions.

*Correspondence: fisekciseher@gmail.com
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Since topological complexity and its related invariants are homotopy invariants, the definition and properties of digital homotopy have gained importance and some features of digital homotopy have been generalized in [26]. For the Lusternik-Schnirelmann category, one of the most important related invariants of the topological complexity, the digital LS-category is defined in [2] and the study is expanded by applying it to digital functions [27]. We refer to studies in [7, 11, 12] for more knowledge about digital topological groundwork.

Projective product space has been introduced by Davis [10] in 2010. This space can be considered a generalization of real projective space, but it is not in general the product of projective spaces. Some bounds of the topological complexity of these spaces have been initiated in [18]. The improvement of this study to finalize the estimating problem about the topological complexity and the LS-category of projective product spaces is included in [17]. Fişekci and Vandembroucq compute the LS-category of projective product space and determine an exact value of the topological complexity for some cases. This leads us to build the digital structure of projective product space and deal with the digital topological complexity and the digital LS-category of these spaces.

This paper is related to topological robotics, more precisely the topological complexity and its related invariant LS-category, and is organized by starting with primary notions and basic facts of digital topology. The digital projective product spaces based on Davis' projective product space is introduced by applying digital topological tools and using digital spheres in [13]. In order to compute the digital topological complexity and the digital LS-category of digital projective product space, we first need to specify these digital homotopy invariants for digital spheres. Besides, these calculations can be used for the digital topological complexity and the digital LS-category of configuration spaces based on spheres. In the process, we present digital projective spaces. Moreover, we define the digital nonsingular map and we calculate the digital topological complexity of digital projective spaces with the digital nonsingular map characterization inspired by the work in [16]. We get new results on digital topological complexity and digital LS-category of digital PPS through an explicit motion planning in digital spheres. In this way, we derive the digital topological complexity and the digital LS-category invariants of special spaces. We determine an upper bound for the digital LS-category and consequently an upper bound for the digital topological complexity of digital projective product space. We also obtain an upper bound for the digital topological complexity of digital projective product space. We give examples for our main results to exhibit the application on specific spaces.

2. Preliminaries

In this section, we give basic definitions, essential facts, useful notations for digital topology and topological robotics.

Given any finite subset $X \subset \mathbb{Z}^n$, which consists of integer points of n-dimensional Euclidean space $\mathbb{R}^n$, (X, κ) is called a digital image, where κ is an adjacency relation on elements of X [3]. Distinct points x and y in $\mathbb{Z}^n$ are c_k- *adjacent* with the properties that there exist at most k indices i such that $|x_i - y_i| = 1$ and for all remaining indices i such that $|x_i - y_i| \neq 1$, $x_i = y_i$, where $0 \leq k \leq n$ [3]. This provides us $c_1 = 2-$adjacency in $\mathbb{Z}$, $c_1 = 4-$ and $c_2 = 8-$adjacencies in $\mathbb{Z}^2$, and $c_1 = 6-$, $c_2 = 18-$ and $c_3 = 26-$adjacencies in $\mathbb{Z}^3$. Let (X_1, κ_1) and (X_2, κ_2) be two digital images. The normal product adjacency and product adjacency on $X_1 \times X_2$ are defined in the following way. The points $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$ are normal product adjacent if one of a), b), or c) holds [1], and are product adjacent if one of a), b), c), or d) holds [7]:

a) $x_1 = x_2$ and y_1, y_2 are κ_2-adjacent.

b) x_1, x_2 are κ_1-adjacent and $y_1 = y_2$.

c) x_1, x_2 are κ_1-adjacent and y_1, y_2 are κ_2-adjacent.

d) $x_1 = x_2$ and $y_1 = y_2$.

If (X, κ) is a digital image in $\mathbb{Z}^n$, then X is called κ*-connected* if for every pair of points $x, y \in X$ with $x \neq y$, there exists $\{x_0, x_1, \ldots, x_l\} \subset X$ such that $x = x_0$, $y = x_l$, x_i and x_{i+1} are κ-adjacent, where $i = 0, 1, \ldots, l-1$ [19]. Given subsets $X_1 \subset \mathbb{Z}^{n_1}$ and $X_2 \subset \mathbb{Z}^{n_2}$, a map $f : (X_1, \kappa_1) \to (X_2, \kappa_2)$ is $(\kappa_1, \kappa_2)-$ *continuous* if for any κ_1-connected subset $U_1 \subset X_1$, $f(U_1)$ is κ_2-connected [3]. Furthermore, f is called a (κ_1, κ_2)*-isomorphism* if f is (κ_1, κ_2)-continuous, bijective, and the inverse f^{-1} is (κ_2, κ_1)-continuous [6].

A digital interval is defined as a set $[a, b]_{\mathbb{Z}} = \{z \in \mathbb{Z} : a \leq z \leq b\}$ [5]. Since $[a, b]_{\mathbb{Z}} \subset \mathbb{Z}$, it has 2-adjacency. We use the notation I_m instead of $[0, m]_{\mathbb{Z}}$ as in [26]. Let (X, κ) be a digital image, then a digital path (κ-path) f in X from x to y is defined by a map $f : I_m \to X$ that is $(2, \kappa)$-continuous with $f(0) = x$ and $f(m) = y$ [5]. The digital path f is called a κ-loop when $f(0) = f(m)$ [5]. Let $f : I_m \to X$ and $g : I_n \to X$ be κ-paths with $f(m) = g(0)$. The product of these two digital paths is defined in [25] as the map $(f * g) : I_{m+n} \to X$ by

$$(f * g)(t) = \begin{cases} f(t), & 0 \leq t \leq m \\ g(t-m), & m \leq t \leq m+n. \end{cases}$$

Let (X, κ_1) and (Y, κ_2) be two digital images. The $(\kappa_1, \kappa_2)-$continuous maps $f, g : X \to Y$ are called digitally $(\kappa_1, \kappa_2)-$homotopic in Y if there exists $m \in \mathbb{Z}^+$ and a map $H : X \times I_m \to Y$ that satisfies the following features [3]:

- for all $x \in X$, $H(x, 0) = f(x)$ and $H(x, m) = g(x)$,
- for all $x \in X$, $H_x : I_m \to Y$, defined by $H_x(t) = H(x, t)$, is $(2, \kappa_2)$-continuous,
- for all $t \in I_m$, $H_t : X \to Y$, defined by $H_t(x) = H(x, t)$, is (κ_1, κ_2)-continuous.

The function H is said to be a digital $(\kappa_1, \kappa_2)-$homotopy between f and g.

A digitally continuous map $f : X \to Y$ is digitally nullhomotopic in Y if f is digitally homotopic to a constant map in Y [3]. A (κ_1, κ_2)-continuous map $f : X \to Y$ is a (κ_1, κ_2)-homotopy equivalence if there exists a $(2, \kappa_1)$-continuous map $g : Y \to X$ such that $g \circ f$ is (κ_1, κ_1)-homotopic to the identity map on X and $f \circ g$ is (κ_2, κ_2)-homotopic to the identity map on Y [4]. A digital image (X, κ) is called κ-contractible if the identity map in X is (κ, κ)-homotopic to a constant map in X [3].

Let X^{I_m} represent the set of all digitally continuous paths $\alpha : I_m \to X$ in X.

We define the digital connectedness on X^{I_m} by using the digital continuity to build a digitally continuous digital motion planning algorithm $s : X \times X \to X^{I_m}$. In order to ensure the continuity of s, an adjacency relation between two digital paths is given: Let $\alpha_1 : I_{m_1} \to X$ and $\alpha_2 : I_{m_2} \to X$ be any two digitally continuous path in X. Then α_1 and α_2 are λ-adjacent on $X^{I_{m_1+m_2}}$, if for all t times, $\alpha_1(t)$ and $\alpha_2(t)$ are λ-adjacent. There is no loss of generality in assuming $m_1 = m_2$, for if, $m_1 < m_2$ then we can extend α_1 by letting $\alpha_1(t) = \alpha_1(m_1)$ for $m_1 < t \leq m_2$. Furthermore, the map $\pi : X^{I_m} \to X \times X$ is a digitally continuous map that assigns any digitally continuous path α in X to the pair of its initial and final points $(\alpha(0), \alpha(m)) \in X \times X$ where $\alpha(m)$ is the final step of α [20].

Theorem 2.1 ([21]) *If (X, κ) is a digital image, the map $\pi : X^{I_m} \to X \times X$ which is defined by $\pi(\alpha) = (\alpha(0), \alpha(m))$ is a fibration.*

We say that a digital motion planning algorithm is a section $s : X \times X \to X^{I_m}$ of fibration π, namely $\pi \circ s = id_{X\times X}$.

Theorem 2.2 ([20]) *Let (X, κ) be a digital image. A digital motion planning algorithm s exists if and only if the space X is κ-contractible.*

The definition of the digital topological complexity in [20] is revised and the normalized version of that is defined in the following way.

Definition 2.3 *Given a digital image (X, κ), the digital topological complexity number d-$TC_\kappa(X)$ is the least integer l such that $l+1$ subsets $U_0, U_1, ..., U_l \subset$ is a cover of $X \times X$ and the map $\pi : X^{I_m} \to X \times X$ admits a digitally continuous map $s_i : U_i \to X^{I_m}$ such that $\pi \circ s_i = id_{U_i}$ with the normal product adjacency on $X \times X$.*

Theorem 2.4 ([20]) *The digital topological complexity number is a homotopy invariant of X.*

We normalize the notion of the digital LS-category in [2] as follows.

Definition 2.5 *The digital LS-category d-cat$_\kappa(X)$ of a space (X, κ) is the least integer l such that there exists a cover of X by $l+1$ subsets $U_0, U_1, \ldots, U_l \subset X$ where each inclusion map $i_i : U_i \hookrightarrow X$ for $i = 0, \ldots, l$ is κ-nullhomotopic.*

Theorem 2.6 ([2]) *The digital LS-category is a homotopy invariant of X as well.*

We prove the following Lemma 2.7 and Theorem 2.8 inspired by [9].

Lemma 2.7 *Let (X, κ) be a path-connected digital image. We have $\mathrm{cat}_\kappa(X) \leq k$ if and only if there exists an increasing sequence of sets*

$$\emptyset = U_{-1} \subset U_0 \subset \ldots \subset U_k = X$$

such that each of the differences $U_i - U_{i-1}$ is κ-contractible in X where $i \in \{0, 1, \ldots, k\}$.

Proof Assume that $\mathrm{cat}_\kappa(X) \leq k$, then there is a cover of X consisting of $k+1$ subsets $V_0, \ldots, V_k$ with each V_i is κ-contractible in X. For $i \in \{0, 1, \ldots, k\}$, we define the sets $U_i = V_0 \cup \ldots \cup V_i$ and clearly we have $U_i - U_{i-1} = V_i$ that are κ-contractible in X. Hence, the sets U_i form an increasing sequence as above.

