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The Fourier spectral method for determining a heat capacity coefficient in a parabolic equation

Durdimurod DURDIEV[1,*], **Dilshod DURDIEV**[2]
[1]Bukhara Branch of the Institute of Mathematics at the Academy of Sciences of the Republic of Uzbekistan, Bukhara, Uzbekistan
[2]Department of Materials Science and Engineering, Faculty of Engineering, University of Erlangen-Nuremberg, Fuerth, Germany

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Abstract: In this paper, the comparison of finite difference and Fourier spectral numerical methods for an inverse problem of simultaneously determining an unknown coefficient in a parabolic equation with the usual initial and boundary conditions is proposed. We represent the detailed description of the methods and their algorithms. The research work conducted in this paper shows that the Fourier spectral method is highly accurate.

Key words: Fourier spectral method, parabolic equation, inverse problem, numerical algorithm, overdetermination condition

1. Introduction

Recent advances in numerical simulations for determining unknown coefficients in a parabolic equation have led to many interesting results. In heat conduction, attention was paid to the unique solvability of one-dimensional inverse problems for the heat equation in the case when the unknown thermal coefficients are constant [2], time-dependent [11, 12], space-dependent [1], or temperature-dependent [8, 10, 14, 15]. Most of these simulations have been carried out with the finite difference method (FDM) [7, 9, 13]. However, the inverse problem of a heat equation with the time-dependent coefficient of heat capacity has mathematically smooth solutions. Accordingly, one expects that the Fourier spectral method (FSM) should be optimal in terms of efficiency and accuracy. Moreover, numerical studies can also be conducted for a coefficient determination problem for a fractional diffusion equation [6].

The spectral method, which belongs to the set of weighted residual methods, was originally developed for solving the spatial part of partial differential equations. For detailed, precise, and numerical information on spectral methods, see the work in [16, 17] and references therein. This allows one to save computational resources when evaluating the differentiation and to employ an efficient algorithm such as the fast Fourier transform when the number of grid nodes is even. The advantage of the spectral method over other numerical methods in solving linear PDEs is its high accuracy; when solutions of PDEs are smooth enough, errors of numerical solutions decrease exponentially as the number of discretization nodes increases, while the finite difference method leads to the algebraically decreasing error in fact.

*Correspondence: d.durdiev@mathinst.uz
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In Figure 1, the order of the spectral accuracy is compared to the order of accuracy of the finite difference. In this case, we use a simple function $u(x) = e^{sin(x)}$ on the periodic domain $[-\pi, \pi]$; hence, $u'(x) = con(x)e^{sin(x)}$. An error is defined as $Error = max|u'(x_i) - u_i^{app}|$, $1 \leq i \leq N$, where u_i^{app} is an approximation of $u'(x_i)$ and N is the number of grid points. According to Figure 1, the error is decreasing exponentially via the Fourier spectral method indeed.

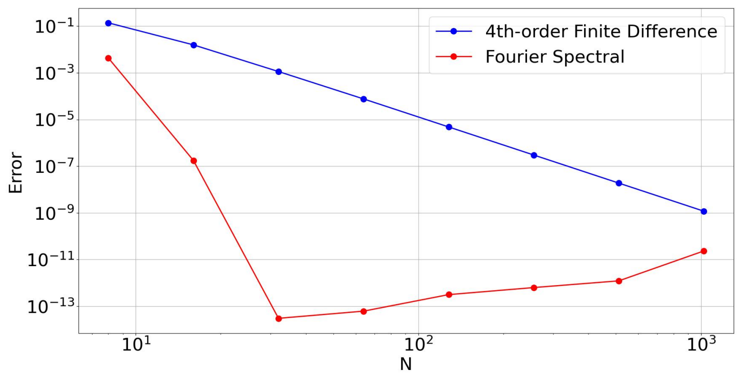

Figure 1. Convergence of fourth-order finite difference and Fourier-spectral method where $u(x) = e^{sin(x)}$. Both axes are on log-scale.