Conversely, assume that there is an increasing sequence of sets

$$\emptyset = U_{-1} \subset U_0 \subset \ldots \subset U_k = X$$

such that each of the difference $U_i - U_{i-1}$ is κ-contractible in X where $i \in \{0, 1, \ldots, k\}$. In this case, the corresponding differences provide us a cover of X by $k+1$ κ-contractible subsets in X, and thus $\mathrm{cat}_\kappa(X) \leq k$. □

Theorem 2.8 *If (X_1, κ_1) and (X_2, κ_2) are path-connected digital image, then*

$$\mathrm{cat}_{\lambda}(X_1 \times X_2) \leq \mathrm{cat}_{\kappa_1}(X_1) + \mathrm{cat}_{\kappa_2}(X_2),$$

where λ is the normal product adjacency on the product space $X_1 \times X_2$.

Proof Suppose that $\mathrm{cat}_{\kappa_1}(X_1) = m$ and $\mathrm{cat}_{\kappa_2}(X_2) = n$. Then there exist increasing sequences of sets

$$\emptyset = U_{-1} \subset U_0 \subset \ldots \subset U_m = X_1$$
$$\emptyset = V_{-1} \subset V_0 \subset \ldots \subset V_n = X_2$$

such that $\tilde{U}_i = U_i - U_{i-1}$ is κ_1-contractible in X_1 for $i \in \{0, \ldots, m\}$ and $\tilde{V}_j = V_j - V_{j-1}$ is κ_2-contractible in X_2 for $j \in \{0, \ldots, n\}$. Namely, inclusion maps $\tilde{U}_i \hookrightarrow X_1$ and $\tilde{V}_j \hookrightarrow X_2$ are κ_1- and κ_2-nullhomotopic, respectively.

We define an increasing sequence

$$\emptyset = W_{-1} \subset W_0 \subset \ldots \subset W_{m+n} = X_1 \times X_2$$

by building subsets of the product $X_1 \times X_2$ as

$$W_r = \bigcup_{i=0}^{r} U_i \times V_{r-i},$$

for $r \geq 0$. Here, $U_i = X_1$ if $i > m$ and $V_j = X_2$ if $j > n$. In addition, we show that

$$\begin{aligned}
W_s - W_{s-1} &= \bigcup_{k=0}^{s} (U_k \times V_{s-k}) - \bigcup_{k=0}^{s-1} (U_k \times V_{s-1-k}) \\
&= \bigcup_{k=0}^{s} \bigcap_{l=0}^{s} (U_k \times V_{s-k}) \cap (U_l \times V_{s-1-l})^{\complement}, \qquad k-1 = l \\
&= \bigcup_{k=0}^{s} \bigcap_{l=0}^{s} \big((U_k - U_l) \times V_{s-k}\big) \cup \big(U_k \times (V_{s-k} - V_{s-1-l})\big) \\
&= \bigcup_{k=0}^{s} (U_k - U_{k-1}) \times (V_{s-k} - V_{s-1-k}) \\
&= \bigcup_{k=0}^{s} \tilde{U}_k \times \tilde{V}_{s-k}
\end{aligned}$$

Furthermore, the subset $\tilde{W}_s = W_s - W_{s-1} \subset X_1 \times X_2$ is λ-contractible in $X_1 \times X_2$. In other words, the inclusion map $\tilde{W}_s \hookrightarrow X_1 \times X_2$ is λ-nullhomotopic.

The normal product adjacency on $X_1 \times X_2$ ensures that

$$(\overline{\tilde{U}_k \times \tilde{V}_{s-k}}) \cap (\tilde{U}_l \times \tilde{V}_{s-l}) = \emptyset$$

$$(\tilde{U}_k \times \tilde{V}_{s-k}) \cap (\overline{\tilde{U}_l \times \tilde{V}_{s-l}}) = \emptyset$$

for $k \neq l$. Therefore, the subsets $\tilde{W}_s$ are disjoint and λ-contractible in $X_1 \times X_2$. Since these λ-contractible subsets are disjoint and the space $X_1 \times X_2$ is digitally path-connected, their union is also λ-contractible. Hence, the union

$$(\tilde{U}_k \times \tilde{V}_{s-k}) \cup (\tilde{U}_l \times \tilde{V}_{s-l})$$

is λ-contractible. We iterate this process one more step and get

$$\begin{aligned}\overline{((\tilde{U}_k \times \tilde{V}_{s-k}) \cup ((\tilde{U}_l \times \tilde{V}_{s-l}))} \cap (\tilde{U}_t \times \tilde{V}_{s-t}) &= ((\overline{\tilde{U}_k \times \tilde{V}_{s-k}}) \cap (\tilde{U}_t \times \tilde{V}_{s-t})) \cup \\ &\quad ((\overline{\tilde{U}_l \times \tilde{V}_{s-l}}) \cap (\tilde{U}_t \times \tilde{V}_{s-t})) \\ &= \emptyset\end{aligned}$$

and

$$((\tilde{U}_k \times \tilde{V}_{s-k}) \cup (\tilde{U}_l \times \tilde{V}_{s-l})) \cap \overline{(\tilde{U}_t \times \tilde{V}_{s-t})} = \emptyset$$

for $k \neq l \neq t$. Therefore, the sets $(\tilde{U}_k \times \tilde{V}_{s-k}) \cup (\tilde{U}_l \times \tilde{V}_{s-l})$ and $\tilde{U}_t \times \tilde{V}_{s-t}$ are disjoint and λ-contractible, and the union $(\tilde{U}_k \times \tilde{V}_{s-k}) \cup (\tilde{U}_l \times \tilde{V}_{s-l}) \cup (\tilde{U}_t \times \tilde{V}_{s-t})$ is λ-contractible in $X_1 \times X_2$. We finally obtain that the subset $W_r - W_{r-1}$ is λ-contractible in $X_1 \times X_2$. Hence, we get the increasing sequence

$$\emptyset = W_{-1} \subset W_0 \subset \ldots \subset W_{m+n} = X_1 \times X_2$$

that each difference $W_r - W_{r-1}$ is λ-contractible in $X_1 \times X_2$. In conclusion, by Lemma 2.7, we have $\text{cat}_\lambda(X_1 \times X_2) \leq m + n = \text{cat}_{\kappa_1}(X_1) + \text{cat}_{\kappa_2}(X_2)$. □

The theorem in [20] is rewritten with the normal product adjacency as follows and the inequality still holds, since the normalized version of the digital topological complexity does not affect the idea of the proof in this case.

Theorem 2.9 *If (X, κ) is a digitally path-connected space, then*

$$\text{d-cat}_\kappa(X) \leq \text{d-}TC_\kappa(X) \leq \text{d-cat}_\lambda(X \times X),$$

where λ is the normal product adjacency on the product space $X \times X$.

As a result of Theorems 2.8 and 2.9, the following corollary is presented.

Corollary 2.10 *Let (X, κ) be a digitally path-connected space. We have*

$$\text{d-cat}_\kappa(X) \leq \text{d-}TC_\kappa(X) \leq 2(\text{d-cat}_\kappa(X)),$$

with the normal product adjacency on the product space $X \times X$.

Throughout this work, we assume that a subset of $\mathbb{Z}^n$ has c_n-adjacency to preserve the adjacency relation on the product of spaces and we consider the normal product adjacency on the product spaces. Additionally, we use the term ”motion planning” instead of ”digital motion planning”. In short, we use the notation $\text{d-cat}(X) = \text{d-cat}_\kappa(X)$ and $\text{d-}TC(X) = \text{d-}TC_\kappa(X)$ and express all the digital terms without indexing adjacency.

3. Main results

Given positive integers $n_1 \leq \ldots \leq n_r$, we denote the r-tuple $(n_1, \ldots, n_r)$ by $\bar{n}$. We use the notation $l(\bar{n}) = r$ for the length of $\bar{n}$. The space $S_{\bar{n}}$ is defined by the product $S^{n_1} \times \cdots \times S^{n_r}$ and the diagonal action of $\mathbb{Z}_2$ on $S_{\bar{n}}$ is given as $\mathbb{Z}_2 \times S_{\bar{n}} \to S_{\bar{n}}$,

$$g \cdot \bar{x} = g \times (x_1, \ldots, x_r) = (gx_1, \ldots, gx_r).$$

Here for each $\bar{x} \in S_{\bar{n}}$, we have $\mathbb{Z}_2\bar{x} = \{g\bar{x} \mid g \in \mathbb{Z}_2\} = \{\bar{x}, -\bar{x}\} \subset S_{\bar{n}}$. Viewing each of these orbits as a single point yields the quotient space

$$P_{\bar{n}} = \frac{S_{\bar{n}}}{\bar{x} \sim -\bar{x}} = \frac{S^{n_1} \times \cdots \times S^{n_r}}{(x_1, \ldots, x_r) \sim (-x_1, \ldots, -x_r)},$$

which is called the projective product space [10]. In the case of r = 1, the space $P_{\bar{n}}$ equals the usual real projective space P^{n_1}.

In order to define the digital projective product space, we use the concept of digital spheres. A digital 0-dimensional sphere is a disconnected digital space $S^0(x, y)$ with two points x and y. The join of two digital 0-dimensional spheres $S^0(x_0, y_0) \oplus S^0(x_1, y_1)$ contains $S^0(x_0, y_0)$, $S^0(x_1, y_1)$ and edges connecting each pair of points except those in the same digital spheres. A minimal digital n-sphere is defined by the join $S^n_{\min} = S^0_0 \oplus S^0_1 \oplus \ldots \oplus S^0_n$ of $n+1$-copies of S^0 [13]. As a result of these definitions, we set the following notations:

- $S^n_{\min} = \left\{x_i = (x_{i_0}, \ldots, x_{i_n}) \in \mathbb{Z}^{n+1}\right\}$, where $|x_{i_j}| = 1$ if $i = j$ and 0 otherwise,
- $S^0_k = \{x = (0, \ldots, 0, x_k, 0, \ldots, 0) \subset \mathbb{Z}^{n+1} : \|x_k\| = 1\}$ for $k \leq n$.

Notice that an n-dimensional digital sphere in $\mathbb{Z}^{n+1}$ has $2n + 2$ vertices.

Example 3.1 *The digital spheres S^1_{min} and S^2_{min} that are modified from figures in [13, 26] are illustrated:*

$$S^1_{min} = \{(1,0), (-1,0), (0,1), (0,-1)\}$$
$$S^2_{min} = \{(1,0,0), (-1,0,0), (0,1,0), (0,-1,0), (0,0,1), (0,0,-1)\}$$

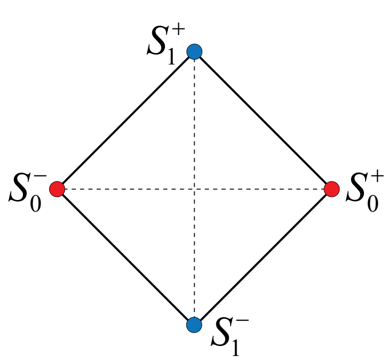

Figure 1. $S^1_{\min}$.

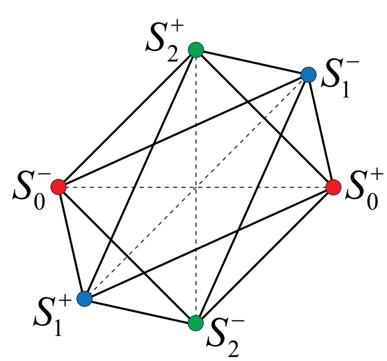

Figure 2. $S^2_{\min}$.

The digital projective space is defined as:

$$\text{d-}P^n = \frac{S^n_{\min}}{x \sim -x}$$

with respect to the diagonal action of $\mathbb{Z}_2$ on $S^n_{\min}$ where $x \in S^n_{\min}$.

We introduce the digital projective product space d-$P_{\bar{n}}$ by the diagonal action of $\mathbb{Z}_2$ on d-$S_{\bar{n}} = S^{n_1}_{\min} \times \cdots \times S^{n_r}_{\min}$ as follows:

$$\text{d-}P_{\bar{n}} = \frac{\text{d-}S_{\bar{n}}}{\bar{x} \sim -\bar{x}} = \frac{S^{n_1}_{\min} \times \cdots \times S^{n_r}_{\min}}{(x_1, \ldots, x_r) \sim (-x_1, \ldots, -x_r)},$$

where $\bar{x} \in$ d-$S_{\bar{n}}$. It is clear that the dimension $\dim(\text{d-}P_{\bar{n}}) = \dim(\text{d-}S_{\bar{n}}) = \sum n_i$. In case of r = 1, the space d-$P_{\bar{n}}$ coincides with the digital projective space d-P^{n_1}.

Theorem 3.2 *Let* d-$P_{\bar{n}}$ *be the digital projective product space where* $\bar{n} = (n_1, \ldots, n_r)$ *with* $n_1 \leq \ldots \leq n_r$, *then the digital LS-category of* d-$P_{\bar{n}}$ *satisfies*

$$\text{d-cat}(\text{d-}P_{\bar{n}}) \leq n_1 + r - 1\,.$$

Proof For $k \leq n$, we use the following notations:

- $S^+_k = \{x = (0, \ldots, 0, x_k, 0, \ldots, 0) \subset \mathbb{Z}^{n+1} : x_k > 0\}$,
- $S^-_k = \{x = (0, \ldots, 0, x_k, 0, \ldots, 0) \subset \mathbb{Z}^{n+1} : x_k < 0\}$,
- $S^0_k = S^+_k \cup S^-_k$,
- $A^+_k = (0, \ldots, 0, \underbrace{1}_{k.\text{ entry}}, 0, \ldots, 0)$ (north pole) and
- $A^-_k = (0, \ldots, 0, \underbrace{-1}_{k.\text{ entry}}, 0, \ldots, 0)$ (south pole).

We define the map $\epsilon_k : S^0_k \to \pm 1$ such that

$$\epsilon_k(0, \ldots, x_k, \ldots, 0) = \begin{cases} 1, & \text{if } x_k > 0 \\ -1, & \text{if } x_k < 0. \end{cases}$$

Note that $\epsilon_k(-x) = -\epsilon_k(x)$ for any $x \in S^0_k$.

Let $\rho(A, B) : I_{m_1} \to S^n_{\min}$ be the digital path from A to B for nonantipodal points of digital sphere $S^n_{\min}$. Notice that $\rho(-A, -B) = -\rho(A, B)$ and that $\rho(A, A)$ is the constant path. We consider the fixed digital path $\sigma_0(A^-_n, A^+_n) : I_{m_2} \to S^n_{\min}$ such that $\sigma_0(0) = A^-_n$ and $\sigma_0(m_2) = A^+_n$.

We define a cover $\bigcup_{i=0}^{n_1+r-1} U_i$ of d-$S_{\bar{n}}$ as follows:

- For $0 \leq i \leq n_1 - 1$, we set $U_i = S^0_i \times \prod_{q=2}^{r} S^{n_q - 1}_{\min}$.

- For a subset $J \subset \{1, \ldots, r\}$, the cardinality of J is denoted by $|J|$ and

$$Q_J = \{\bar{u} \in \text{d-}S_{\bar{n}} : u_j \in S^0_{n_j} \text{ if } j \in J, \; u_j \in S^{n_j}_{\min} \text{ if } j \notin J\},$$

  which is inspired by [8]. Realize that for $J \neq J'$ with $|J| = |J'|$, the sets Q_J and $Q_{J'}$ are disjoint.

  For $i = n_1 - 1 + k$, where $k = 1, \ldots, r-1$, we set $U_i = U_{n_1-1+k} = \bigcup_{|J|=k} Q_J$.

- For $i = n_1 + r - 1$, we set $U_i = U_{n_1+r-1} = S^0_{n_1} \times \cdots \times S^0_{n_r} = S^0_{\bar{n}}$.

Hence, we have a cover of $\text{d-}S_{\bar{n}} = U_0 \cup \ldots \cup U_{n_1+r-1}$ by subsets of $\text{d-}S_{\bar{n}}$. We now define a digital homotopy function as:

$$h_i : U_i \to (\text{d-}P_{\bar{n}})^{I_{m_1+m_2}},$$

where $0 \leq i \leq n_1 + r - 1$. The class of an element $\bar{u} \in \text{d-}S_{\bar{n}}$ in $\text{d-}P_{\bar{n}}$ is denoted by $[\bar{u}]$.

For $0 \leq i \leq n_1 - 1$: For $\bar{u} = (u_1, \ldots, u_r) \in S^0_i \times \prod_{q=2}^r S^{n_q-1}_{\min}$ and $t \in I_{m_1+m_2}$, we set:

$$h_i(\bar{u}, t) = \left[\rho(u_1, A^i_{\epsilon_i(u_1)})(t), \rho(u_2, A^{n_2}_{\epsilon_i(u_1)})(t), \ldots, \rho(u_r, A^{n_r}_{\epsilon_i(u_1)})(t)\right].$$

For $n_1 \leq i \leq n_1 + r - 2$: Recall that we write $i = n_1 - 1 + k$ with $k = 1, \ldots, r-1$ and that $U_i = U_{n_1-1+k} = \bigcup_{|J|=k} Q_J$. We construct a digital homotopy function by:

$$h_J : Q_J \to (\text{d-}P_{\bar{n}})^{I_{m_1+m_2}}.$$

Set $j_0 \in J$ as the smallest element of J. If $\bar{u} \in Q_J$, then $u_{j_0} \in S^0_{n_{j_0}}$. For $\bar{u} \in Q_J$ and $t \in I_{m_1+m_2}$, $0 \leq t \leq m_1$, we set:

$$h_J(\bar{u}, t) = [\rho(u_1, B(u_1))(t), \ldots, \rho(u_q, B(u_q))(t), \ldots, \rho(u_r, B(u_r))(t)],$$

where, for $1 \leq q \leq r$, $B(u_q) = \begin{cases} A^{n_q}_{\epsilon(u_q)}, & \text{if } q \in J \\ A^{n_q}_{\epsilon(u_{j_0})}, & \text{if } q \notin J. \end{cases}$

We distinguish two cases for $t \in I_{m_1+m_2}$, $m_1 \leq t \leq m_1 + m_2$.
If $\epsilon(u_{j_0}) = 1$, then $h_J(\bar{u}, t) = [\omega_1, \ldots, \omega_r]$ where, for $1 \leq q \leq r$,

$$\omega_q = \begin{cases} \sigma_0(t - m_1), & \text{if } \epsilon(u_q) = -1 \text{ and } q \in J \\ A^{n_q}_{\epsilon(u_{j_0})}, & \text{otherwise.} \end{cases}$$

If $\epsilon(u_{j_0}) = -1$, then $h_J(\bar{u}, t) = [\omega_1, \ldots, \omega_r]$ where, for $1 \leq q \leq r$,

$$\omega_q = \begin{cases} \sigma_0(t - m_1), & \text{if } \epsilon(u_q) = 1 \text{ and } q \in J \\ A^{n_q}_{-\epsilon(u_{j_0})}, & \text{otherwise.} \end{cases}$$

We obtain $\left[A^{n_1}_{\epsilon(u_1)}, \ldots, A^{n_j}_{\epsilon(u_j)}, \ldots, A^{n_r}_{\epsilon(u_r)}\right] = \left[A^{n_1}_{-\epsilon(u_1)}, \ldots, A^{n_j}_{-\epsilon(u_j)}, \ldots, A^{n_r}_{-\epsilon(u_r)}\right]$ for $t = m_1$. This provides us a well-defined digitally continuous map on $Q_J \times I_{m_1+m_2}$.

We define h_i on $U_i = U_{n_1-1+k} = \bigcup_{|J|=k} Q_J$ by setting $h_i|_{Q_J \times I_{m_1+m_2}} = h_J$, for $k = n_1 - 1 + k$.

For $i = n_1 + r - 1$: For $\bar{u} \in S^0_{n_1} \times \cdots \times S^0_{n_r} = S^0_{\bar{n}}$ and $t \in I_{m_1+m_2}$, we set:

$$h_i(\bar{u}, t) = \left[\rho(u_1, A^{n_1}_{\epsilon(u_1)})(t), \ldots, \rho(u_r, A^{n_r}_{\epsilon(u_r)})(t)\right].$$

We seperate two cases for $t \in I_{m_1+m_2}$ and $m_1 \leq t \leq m_1 + m_2$.
If $\epsilon(u_1) = 1$, then $h_i(\bar{u}, t) = [A^{n_1}_{\epsilon(u_1)}, \omega_2, \ldots, \omega_r]$ where, for $2 \leq q \leq r$,

$$\omega_q = \begin{cases} A^{n_q}_{\epsilon(u_q)}, & \text{if } \epsilon(u_q) = 1 \\ \sigma_0(t - m_1), & \text{if } \epsilon(u_q) = -1 \end{cases}$$

If $\epsilon(u_1) = -1$, then $h_i(\bar{u}, t) = [A^{n_1}_{-\epsilon(u_1)}, \omega_2, \ldots, \omega_r]$ where, for $2 \leq q \leq r$,

$$\omega_q = \begin{cases} A^{n_q}_{-\epsilon(u_q)}, & \text{if } \epsilon(u_q) = -1 \\ \sigma_0(t - m_1), & \text{if } \epsilon(u_q) = 1 \end{cases}$$

For $t = m_1$, $\left[A^{n_1}_{\epsilon(u_1)}, \ldots, A^{n_r}_{\epsilon(u_r)}\right] = \left[A^{n_1}_{-\epsilon(u_1)}, \ldots, A^{n_r}_{-\epsilon(u_r)}\right]$. This gives a well-defined digitally continuous map on $S^0_{\bar{n}} \times I_{m_1+m_2}$.