In this paper, we investigate the inverse problem for simultaneous determination of a time-dependent coefficient in the one-dimensional heat equation. The paper is organized as follows. In the next section, we give the mathematical formulations of the inverse problem. The numerical setup is presented in Section 3. The numerical finite-difference discretization of the direct problem is described in Subsection 3.1, whilst Subsection 3.2 introduces the numerical implementation of the direct problem using the Fourier spectral method. In Section 4, we provide numerical results and discussion. Finally, conclusions are presented in Section 5.

2. Mathematical formulation of the inverse problem

Consider the linear one-dimensional parabolic equation on a periodic domain with a time-dependent coefficient:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} - q(t)u(x,t), \quad (x,t) \in (0,l) \times (0,T] =: \Omega_T, \tag{2.1}$$

where $u(x,t)$ represents the temperature in a finite slab of length $l > 0$ over time interval $(0,T]$ with $T > 0$, $q(t)$ represents the coefficient of heat capacity.

We study the inverse problem to find the coefficient q together with the solution of u in Eq. (2.1) under the following conditions: initial condition:

$$u(x,0) = \varphi(x), \quad x \in [0,l], \tag{2.2}$$

boundary and overdetermination conditions:

$$u(0,t) = u(l,t) = 0, \quad \frac{\partial u(0,t)}{\partial x} = h(t). \tag{2.3}$$

The first conditions of (2.3) represent the specification of the boundary temperature.

Definition 2.1 *The pair* $\{q(t), u(x,t)\}$ *from the class* $C[0,T] \times C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T)$, *for which equalities* (2.1)–(2.3) *are satisfied and* $q(t) \geq 0$ *on the* $[0,T]$, *is called a classical solution of the inverse problem* (2.1)–(2.3).

We assume that the data of the problem (2.1)–(2.3) satisfy the following conditions:
$\left(A_1\right)$ $\varphi(x) \in C^3[0,l],\ \varphi(0) = \varphi(l) = 0,\ \varphi''(0) = \varphi''(l) = 0,\ \varphi'(0) = h(0);$
$\left(A_2\right)$ $h(t) \in C^1[0,T],\ |h(t)| \geq h_0 > 0,\ h_0 = const.$

Theorem 2.2 *Let the assumptions* $\left(A_1\right)$ *and* $\left(A_2\right)$ *be valid. Then, the inverse problem* (2.1)–(2.3) *has a unique classical solution in* Ω_{T_0}, *where the number* T_0 $\left(0 < T_0 < T\right)$ *is determined by the data of the problem.*

In order to show the existence of a unique solution to inverse problem (2.1)–(2.3), for convenience, introducing a new function by formula $v(x,t) = \frac{\partial}{\partial x}u(x,t)$, we reduce the inverse problem (2.1)–(2.3) to the following form:

$$\frac{\partial v(x,t)}{\partial t} = \frac{\partial^2 v(x,t)}{\partial x^2} - q(t)v(x,t), \quad (x,t) \in \Omega_T, \tag{2.4}$$

$$v(x,0) = \varphi'(x), \quad x \in [0,l], \tag{2.5}$$

$$v_x(0,t) = v_x(l,t) = 0, \quad t \in [0,T], \tag{2.6}$$

$$v(0,t) = h(t), \quad t \in [0,T]. \tag{2.7}$$

After determining $\{q, v\}$, the pair $\{q, u\}$ will be the classical solution to the inverse problem (2.1)–(2.3), where $u(x,t) = \int_0^x v(\xi,t)d\xi$. Firstly, we study the direct problem (2.4)–(2.6). At the same time, function $q(t)$ will be considered known and $q(t) \in C[0,T]$. The function $v(x,t)$, as a solution to the second initial-boundary value problem (2.4)–(2.6), satisfies the integral equation:

$$v(x,t) = \int_0^l G(x,\xi,t)\varphi'(\xi)d\xi - \int_0^t q(\tau)\int_0^l G(x,\xi,t-\tau)v(\xi,\tau)d\xi d\tau, \quad (x,t) \in \overline{\Omega}_T, \tag{2.8}$$