We have $h_i(\bar{u}, t) = h_i(-\bar{u}, t)$ for any $t \in I_{m_1+m_2}$ and $\bar{u}, -\bar{u} \in U_i$ for any i. According to the maps, we obtain $h_i(\bar{u}, 0) = [\bar{u}]$ for $0 \leq i \leq n_1 + r - 1$,

$$h_i(\bar{u}, m_1 + m_2) = \begin{cases} [A^i_+, A^{n_2}_+, \ldots, A^{n_r}_+], & 0 \leq i \leq n_1 - 1 \\ [A^{n_1}_+, A^{n_2}_+, \ldots, A^{n_r}_+], & n_1 \leq i \leq n_1 + r - 1. \end{cases}$$

For any i and $\bar{u} \in U_i$, $h_{i_{\bar{u}}} : I_{m_1+m_2} \to$ d-$P_{\bar{n}}$ given by $h_{i_{\bar{u}}}(t) = h_i(\bar{u}, t)$ is digitally continuous. Moreover, for any $t \in I_{m_1+m_2}$, $h_{i_t} : U_i \to$ d-$P_{\bar{n}}$ defined by $h_{i_t}(\bar{u}) = h_i(\bar{u}, t)$ is digitally continuous as well. Hence, for any i, $V_i = U_i/\sim$ is a subset of d-$P_{\bar{n}}$ and we get a digital homotopy function $\bar{h}_i : V_i \to (\text{d-}P_{\bar{n}})^{I_{m_1+m_2}}$ such that $\bar{h}_i([\bar{u}], 0) = [\bar{u}]$ for $0 \leq i \leq n_1 + r - 1$ and

$$\bar{h}_i([\bar{u}], m_1 + m_2) = \begin{cases} [A^i_+, A^{n_2}_+, \ldots, A^{n_r}_+], & 0 \leq i \leq n_1 - 1 \\ [A^{n_1}_+, A^{n_2}_+, \ldots, A^{n_r}_+], & n_1 \leq i \leq n_1 + r - 1. \end{cases}$$

Similarly, for any i and $[\bar{u}] \in V_i$, $\bar{h}_{i_{[\bar{u}]}} : I_{m_1+m_2} \to$ d-$P_{\bar{n}}$ presented by $\bar{h}_{i_{[\bar{u}]}}(t) = \bar{h}_i([\bar{u}], t)$ is also digitally continuous. Furthermore, for any $t \in I_{m_1+m_2}$, $\bar{h}_{i_t} : V_i \to$ d-$P_{\bar{n}}$ given by $\bar{h}_{i_t}([\bar{u}]) = \bar{h}_i([\bar{u}], t)$ is also digitally continuous. In addition, we obtain a cover $\bigcup_{i=0}^{n_1+r-1} V_i$ of d-$P_{\bar{n}}$ and each inclusion map $V_i \hookrightarrow$ d-$P_{\bar{n}}$ is digitally nullhomotopic. Therefore, we prove that d-cat(d-$P_{\bar{n}}$) $\leq n_1 + r - 1$. □

Combining Theorem 3.2 and Corollary 2.10 leads to the following result.

Corollary 3.3 d-TC(d-$P_{\bar{n}}$) ≤ 2(d-cat(d-$P_{\bar{n}}$)) $\leq 2(n_1 + r - 1)$.

Example 3.4 *The construction of Theorem 3.2 is specified for $n_1 = 1$, $n_2 = 2$ and $r = 2$. In other words, we show that:*

$$\textit{d-cat}(\textit{d-}P) = \textit{d-cat}\left(\frac{S^1_{min} \times S^2_{min}}{\sim}\right) \leq n_1 + r - 1 = 2.$$

Therefore, $d\text{-}TC(d\text{-}P) \le 2(d\text{-}cat(d\text{-}P)) \le 4$ *by Corollary 3.3.*

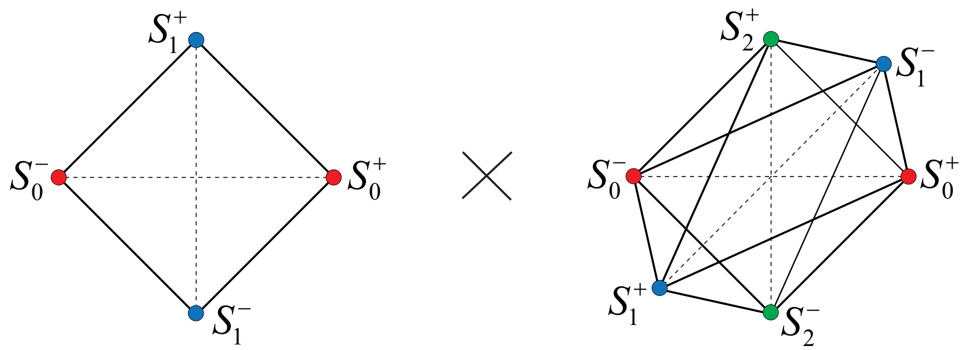

Figure 3. $S^1_{\min} \times S^2_{\min}$.

As before, assume that $\rho(A,B) : I_{m_1} \to S^n_{min}$ *is the digital path from* A *to* B *for nonantipodal points* $A, B \in S^n_{min}$ *with* $A \neq B$. *Note that* $\rho(-A,-B) = -\rho(A,B)$. *Moreover, let* $\sigma_0 : I_{m_2} \to S^n_{min}$ *be the digital path from* A^n_- *to* A^n_+ *with* $\sigma_0(0) = A^n_-$ *and* $\sigma_0(1) = A^n_+$.

Fix the set U_i *in the following notation:*

$$U_i = \begin{cases} S^0_0 \times S^1_{min}, & i = 0 \\ S^0_1 \times S^1_{min} \cup S^0_{min} \times S^0_2, & i = 1 \\ S^0_1 \times S^0_2, & i = 2 \end{cases}$$

and here we have $S^1_{min} \times S^2_{min} \subset \bigcup_{i=0}^{2} U_i$.

For $i = 0$, *the digital homotopy function* $h_0 : U_0 \times I_{m_1+m_2} \to d\text{-}P = (S^1_{min} \times S^2_{min})/\sim$ *is defined by:*

$$h_0(u_1,u_2,t) = \begin{cases} \left[\rho(u_1, A^0_+)(t), \rho(u_2, A^2_+)(t)\right], & (u_1,u_2,t) \in S^+_0 \times S^1_{min} \times I_{m_1+m_2} \\ \left[\rho(u_1, A^0_-)(t), \rho(u_2, A^2_-)(t)\right], & (u_1,u_2,t) \in S^-_0 \times S^1_{min} \times I_{m_1+m_2}. \end{cases}$$

For $i = 1$, *the digital homotopy function* $h_1 : U_1 \times I_{m_1+m_2} \to d\text{-}P = (S^1_{min} \times S^2_{min})/\sim$ *is given by:*

$$h_1(u_1,u_2,t) = \begin{cases} \left[\rho(u_1, A^1_+)(t), \rho(u_2, A^2_+)(t)\right], & (u_1,u_2,t) \in S^+_1 \times S^1_{min} \times I_{m_1+m_2} \\ \left[\rho(u_1, A^1_-)(t), \rho(u_2, A^2_-)(t)\right], & (u_1,u_2,t) \in S^-_1 \times S^1_{min} \times I_{m_1+m_2} \\ \left[\rho(u_1, A^1_+)(t), \rho(u_2, A^2_+)(t)\right], & (u_1,u_2,t) \in S^0_{min} \times S^+_2 \times I_{m_1+m_2} \\ \left[\rho(u_1, A^1_-)(t), \rho(u_2, A^2_-)(t)\right], & (u_1,u_2,t) \in S^0_{min} \times S^-_2 \times I_{m_1+m_2}. \end{cases}$$

For $i = 2$, *the digital homotopy function* $h_2 : U_2 \times I_{m_1+m_2} \to d\text{-}P = (S^1_{min} \times S^2_{min})/\sim$ *is considered as follows:*

$$h_1(u_1,u_2,t) = \begin{cases} \left[\rho(u_1, A^1_+)(t), \rho(u_2, A^2_+)(t)\right], & (u_1,u_2,t) \in S^+_1 \times S^+_2 \times I_{m_1+m_2} \\ \left[\rho(u_1, A^1_-)(t), \rho(u_2, A^2_-)(t)\right], & (u_1,u_2,t) \in S^-_1 \times S^-_2 \times I_{m_1+m_2} \\ \left[\rho(u_1, A^1_+)(t), \rho(u_2, A^2_+)(t)\right], & (u_1,u_2,t) \in S^+_1 \times S^-_2 \times I_{m_1+m_2} \\ & \text{and } 0 \le t \le m_1, \\ \left[\rho(u_1, A^1_-)(t), \rho(u_2, A^2_-)(t)\right], & (u_1,u_2,t) \in S^+_1 \times S^-_2 \times I_{m_1+m_2} \\ & \text{and } m_1 \le t \le m_1 + m_2. \end{cases}$$

Notice that h_2 *is well-defined digitally continuous on* $S_1^+ \times S_2^- \times I_{m_1+m_2}$ *since there exists* $[A_+^1, A_-^2] = [A_+^1, A_-^2]$ *at* $t = m_1$.

If $(u_1, u_2, t) \in S_1^- \times S_2^+ \times I_{m_1+m_2}$, *then we get:*

$$h_2(u_1,u_2,t) = \begin{cases} [\rho(u_1, A_-^1)(t), \rho(u_2, A_+^2)(t)], & 0 \le t \le m_1 \\ [A_+^1, \sigma_0(t-m_1)], & m_1 \le t \le m_1+m_2. \end{cases}$$

In this case, h_2 *is well-defined digitally continuous on* $S_1^- \times S_2^+ \times I_{m_1+m_2}$ *since we have* $[A_-^1, A_+^2] = [A_+^1, A_-^2]$ *at* $t = m_1$.