where $G(x,\xi,t) = \frac{2}{l}\left\{\frac{1}{2} + \sum_{n=1}^{\infty} \exp\left[-\left(\frac{\pi n}{l}\right)^2 t\right] \cos\frac{\pi n}{l}x \cos\frac{\pi n}{l}\xi\right\}$ is Green's function of the second initial boundary value problem for the heat equation. By definition, it satisfies (in the generalized sense) the following equations:

$$\frac{\partial G}{\partial t} = \frac{\partial^2 G}{\partial x^2},$$

$$\lim_{t \to 0} G(x,\xi,t) = \delta(x-\xi),\ \frac{\partial G}{\partial x}(0,\xi,t) = \frac{\partial G}{\partial x}(l,\xi,t) = 0,$$

where $\delta(\cdot)$ is Dirac's delta function.

We shall now demonstrate that Eq. (2.8) determines a single continuous solution within the domain Ω_T. For this purpose, the method of successive approximation will be used, presenting $v(x,t)$ as a series:

$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t), \tag{2.9}$$

where $v_0(x,t)=\int_0^l G(x,\xi,t)\varphi'(\xi)d\xi$, and $v_n(x,t),\ n\geq 1$, are obtained by:

$$v_n(x,t)=-\int_0^t q(\tau)\int_0^l G(x,\xi,t-\tau)v_{n-1}(\xi,\tau)d\xi d\tau, \quad n\geq 1 \quad (x,t)\in\overline{\Omega}_T. \tag{2.10}$$

Denote $\varphi_0=\max\limits_{x\in[0,l]}|\varphi'(x)|, \quad q_0=\max\limits_{t\in[0,T]}|q(t)|$. According to the formula (2.10), we estimate $u_n(x,t)$ for $(x,t)\in\overline{\Omega}_T$ as follows:

$$|v_0(x,t)|\leq\varphi_0, \quad |v_n(x,t)|\leq\varphi_0\frac{(q_0t)^n}{n!}, \quad n\geq 1. \tag{2.11}$$

Estimates (2.11) show that series (2.9) converges uniformly in the domain Ω_T, since it is majorized in $\overline{\Omega}_T$ by convergent numerical series $\varphi_0\sum_{n=0}^{\infty}(q_0T)^n/n!$. Thus, it determines continuously within the domain $\overline{\Omega}_T$ of function $\vartheta(x,t)$, which is the solution to (2.8). This solution is unique since the uniform equation corresponding to (2.8):

$$v(x,t)=-\int_0^t q(\tau)\int_0^l G(x,\xi,t-\tau)v(\xi,\tau)d\xi d\tau \tag{2.12}$$

has only the zero solution in the class of continuous in $\overline{\Omega}_T$ functions.

Indeed, if

$$v(t)=\max_{0\leq x\leq l}v(x,t),$$

then, (2.12) yields:

$$v(t)\leq q_0\int_0^t v(\tau)d\tau \quad t\in[0,T].$$

It is known that this integral inequality has the only solution, which is $v(t)=0$; hence, $v(x,t)=0$ for $(x,t)\in\overline{\Omega}_T$. Under the conditions of A_1, the series obtained by differentiating once by x is uniformly convergent in $\overline{\Omega}_T$. Therefore, its sum $v_x(x,t)$, like $v(x,t)$, is convergent in $\overline{\Omega}_T$. From this and in view of that $G(x,\xi,t)$ is infinitely continuously differentiable in Ω_T, we have $v(x,t)\in C^{2,1}(\Omega_T)\cap C^{1,0}(\overline{\Omega}_T)$.

In addition, $v_t(x,t), v_{xx}(x,t)$ are continuous in $\overline{\Omega}_T$. Indeed, by differentiating (2.8) in t and using relation $\lim_{t\to 0}G(x,\xi,t)=\delta(x-\xi)$, we obtain:

$$v_t(x,t)=\frac{\partial}{\partial t}\int_0^l G(x,\xi,t)\varphi'(\xi)d\xi-q(t)v(x,t)-\int_0^t q(\tau)\int_0^l\frac{\partial}{\partial t}G(x,\xi,t-\tau)v(\xi,\tau)d\xi d\tau.$$

Using equalities $\frac{\partial}{\partial t}G=\frac{\partial^2}{\partial x^2}G, \quad G_\xi(x,0,t)=G_\xi(x,l,t)=0,$ and integrating by parts, we find:

$$\frac{\partial}{\partial t}\int_0^l G(x,\xi,t)\varphi'(\xi)d\xi=\int_0^l G_{\xi\xi}(x,\xi,t)\varphi(\xi)d\xi=G_\xi(x,\xi,t)\varphi'(\xi)\Big|_0^l-$$

$$-\int_0^l G_\xi(x,\xi,t)\varphi''(\xi)d\xi=-\int_0^l G_\xi(x,\xi,t)\varphi''(\xi)d\xi=\int_0^l G(x,\xi,t)\varphi'''(\xi)d\xi.$$

Transform now the inner integral of the second term in (2.15) by a similar way. Taking into account also conditions (2.6), we get:

$$v_t(x,t) = \int_0^l G(x,\xi,t)\varphi'''(\xi)d\xi + \int_0^t q(\tau)\int_0^l G(x,\xi,t-\tau)v_{\xi\xi}(\xi,\tau)d\xi d\tau - q(t)v(x,t). \tag{2.13}$$

Note also that function $v(x,t)$ satisfies the (2.4), i.e. $v_{xx}(x,t) = v_t(x,t) + q(t)v(x,t)$. In view of this equality, we rewrite (2.13) in the following form:

$$v_t(x,t) = \int_0^l G(x,\xi,t)\varphi'''(\xi)d\xi - \int_0^t q(\tau)\int_0^l G(x,\xi,t-\tau)\big[v_t(\xi,\tau) + q(\tau)v(\xi,\tau)\big]d\xi d\tau - q(t)v(x,t).$$

The last equation can be considered the Volterra type integral equation with respect to v_t with continuous free term and kernel (q and v are known functions). As usually, this equation determines the continuous within the domain $\overline{\Omega}_T$ function $v_t(x,t)$. Since the right side of $v_{xx}(x,t) = v_t(x,t) + q(t)v(x,t)$ is continuous in $\overline{\Omega}_T$, then $v_{xx}(x,t) \in C(\overline{\Omega}_T)$.

Now we will start studying the inverse problem (2.4)–(2.7). In (2.13), we set $x = 0$ and use the overdetermination condition (2.7). Resolving the resulting equality with respect to $q(t)$, we obtain:

$$q(t) = -\frac{h'(t)}{h(t)} + \frac{1}{h(t)}\int_0^l G(0,\xi,t)\varphi'''(\xi)d\xi - \frac{1}{h(t)}\int_0^t q(\tau)\int_0^l G(0,\xi,t-\tau)v_{\xi\xi}(\xi,\tau)d\xi d\tau, \quad t \in [0,T]. \tag{2.14}$$

Considering (2.14) and equation for v_{xx}, which is obtained by differentiating twice the equality (2.8) in the variable x and integrating by part:

$$v_{xx}(x,t) = \int_0^l G(x,\xi,t)\varphi'''(\xi)d\xi - \int_0^t q(\tau)\int_0^l G(x,\xi,t-\tau)v_{\xi\xi}(\xi,\tau)d\xi d\tau, \quad (x,t) \in \overline{\Omega}_T, \tag{2.15}$$

we see that these equations constitute a closed system of integral equations of the Volterra type with respect to unknown functions q, v_{xx}. The proof of Theorem 2.2 is completed by application of the fixed point principle (Banach's theorem) to the system of integral equations (2.14), (2.15). On the application of the fixed point argument to solving of inverse problems for parabolic equations, see [3–5]. By found function q, v_{xx}, function v is determined via formula:

$$v(x,t) = v(0,t) + \int_0^x (x-\xi)v_{\xi\xi}(\xi,t)d\xi,$$

where $v(0,t)$ is the value of the solution of integral equation (2.8) with known function $q(t)$ at $x = 0$.