Given any $(u_1, u_2) \in U_i$, *we have* $(-u_1, -u_2) \in U_i$ *and* $h_i(u_1, u_2, t) = h_i(-u_1, -u_2, t)$ *for any* $t \in I_{m_1+m_2}$ *and any* $0 \le i \le 2$. *We obtain* $h_i(u_1, u_2, 0) = [u_1, u_2]$ *for* $0 \le i \le 2$, *and*

$$h_i(u_1,u_2,m_1+m_2) = \begin{cases} [A_+^0, A_+^2], & i = 0 \\ [A_+^1, A_+^2], & i = 1,2. \end{cases}$$

The map $h_{i_{(u_1,u_2)}} : I_{m_1+m_2} \to d\text{-}P$ *defined by* $h_{i_{(u_1,u_2)}}(t) = h_i((u_1,u_2),t)$ *is digitally continuous for any* i *and* $(u_1, u_2) \in U_i$, *and the map* $h_{i_t} : U_i \to d\text{-}P$ *given by* $h_{i_t}((u_1,u_2)) = h_i((u_1,u_2),t)$ *is digitally continuous for any* $t \in I_{m_1+m_2}$. *These yield a digital homotopy function* $\bar{h}_i : V_i \times I_{m_1+m_2} \to d\text{-}P$ *such that* $\bar{h}_i([u_1,u_2],0) = [u_1,u_2]$ *for* $0 \le i \le 2$, *and*

$$\bar{h}_i([u_1,u_2],m_1+m_2) = \begin{cases} [A_+^0, A_+^2], & i = 0 \\ [A_+^1, A_+^2], & i = 1,2. \end{cases}$$

Thus, we obtain a cover of d-P as $\bigcup_{i=0}^{n_1+1} V_i$ *where* $V_i = U_i/\sim$ *and the digital homotopy function* $\bar{h}_i$ *provides that each inclusion* $V_i \hookrightarrow d\text{-}P$ *is nullhomotopic. Hence,* $d\text{-}cat(d\text{-}P) \le 2$. *Consequently,* $d\text{-}TC(d\text{-}P) \le 2(d\text{-}cat(d\text{-}P)) \le 4$ *by Corollary 3.3.*

Definition 3.5 *A digitally continuous map* $f : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}^k$ *is called a digital nonsingular map, if the following conditions hold:*

- $f(ax, by) = ab \cdot f(x, y)$ *for every* $x, y \in \mathbb{Z}^n$ *and* $a, b \in \mathbb{Z}$.
- $f(x, y) = 0$ *implies that either* $x = 0$ *or* $y = 0$.

Proposition 3.6 *Let* $f : \mathbb{Z}^{n+1} \times \mathbb{Z}^{n+1} \to \mathbb{Z}^{k+1}$ *be a digital nonsingular map where* $n+1 \le k$, *then the digital projective space* d-P^n *has a motion planner with* k *local acts, which is:*

$$d\text{-}TC(\text{d-}P^n) \le k.$$

Proof We assume that $\theta : \mathbb{Z}^{n+1} \times \mathbb{Z}^{n+1} \to \mathbb{Z}$ is a scalar digitally continuous map with $\theta(au, bv) = ab \cdot \theta(u, v)$ for all $(u, v) \in S_{\min}^n \times S_{\min}^n$ and $a, b \in \mathbb{Z}$.

Let $U_\theta \subset$ d-$P^n \times$ d-P^n represent the set of all pairs (u, v) of points in $S_{\min}^n$ such that $u \ne v$ and $\theta(u, v) \ne 0$.

We assert that there is a continuous motion planning in U_θ. Namely, there exists a digitally continuous map s defined on U_θ with values in the space of digitally continuous paths in d-P^n such that for each pair

$(u,v) \in U_\theta$ the digital path $s(u,v)(t)$, $t \in I_m$, begins at point u and finishes at point v. The construction of d-P^n may lead to the existence of points in $S^n_{\min}$ such that $\theta(u,v) > 0$. In this manner, we may take $-u, -v$ instead of u, v. Notice that u, v and equivalently $-u, -v$ dictate the same orientation of the plane based on these points. The intended motion planning map s occurs in rotating u to v in the plane, in the positive direction determined by the orientation.

Furthermore, the map $\theta : \mathbb{Z}^{n+1} \times \mathbb{Z}^{n+1} \to \mathbb{Z}$ is called positive if $\theta(u,u) > 0$ for any $u \in \mathbb{Z}^{n+1}$. Consider the set $U'_\theta \subset$ d-$P^n \times$ d-P^n of all pairs $u, v \in S^n_{\min}$ with $\theta(u,v) \neq 0$. Here, U'_θ contains all pairs (u,u) and so $U_\theta \subset U'_\theta$. We describe the digital path from u to v for $u \neq v$ as rotating from u to v in the plane, based on u and v in the positive direction determined by the orientation. At point u, we fix the constant digital path. Therefore, the digital continuity is preserved.

A digital nonsingular map $f : \mathbb{Z}^{n+1} \times \mathbb{Z}^{n+1} \to \mathbb{Z}^k$ admits k scalar maps $\theta_1, \ldots, \theta_k : \mathbb{Z}^{n+1} \times \mathbb{Z}^{n+1} \to \mathbb{Z}$ and U_{θ_i} cover the product d-$P^n \times$ d-P^n except the diagonal. Since $n+1 < k$, we may use such an f as the initial digital nonsingular map such that for any $u \in \mathbb{Z}^{n+1}$, the first coordinate $\theta_1(u,u)$ is positive. The sets $U'_{\theta_1}, U_{\theta_2}, \ldots, U_{\theta_k}$ form a cover of d-$P^n \times$ d-P^n. We have described explicit motion planning instructions over each of these sets; hence, we get the inequality d-TC(d-P^n) $\leq k$. □

Theorem 3.7 *If* d-$P_{\bar{n}}$ *is the digital projective product space where* $\bar{n} = (n_1, \ldots, n_r)$ *and* $n_1 \leq \ldots \leq n_r$, *then we have:*

$$\text{d-}TC(\text{d-}P_{\bar{n}}) \leq \text{d-}TC(\text{d-}P^{n_1}) + \textstyle\sum_{q=2}^{r} \text{d-}TC(S^{n_q}_{min}).$$

Proof We consider the cartesian product of two digital projective product spaces d-$P_{\bar{n}} \times$ d-$P_{\bar{n}}$ = (d-$S_{\bar{n}} \times$ d-$S_{\bar{n}}$)$/\sim$ as the quotient of $(S^{n_1}_{\min} \times S^{n_1}_{\min}) \times \cdots \times (S^{n_r}_{\min} \times S^{n_r}_{\min})$ by using the well-known isomorphism for the relation

$$(u_1, v_1, \ldots, u_r, v_r) \sim (u'_1, v'_1, \ldots, u'_r, v'_r) \Leftrightarrow \begin{cases} \forall i \quad u_i = u'_i \text{ and } v_i = v'_i \\ \text{or } \forall i \quad u_i = -u'_i \text{ and } v_i = v'_i \\ \text{or } \forall i \quad u_i = u'_i \text{ and } v_i = -v'_i \\ \text{or } \forall i \quad u_i = -u'_i \text{ and } v_i = -v'_i. \end{cases} \tag{1}$$

We set the construction of motion planning in the digital projective space d-P^{n_1} and a digital sphere $S^{n_q}_{\min}$, which is inspired by [16] and [14], respectively. We will get a motion planning on d-$P_{\bar{n}}$ by gathering them.

Assume that d-TC(d-P^{n_1}) $= k$. Thus, there exists a digital nonsingular map $\theta = (\theta_0, \cdots, \theta_k)$: $\mathbb{Z}^{n_1+1} \times \mathbb{Z}^{n_1+1} \to \mathbb{Z}^{k+1}$ by Proposition 3.6. The $k+1$ scalar maps $\theta_0, \ldots, \theta_k : \mathbb{Z}^{n_1+1} \times \mathbb{Z}^{n_1+1} \to \mathbb{Z}$ satisfy $\theta_i(au_1, bv_1) = ab \cdot \theta(u_1, v_1)$ for $(u_1, v_1) \in S^{n_1}_{\min} \times S^{n_1}_{\min}$ with $a, b \in \mathbb{Z}$ and they do not become zero simultaneously. We suppose that $\theta_0(u_1, u_1) > 0$ for any $u_1 \in S^{n_1}_{\min}$ by the definition of the digital sphere for $n_1 + 1 < k$. Let

$$U_0 = \{(u_1, v_1) \in S^{n_1}_{\min} \times S^{n_1}_{\min} : \theta_0(u_1, v_1) \neq 0\}$$

$$U_i = \{(u_1, v_1) \in S^{n_1}_{\min} \times S^{n_1}_{\min} : \text{for all } 0 \leq n < i, \theta_n(u_1, v_1) = 0 \text{ and } \theta_i(u_1, v_1) \neq 0\},$$

where $1 \leq i \leq k-1$.

Notice that all the sets are compatible with the equivalence relation on $S^{n_1}_{\min} \times S^{n_1}_{\min}$ deduced by the antipodal relation $u_1 \sim -u_1$ on $S^{n_1}_{\min}$. Moreover, all the sets U_i are disjoint and $\bigcup_{i=0}^{k} U_i$ is a cover of $S^{n_1}_{\min} \times S^{n_1}_{\min}$.

Let $\rho(A,B) : I_{m_1} \to S^n_{\min}$ be the digital path from A to B except for antipodal points A, B of digital sphere $S^n_{\min}$. Recall that $\rho(-A,-B) = -\rho(A,B)$ and $\rho(A,A)$ is the constant digital path.

For $0 \leq i \leq k$, we define the map ${}_i : U_i \to (\text{d-}P^{n_1})^{I_{m_1+m_2}}$ by

$$\psi_i(u_1, v_1) = \begin{cases} [\rho(u_1, v_1)], & \text{if } \theta_i(u_1, v_1) > 0 \\ [\rho(-u_1, v_1)], & \text{if } \theta_i(u_1, v_1) < 0. \end{cases}$$

We obtain $\theta_0(u_1, u_1) > 0$ for any $u_1 \in S^n_{\min}$ and we get $\theta_0(u_1, -u_1) = \theta_0(-u_1, u_1) < 0$, consequently. Therefore, we have $(u_1, u_1) \in U_0$ or $(-u_1, u_1) \in U_0$, equivalently. This allows us to assure that ψ_i is well-defined on pairs of antipodal points. The map ψ is digitally continuous on U_i and satisfies $\psi_i(u_1, v_1) = \psi_i(\pm u_1, \pm v_1)$ for $0 \leq i \leq k$ and the induced map $\overline{\psi}_i : U_i/\sim \to (\text{d-}P^{n_1})^{I_{m_1+m_2}}$ admits an explicit motion planning in the digital projective space d-P^{n_1}.

For $2 \leq q \leq r$, we use the following subsets of $S^{n_q}_{\min} \times S^{n_q}_{\min}$. In the case that n_q is odd, we take subsets

$$V_0 = \left\{(u_q, v_q) \in S^{n_q}_{\min} \times S^{n_q}_{\min} : v_q \neq \pm u_q\right\}$$

$$V_1 = \left\{(u_q, v_q) \in S^{n_q}_{\min} \times S^{n_q}_{\min} : v_q = \pm u_q\right\}.$$

In the case that n_q is even, we consider subsets

$$V_0 = \left\{(u_q, v_q) \in S^{n_q}_{\min} \times S^{n_q}_{\min} : v_q \neq \pm u_q\right\}$$

$$V_1 = \left\{(u_q, v_q) \in S^{n_q}_{\min} \times S^{n_q}_{\min} : v_q = \pm u_q, u_q \neq \pm a_q\right\}$$

$$V_2 = \left\{(u_q, v_q) \in S^{n_q}_{\min} \times S^{n_q}_{\min} : v_q = \pm u_q, u_q = \pm a_q\right\}.$$

Here, the fixed element $a_q = (0, \ldots, 0, 1) \in S^{n_q}_{\min}$ corresponds to the vanishing point of even dimensional spheres.

We describe the motion planning in a digital sphere by the paths as follows:

- For $(u_q, v_q) \in V_0$ and $(u_q, u_q) \in V_1 \cup V_2$, we consider the digital path $\rho(u_q, v_q)$.
- For $(u_q, -u_q) \in V_1$, we choose the digital path $\sigma : I_{m_2} \to S^{n_q}_{\min}$ path $\sigma(u_q, -u_q)$ from u_q to $-u_q$ in the positive direction, which is symmetrical to the digital path from $-u_q$ to u_q.
- For $(a_q, -a_q) \in V_2$, we fix the digital path $\sigma_0 : I_{m_2} \to S^{n_q}_{\min}$ from a_q to $-a_q$ and we set $\sigma_0(-a_q, a_q) = -\sigma_0(a_q, -a_q)$.

We combine these motion plannings in the following way:

Given $i \in \{0, 1, \ldots, k\}$ and $2 \leq q \leq r$, let $j_q \in \{0, 1\}$ when n_q is odd; or $j_q \in \{0, 1, 2\}$ when n_q is even. We define the map

$${}_{(i,j_2,\ldots,j_r)} : U_i \times \prod_{q=2}^{r} V_{j_q} \to (\text{d-}P_{\bar{n}})^{I_{m_1+m_2}}$$

by

$$\psi_{(i,j_2,\ldots,j_r)}(u_1,v_1,u_2,v_2,\ldots,u_r,v_r)=\begin{cases}[\rho(u_1,v_1),\omega_2,\ldots,\omega_r], & \theta_i(u_1,v_1)>0\\ [\rho(u_1,v_1),\omega'_2,\ldots,\omega'_r], & \theta_i(u_1,v_1)<0,\end{cases}$$

where, for n_q odd,

$$\omega_q=\begin{cases}\sigma(u_q,v_q), & \text{if } v_q=-u_q\\ \rho(u_q,v_q), & \text{otherwise,}\end{cases}$$

$$\omega'_q=\begin{cases}\sigma(-u_q,v_q), & \text{if } v_q=u_q\\ \rho(-u_q,v_q), & \text{otherwise,}\end{cases}$$

and, for n_q even,

$$\omega_q=\begin{cases}\sigma(u_q,v_q), & \text{if } v_q=-u_q, u_q\neq\pm a_q\\ \sigma_0(u_q,v_q), & \text{if } v_q=-u_q, u_q=\pm a_q\\ \rho(u_q,v_q), & \text{otherwise,}\end{cases}$$

$$\omega'_q=\begin{cases}\sigma(-u_q,v_q), & \text{if } v_q=u_q, u_q\neq\pm a_q\\ \sigma_0(-u_q,v_q), & \text{if } v_q=u_q, u_q=\pm a_q\\ \rho(-u_q,v_q), & \text{otherwise.}\end{cases}$$

The map $\psi_{(i,j_2,\ldots,j_r)}$ is the well-defined digitally continuous on $U_i\times\prod_{q=2}^{r}V_{j_q}$. Moreover, the compatibility of this map does not conflict with the equivalence relation (1).

For $i\in\{0,...,k\}$ and $2\leq q\leq r$, let $j_q\in\{0,1\}$ when n_q is odd; and $j_q\in\{0,1,2\}$ when n_q is even. We obtain a digitally continuous map with respect to antipodal relation

$$\bar{\psi}_{(i,j_2,\ldots,j_r)}:\frac{U_i\times\prod_{q=2}^{r}V_{j_q}}{\sim}\to(\text{d-}P_{\bar{n}})^{I_{m_1+m_2}}$$

that satisfies $\bar{\psi}_{(i,j_2,\ldots,j_q)}(u_1,v_1,u_2,v_2,\ldots,u_r,v_r)=[\rho(u_1,v_1),\bar{\omega}_2,\ldots,\bar{\omega}_r]$, where for n_q odd,

$$\bar{\omega}_q=\begin{cases}\sigma(u_q,v_q), & \text{if } v_q=-u_q\\ \rho(u_q,v_q), & \text{otherwise,}\end{cases}$$

and, for n_q even,

$$\bar{\omega}_q=\begin{cases}\sigma(u_q,v_q), & \text{if } v_q=-u_q, u_q\neq\pm a_q\\ \sigma_0(u_q,v_q), & \text{if } v_q=-u_q, u_q=\pm a_q\\ \rho(u_q,v_q), & \text{otherwise,}\end{cases}$$

For $i\in\{0,1,\ldots,k\}$ and $2\leq q\leq r$, let $j_q\in\{0,1\}$ when n_q is odd; and $j_q\in\{0,1,2\}$ when n_q is even. The disjoint union W_l is presented by

$$W_l=\bigcup_{l=i+\sum_{q=2}^{r}j_q}\left(U_i\times\prod_{q=2}^{r}V_q\right)\subset(S^{n_1}_{\min}\times S^{n_1}_{\min})\times\cdots\times(S^{n_r}_{\min}\times S^{n_r}_{\min})$$

where $l=0,\ldots,k+\sum_{q=2}^{r}\text{d-}TC(S^{n_q}_{\min})=\text{d-}TC(\text{d-}P^{n_1})+\sum_{q=2}^{r}\text{d-}TC(S^{n_q}_{\min})$. The cartesian product $S^{n_1}_{\min}\times S^{n_1}_{\min}\times S^{n_2}_{\min}\times S^{n_2}_{\min}\times\cdots\times S^{n_r}_{\min}\times S^{n_r}_{\min}\cong\text{d-}S_{\bar{n}}\times\text{d-}S_{\bar{n}}$ contains all subsets W_l concerning the relation (1).

In the quotient space,

$$\bar{W}_l=\bigcup_{i+\sum_{q=2}^{r}j_q=l}\frac{(U_i\times\prod_{q=2}^{r}V_q)}{\sim}\subset\text{d-}P_{\bar{n}}\times\text{d-}P_{\bar{n}}$$

is a disjoint union where $l = 0, \ldots, k + \sum_{q=2}^{r} \text{d-}TC(S^{n_q}_{\min})$. The induced maps$\psi^{-}_{(i,j_2,\ldots,j_r)}$ provide an explicit motion planning in $\bar{W}_l$ and $\bigcup_{l=0}^{k+\sum_{q=2}^{r}\text{d-}TC(S^{n_q}_{\min})} \bar{W}_l$ covers d-$P_{\bar{n}} \times$ d-$P_{\bar{n}}$. Hence, we conclude that

$$\text{d-}TC(\text{d-}P_{\bar{n}}) \leq k + \textstyle\sum_{q=2}^{r} \text{d-}TC(S^{n_q}_{\min}) = \text{d-}TC(\text{d-}P^{n_1}) + \text{d-}TC(S^{n_q}_{\min}).$$

□

Example 3.8 *We analyze the digital topological complexity of digital projective product space d-$P = (S^2_{min} \times S^2_{min})/\sim$. We state that*

$$d\text{-}TC(d\text{-}P) = d\text{-}TC\left(\frac{S^2_{min} \times S^2_{min}}{\sim}\right) \leq d\text{-}TC(d\text{-}P^2) + d\text{-}TC(S^2_{min}),$$

where $d\text{-}P^2 = S^2_{min}/\sim$.

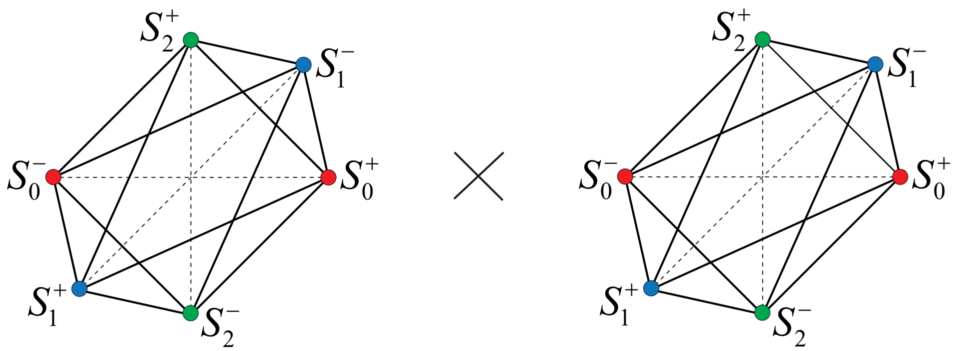

Figure 4. $S^2_{\min} \times S^2_{\min}$.

We construct explicitly motion plannings in the digital projective space $d\text{-}P^2$ and digital sphere S^2_{min}. The case of $d\text{-}P^2$, we use the characterization of digital nonsingular maps.

We build a cover of $S^2_{min} \times S^2_{min} \subset \mathbb{Z}^3 \times \mathbb{Z}^3$ by processing similarly as in [16]. The digital nonsingular map $\mathbb{Z}^4 \times \mathbb{Z}^4 \to \mathbb{Z}^4$ has a restriction onto $\mathbb{Z}^3 \subset \mathbb{Z}^4$. This provides us the digital nonsingular map $\theta : \mathbb{Z}^3 \times \mathbb{Z}^3 \to \mathbb{Z}^4$ with the formula:

$$\theta(u_1, v_1) = \langle u_1, v_1 \rangle - \begin{vmatrix} u_{1_1} & u_{1_2} \\ v_{1_1} & v_{1_2} \end{vmatrix} i - \begin{vmatrix} u_{1_1} & u_{1_3} \\ v_{1_1} & v_{1_3} \end{vmatrix} j - \begin{vmatrix} u_{1_2} & u_{1_3} \\ v_{1_2} & v_{1_3} \end{vmatrix} k,$$

where $u_1 = (u_{1_1}, u_{1_2}, u_{1_3}), v_1 = (v_{1_1}, v_{1_2}, v_{1_3}) \in \mathbb{Z}^3$, $i, j, k \in \mathbb{Z}^4$ are the imaginary units and $\langle u_1, v_1 \rangle$ represents the scalar product of u_1 and v_1 such that

$$\theta(u_1, v_1) = \theta_0(u_1, v_1) + \theta_1(u_1, v_1)i + \theta_2(u_1, v_1)j + \theta_3(u_1, v_1)k.$$

We indicate that the following subsets are compatible with the antipodal relation on S^2_{min}:

$$U_0 = \left\{(u_1, v_1) \in S^2_{min} \times S^2_{min} : \theta_0(u_1, v_1) \neq 0\right\}$$

$$U_1 = \left\{((u_1, v_1) \in S^2_{min} \times S^2_{min} : \theta_0(u_1, v_1) = 0, \theta_1(u_1, v_1) \neq 0\right\}$$

$$U_2 = \left\{(u_1, v_1) \in S^2_{min} \times S^2_{min} : \theta_0(u_1, v_1) = 0, \theta_1(u_1, v_1) = 0, \theta_2(u_1, v_1) \neq 0\right\}$$

$$U_3 = \left\{(u_1, v_1) \in S^2_{min} \times S^2_{min} : \theta_0(u_1, v_1) = 0, \theta_1(u_1, v_1) = 0, \theta_2(u_1, v_1) = 0, \theta_3(u_1, v_1) \neq 0\right\}.$$

Notice that $\bigcup_{i=0}^{3} U_i$ *includes the disjoint subsets of* $S^2_{min} \times S^2_{min}$ *and this gives us a cover of* $S^2_{min} \times S^2_{min}$.

We consider the digital path $\rho(A,B) : I_{m_1} \to S^n_{min}$ *as mentioned before.*

We set the map $\quad_i : U_i \to (d\text{-}P^2)^{I_{m_1+m_2}}$ *for* $0 \leq i \leq 3$ *by*

$$\psi_i(u_1,v_1) = \begin{cases} [\rho(u_1,v_1)], & \text{if } \theta_i(u_1,v_1) > 0 \\ [\rho(-u_1,v_1], & \text{if } \theta_i(u_1,v_1) < 0. \end{cases}$$

For any $u_1 \in S^2_{min}$, *we have* $\theta_0(u_1,u_1) = \theta_0(-u_1,-u_1) > 0$ *and* $\theta_0(u_1,-u_1) = \theta_0(-u_1,u_1) < 0$. *Accordingly, we have pairs* $(u_1,u_1),(-u_1,-u_1) \in U_0$; *hence,* ψ_i *is well-defined on pairs of antipodal points. This map is digitally continuous on* U_i *and satisfies* $\psi_i(u_1,v_1) = \psi_i(\pm u_1, \pm v_1)$ *for* $0 \leq i \leq k$. *Furthermore, the induced map* $\bar{\psi}_i : U_i/\sim \to (d\text{-}P^2)^{I_{m_1+m_2}}$ *gives us an explicit motion planning in* $d\text{-}P^2$.

In the case that n_q *is even, we use the following subsets of* $S^2_{min} \times S^2_{min}$:

$$V_0 = \left\{(u_2,v_2) \in S^2_{min} \times S^2_{min} : v_2 \neq \pm u_2\right\}$$
$$V_1 = \left\{(u_2,v_2) \in S^2_{min} \times S^2_{min} : v_2 = \pm u_2, u_2 \neq \pm a_2\right\}$$
$$V_2 = \left\{(u_2,v_2) \in S^2_{min} \times S^2_{min} : v_2 = \pm u_2, u_2 = \pm a_2\right\}.$$

Here, $a_2 = (0,0,1) \in S^2_{min}$ *corresponds to the vanishing point of even dimensional spheres.*

We present the motion planning in even dimensional digital sphere S^2_{min} *as below.*

- *For* $(u_2,v_2) \in V_0$ *and for* $(u_2,v_2) \in V_1 \cup V_2$, *we use the digital path* $\rho(u_2,v_2)$.
- *For* $(u_2,-u_2) \in V_1$, *we consider the digital path* $\sigma(u_2,-u_2) : I_{m_2} \to S^2_{min}$ *from* u_2 *to* $-u_2$ *in the positive direction.*
- *For* $(a_2,-a_2)$, *we fix the digital path* $\sigma_0 : I_{m_2} \to S^2_{min}$ *from* a_2 *to* $-a_2$ *and we set* $\sigma_0(-a_2,a_2) = -\sigma_0(a_2,-a_2)$.

We gather these motion planners on $U_i \times V_j \subset S^2_{min} \times S^2_{min} \times S^2_{min} \times S^2_{min}$ *for* $i \in \{0,1,2,3\}$ *and* $j \in \{0,1,2\}$ *as follows:*

<u>*The motion planning in* $U_i \times V_0$:</u> *We define the map* $\quad_{(i,0)} : U_i \times V_0 \to (d\text{-}P)^{I_{m_1+m_2}}$ *by*

$$\psi_{(i,0)}(u_1,v_1,u_2,v_2) = \begin{cases} [\rho(u_1,v_1),\rho(u_2,v_2)] & \text{if } \theta_i(u_1,v_1) > 0 \\ [\rho(-u_1,v_1),\rho(-u_2,v_2)] & \text{if } \theta_i(u_1,v_1) < 0, \end{cases}$$

where $(u_1,v_1) \in U_i$ *and* $(u_2,v_2) \in V_0$. *As in the proof of Theorem 3.7, we consider the image of* (u_1,v_1,u_2,v_2) *under the isomorphism and specify the image of* $\psi_{(i,0)}$ *as below.*

$*$ *If* $(u_1,u_2,v_1,v_2) \in U_i \times V_0$ *with* $\theta_i(u_1,v_1) > 0$ *and* $v_2 \neq \pm u_2$, *then*

$$_{(i,0)}(u_1,u_2,v_1,v_2) = [\rho(u_1,v_1),\rho(u_2,v_2)].$$

$*$ *For* $(u_1,u_2,-v_1,-v_2) \in U_i \times V_0$, *we have* $\theta_i(u_1,v_1) < 0$ *and* $v_2 \neq \pm u_2$. *Thus, we get*

$$ {}_{(i,0)}(u_1, u_2, -v_1, -v_2) = [\rho(-u_1, -v_1), \rho(-u_2, -v_2)] = [\rho(u_1, v_1), \rho(u_2, v_2)]\,. $$

* *For* $(-u_1, -u_2, v_1, v_2) \in U_i \times V_0$ *, we have* $\theta_i(u_1, v_1) < 0$ *and* $v_2 \neq \pm u_2$ *. In that case, we have*

$$ {}_{(i,0)}(-u_1, -u_2, v_1, v_2) = [\rho(u_1, v_1), \rho(u_2, v_2)]\,. $$

* *For* $(-u_1, -u_2, -v_1, -v_2) \in U_i \times V_0$ *, we have* $\theta_i(u_1, v_1) > 0$ *and* $v_2 \neq \pm u_2$ *. Thus, we obtain*

$$ {}_{(i,0)}(-u_1, -u_2, -v_1, -v_2) = [\rho(-u_1, -v_1), \rho(-u_2, -v_2)] = [\rho(u_1, v_1), \rho(u_2, v_2)]\,. $$

The motion planning in $U_i \times V_1$*: We give the map* $\psi_{(i,1)} : U_i \times V_1 \to (d\text{-}P)^{I_{m_1+m_2}}$ *by*

$$ \psi_{(i,1)}(u_1, v_1, u_2, v_2) = \begin{cases} [\rho(u_1, v_1), \sigma(u_2, v_2)] & \text{if } \theta_i(u_1, v_1) > 0, v_2 = -u_2, u_2 \neq \pm a_2 \\ [\rho(u_1, v_1), \rho(u_2, v_2)] & \text{if } \theta_i(u_1, v_1) > 0, v_2 = u_2, u_2 \neq \pm a_2 \\ [\rho(-u_1, v_1), \rho(-u_2, v_2)] & \text{if } \theta_i(u_1, v_1) < 0, v_2 = -u_2, u_2 \neq \pm a_2 \\ [\rho(-u_1, v_1), \sigma(-u_2, v_2)] & \text{if } \theta_i(u_1, v_1) < 0, v_2 = u_2, u_2 \neq \pm a_2. \end{cases} $$

* *Let* $(u_1, u_2, v_1, v_2) \in U_i \times V_1$ *with* $\theta_i(u_1, v_1) > 0$*,* $v_2 = -u_2$ *and* $u_2 \neq \pm a_2$*. After that, we get*

$$ {}_{(i,1)}(u_1, u_2, v_1, v_2) = \psi_{(i,1)}(u_1, u_2, v_1, -u_2) = [\rho(u_1, v_1), \sigma(u_2, -u_2)]\,. $$

* *For* $(u_1, u_2, -v_1, -v_2) \in U_i \times V_1$ *, we have* $\theta_i(u_1, v_1) < 0$*,* $v_2 = -u_2$ *and* $u_2 \neq \pm a_2$*. Thus, we have*

$$ \begin{aligned} {}_{(i,1)}(u_1, u_2, -v_1, -v_2) = \quad {}_{(i,1)}(u_1, u_2, -v_1, u_2) &= [\rho(-u_1, -v_1), \sigma(-u_2, u_2)] \\ &= [\rho(u_1, v_1), \sigma(u_2, -u_2)]. \end{aligned} $$

* *For* $(-u_1, -u_2, v_1, v_2) \in U_i \times V_1$ *, we have* $\theta_i(u_1, v_1) < 0$*,* $v_2 = -u_2$ *and* $u_2 \neq \pm a_2$*. Hence, we obtain*

$$ {}_{(i,1)}(-u_1, -u_2, v_1, v_2) = \quad {}_{(i,1)}(-u_1, -u_2, v_1, -u_2) = [\rho(u_1, v_1), \sigma(u_2, -u_2)]\,. $$

* *For* $(-u_1, -u_2, -v_1, -v_2) \in U_i \times V_1$ *, we have* $\theta_i(u_1, v_1) > 0$*,* $v_2 = -u_2$ *and* $u_2 \neq \pm a_2$*. Thus, we get*

$$ \begin{aligned} {}_{(i,1)}(-u_1, -u_2, -v_1, -v_2) = \quad {}_{(i,1)}(-u_1, -u_2, -v_1, u_2) &= [\rho(-u_1, -v_1), \sigma(-u_2, u_2)] \\ &= [\rho(u_1, v_1), \sigma(u_2, -u_2)]. \end{aligned} $$

* *Let* $(u_1, u_2, v_1, v_2) \in U_i \times V_1$ *with* $\theta_i(u_1, v_1) > 0$*,* $v_2 = u_2$ *and* $u_2 \neq \pm a_2$*. Thus, there exists:*

$$ {}_{(i,1)}(u_1, u_2, v_1, v_2) = \psi_{(i,1)}(u_1, u_2, v_1, u_2) = [\rho(u_1, v_1), \rho(u_2, u_2)]\,. $$

* *For* $(u_1, u_2, -v_1, -v_2) \in U_i \times V_1$ *, we have* $\theta_i(u_1, v_1) < 0$*,* $v_2 = u_2$ *and* $u_2 \neq \pm a_2$*. At that case, this satisfies:*

$$ \begin{aligned} {}_{(i,1)}(u_1, u_2, -v_1, -v_2) = \psi_{(i,1)}(u_1, u_2, -v_1, -u_2) &= [\rho(-u_1, -v_1), \rho(-u_2, -u_2)] \\ &= [\rho(u_1, v_1), \rho(u_2, u_2)]. \end{aligned} $$

* *For* $(-u_1,-u_2,v_1,v_2)\in U_i\times V_1$, *we have* $\theta_i(u_1,v_1)<0$, $v_2=u_2$ *and* $u_2\neq\pm a_2$. *Afterwards, this gives that:*

$$ {}_{(i,1)}(-u_1,-u_2,v_1,v_2)=\psi_{(i,1)}(-u_1,-u_2,v_1,u_2)=[\rho(u_1,v_1),\rho(u_2,u_2)]. $$

* *For* $(-u_1,-u_2,-v_1,-v_2)\in U_i\times V_1$, *we have* $\theta_i(u_1,v_1)>0$, $v_2=u_2$ *and* $u_2\neq\pm a_2$. *Therefore, this provides that:*

$$ \begin{aligned} {}_{(i,1)}(-u_1,-u_2,-v_1,-v_2)=\psi_{(i,1)}(-u_1,-u_2,-v_1,-u_2)&=[\rho(-u_1,-v_1),\rho(-u_2,-u_2)]\\ &=[\rho(u_1,v_1),\rho(u_2,u_2)]. \end{aligned} $$

The motion planning in $U_i\times V_2$*:* *We present the map* ${}_{(i,2)}:U_i\times V_2\to(d\text{-}P)^{I_{m_1+m_2}}$ *by:*

$$ \psi_{(i,2)}(u_1,v_1,u_2,v_2)=\begin{cases}[\rho(u_1,v_1),\sigma_0(u_2,v_2)] & \textit{if } \theta_i(u_1,v_1)>0, v_2=-u_2, u_2=\pm a_2\\ [\rho(u_1,v_1),\rho(u_2,v_2)] & \textit{if } \theta_i(u_1,v_1)>0, v_2=u_2, u_2=\pm a_2\\ [\rho(-u_1,v_1),\rho(-u_2,v_2)] & \textit{if } \theta_i(u_1,v_1)<0, v_2=-u_2, u_2=\pm a_2\\ [\rho(-u_1,v_1),\sigma_0(-u_2,v_2)] & \textit{if } \theta_i(u_1,v_1)<0, v_2=u_2, u_2=\pm a_2.\end{cases} $$

* *Let* $(u_1,u_2,v_1,v_2)\in U_i\times V_2$ *with* $\theta_i(u_1,v_1)>0$, $v_2=-u_2$ *and* $u_2=\pm a_2$. *Then we have:*

$$ {}_{(i,2)}(u_1,u_2,v_1,v_2)=\psi_{(i,2)}(u_1,\pm a_2,v_1,-u_2)=[\rho(u_1,v_1),\sigma_0(\pm a_2,\mp a_2)]. $$

* *For* $(u_1,u_2,-v_1,-v_2)\in U_i\times V_2$, *we have* $\theta_i(u_1,v_1)<0$, $v_2=-u_2$ *and* $u_2=\pm a_2$. *Hence, we obtain:*

$$ \begin{aligned} {}_{(i,2)}(u_1,u_2,-v_1,-v_2)= {}_{(i,2)}(u_1,\pm a_2,-v_1,\pm a_2)&=[\rho(-u_1,-v_1),\sigma_0(\mp a_2,\pm a_2)]\\ &=[\rho(u_1,v_1),\sigma_0(\pm a_2,\mp a_2)]. \end{aligned} $$

* *For* $(-u_1,-u_2,v_1,v_2)\in U_i\times V_2$, *we have* $\theta_i(u_1,v_1)<0$, $v_2=-u_2$ *and* $u_2=\pm a_2$. *Thus, we get:*

$$ {}_{(i,2)}(-u_1,-u_2,v_1,v_2)=\psi_{(i,2)}(-u_1,\mp a_2,v_1,\mp a_2)=[\rho(u_1,v_1),\sigma_0(\pm a_2,\mp a_2)]. $$

* *For* $(-u_1,-u_2,-v_1,-v_2)\in U_i\times V_2$, *we have* $\theta_i(u_1,v_1)>0$, $v_2=-u_2$ *and* $u_2=\pm a_2$. *Thus, we state that:*

$$ \begin{aligned} {}_{(i,2)}(-u_1,-u_2,-v_1,-v_2)&=\psi_{(i,2)}(-u_1,\mp a_2,-v_1,\pm a_2)\\ &=[\rho(-u_1,-v_1),\sigma_0(\mp a_2,\pm a_2)]\\ &=[\rho(u_1,v_1),\sigma_0(\pm a_2,\mp a_2)]. \end{aligned} $$

* *Let* $(u_1,u_2,v_1,v_2)\in U_i\times V_2$ *with* $\theta_i(u_1,v_1)>0$, $v_2=u_2$ *and* $u_2=\pm a_2$. *Then*

$$ {}_{(i,2)}(u_1,u_2,v_1,v_2)=\psi_{(i,2)}(u_1,\pm a_2,v_1,\pm a_2)=[\rho(u_1,v_1),\rho(\pm a_2,\pm a_2)]. $$

* *For* $(u_1,u_2,-v_1,-v_2)\in U_i\times V_2$, *we have* $\theta_i(u_1,v_1)<0$, $v_2=u_2$ *and* $u_2=\pm a_2$. *Next, this gives that:*

$$ \begin{aligned} {}_{(i,2)}(u_1,u_2,-v_1,-v_2)=\psi_{(i,2)}(u_1,\pm a_2,-v_1,\mp a_2)&=[\rho(-u_1,-v_1),\rho(\mp a_2,\mp a_2)]\\ &=[\rho(u_1,v_1),\rho(\pm a_2,\pm a_2)]. \end{aligned} $$

∗ *For* $(-u_1, -u_2, v_1, v_2) \in U_i \times V_2$, *we have* $\theta_i(u_1, v_1) < 0$, $v_2 = u_2$ *and* $u_2 = \pm a_2$. *Hence, this provides that:*

$$_{(i,2)}(-u_1, -u_2, v_1, v_2) = \psi_{(i,2)}(-u_1, \mp a_2, v_1, \pm a_2) = [\rho(u_1, v_1), \rho(\pm a_2, \pm a_2)].$$

∗ *For* $(-u_1, -u_2, -v_1, -v_2) \in U_i \times V_2$, *we have* $\theta_i(u_1, v_1) > 0$, $v_2 = u_2$ *and* $u_2 = \pm a_2$. *Accordingly, this satisfies:*

$$\begin{aligned} _{(i,2)}(-u_1, -u_2, -v_1, -v_2) &= \psi_{(i,2)}(-u_1, \mp a_2, -v_1, \mp a_2) \\ &= [\rho(-u_1, -v_1), \rho(\mp a_2, \mp a_2)] \\ &= [\rho(u_1, v_1), \rho(\pm a_2, \pm a_2)]. \end{aligned}$$

These constructions yield the following maps by considering the quotient space, since the equivalence classes are the same.

- $\bar{\psi}_{(i,0)} : \frac{(U_i \times V_0)}{\sim} \to (d\text{-}P)^{I_{m_1+m_2}}$ *is defined by:*

$$\bar{\psi}_{(i,0)}([u_1, v_1, u_2, v_2]) = [\rho(u_1, v_1), \rho(u_2, v_2)]$$

- $\bar{\psi}_{(i,1)} : \frac{(U_i \times V_1)}{\sim} \to (d\text{-}P)^{I_{m_1+m_2}}$ *is given by:*

$$\bar{\psi}_{(i,1)}([u_1, v_1, u_2, v_2]) = \begin{cases} [\rho(u_1, v_1), \sigma(u_2, v_2)] & \text{if } \theta_i(u_1, v_1) \neq 0, v_2 = -u_2, u_2 \neq \pm a_2 \\ [\rho(u_1, v_1), \rho(u_2, v_2)] & \text{if } \theta_i(u_1, v_1) \neq 0, v_2 = u_2, u_2 \neq \pm a_2 \end{cases}$$

- $\bar{\psi}_{(i,2)} : \frac{(U_i \times V_2)}{\sim} \to (d\text{-}P)^{I_{m_1+m_2}}$ *is set by:*

$$\bar{\psi}_{(i,2)}([u_1, v_1, u_2, v_2]) = \begin{cases} [\rho(u_1, v_1), \sigma_0(u_2, v_2)] & \text{if } \theta_i(u_1, v_1) \neq 0, v_2 = -u_2, u_2 \neq \pm a_2 \\ [\rho(u_1, v_1), \rho(u_2, v_2)] & \text{if } \theta_i(u_1, v_1) \neq 0, v_2 = u_2, u_2 \neq \pm a_2, \end{cases}$$

where $i \in \{0, 1, 2, 3\}$ *and* $d\text{-}P = \frac{S^2_{min} \times S^2_{min}}{\sim}$.

We obtain an explicit construction of $\frac{S^2_{min} \times S^2_{min}}{\sim} = \frac{U_0}{\sim} \cup \frac{U_1}{\sim} \cup \frac{U_2}{\sim} \cup \frac{U_3}{\sim}$ *and* $S^2_{min} \times S^2_{min} = V_0 \cup V_1 \cup V_2$. *We define the disjoint union:*

$$W_l = \bigcup_{i+j=l} (U_i \times V_j) \subset S^2_{min} \times S^2_{min} \times S^2_{min} \times S^2_{min},$$

where $l = 0, \dots, 3 + d\text{-}TCS^2_{min}$. *All subsets of* $S^2_{min} \times S^2_{min} \times S^2_{min} \times S^2_{min}$ *are compatible with respect to the equivalence relation (1). In the quotient space,*

$$\bar{W}_l = \bigcup_{i+j=l} \frac{U_i \times V_j}{\sim} \subset d\text{-}P \times d\text{-}P$$

is the disjoint union, where $l = 0, \ldots, 3 + d\text{-}TCS^2_{min}$. *We describe a motion planning strategy over each* $\bar{W}_l$ *and* $\bigcup_{l=0}^{5} \bar{W}_l$ *is a cover of* $d\text{-}P \times d\text{-}P$. *Therefore, we conclude that:*

$$d\text{-}TC(d\text{-}P) \leq d\text{-}TC(d\text{-}P^2) + d\text{-}TC(S^2_{min}).$$

4. Conclusion

The combination of topological structures with robotics forms a new area called topological robotics. Although robotics is a practical discipline, there is a theoretical side of the subject. The theoretical idea of robotics is associated with many branches of mathematics. Topology plays a key role in implementing great ideas. For instance, researchers discuss topological problems inspired by robotics and study motion planning problem, as well as the concept of Farber's topological complexity in detail. When the digital topological tools, more specifically, the notion of the digital topological complexity and related invariants are utilized in finding solutions to problems, interdisciplinary interaction will increase and hence this will open new windows in the field.

In this paper, we aim to introduce the digital projective product space and the digital projective spaces by using the digital spheres [13]. The main goal is to deal with the digital topological complexity and the digital LS-category of these spaces. We determine an upper bound for the digital LS-category and ultimately an upper bound for the digital topological complexity of the digital projective product spaces. Additionally, the digital nonsingular map characterization is used to measure the digital topological complexity of the digital projective spaces. This study reveals the digital topological complexity of the digital PPS in terms of the digital topological complexity of the digital projective space associated with the first digital sphere and the digital topological complexity of the remaining digital spheres. We accomplish this by constructing an explicit motion planning in these spaces. In this context, the advantages of more direct methods in the digital sense provide the results in [17] apart from requiring cohomological operational lower bound properties. In particular, we give examples on specific spaces to clarify our results.

This leads us to work on the digital higher topological complexity and related invariants of the digital projective product spaces, which is an open problem.

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