3. Numerical procedure

In this section, we represent the finite difference and Fourier spectral methods for the numerical solution of Eq. (2.1) with initial boundary (2.2) and overdetermination (2.3) conditions in a line segment $\Omega = [l_x, r_x]$. Let N_x be positive even integer and $L_x = r_x - l_x$ be the length of a line segment; hence, define $\Delta x = L_x/N_x$ as the spatial step size. We denote discretized points as $x_j = l_x + j\Delta x$ where $0 \le j \le N_x$ is integer. Let u_j^n be an approximation of $u(x_j, t_n)$, where $t_n = n\Delta t$ and $\Delta t = T/N_t$ is the temporal step size, N_t is the number of time steps.

3.1. Finite difference method

By utilizing the forward difference for the time derivative and centered second-order finite difference for the spatial derivative, Eq. (2.1) takes the following form:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} - q^n u_j^n, \quad 0 \le j \le N_x, \quad 0 \le n \le N_t.$$

With forward Euler time marching, the u_j^{n+1} at grid point j for the time step $n+1$ results in:

$$u_j^{n+1} = u_j^n(1 - \Delta t q^n) + \Delta t\Big(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}\Big), \quad 0 \le j \le N_x, \quad 0 \le n \le N_t. \tag{3.1}$$

Because the u_j^n is known at the time step $n = 0$, due to the initial boundary condition (2.2), the explicit Euler time marching scheme enables the direct solution of Eq. (3.1). The overdetermination condition (2.3) has been used to compute the unknown coefficient q^n by applying the forward difference for the spatial derivative in the left-hand side of Eq. (2.3) at $j = 0$,

$$\frac{u_1^{n+1} - u_0^{n+1}}{\Delta x} = h^{n+1}, \quad 0 \le n \le N_t. \tag{3.2}$$

Eq. (3.1) can be represented as:

$$u_j^{n+1} = u_j^n(1 - \Delta t q^n) + \Delta t A_j^n, \quad 0 \le j \le N_x, \quad 0 \le n \le N_t,$$

where

$$A_j^n = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}, \quad 0 \le j \le N_x, \quad 0 \le n \le N_t, \tag{3.3}$$

at the discretized point $j = 0$ and $j = 1$,

$$u_0^{n+1} = u_0^n(1 - \Delta t q^n) + \Delta t A_0^n, \quad 0 \le n \le N_t,$$

$$u_1^{n+1} = u_1^n(1 - \Delta t q^n) + \Delta t A_1^n, \quad 0 \le n \le N_t.$$

The q^n can be obtained by substituting u_0^{n+1} and u_1^{n+1} in Eq. (3.2), such as,

$$\frac{(u_1^n - u_0^n)(1 - \Delta t q^n) + \Delta t(A_1^n - A_0^n)}{\Delta x} = h^{n+1}, \quad 0 \le n \le N_t$$

and q^n,

$$q^n = \frac{1}{\Delta t}\Big(1 - \frac{\Delta x h^{n+1} - \Delta t(A_1^n - A_0^n)}{u_1^n - u_0^n}\Big), \quad 0 \le n \le N_t. \tag{3.4}$$

Since $u_0^n = 0$ and $A_0^n = 0$, due to the boundary condition (2.2), Eq. (3.4) can be reduced to the following form:

$$q^n = \frac{1}{\Delta t}\Big(1 - \frac{\Delta x h^{n+1} - \Delta t A_1^n}{u_1^n}\Big), \quad 0 \le n \le N_t. \tag{3.5}$$

Thus, for (2.1), we now have the following solution steps:

1) for the known u_j^n, compute A_j^n using (3.3) at $n = 0$;

2) at $j = 1$, evaluate q^n using (3.4);

3) for the known u_j^n and q^n, find u_j^{n+1} using (3.1).

3.2. Fourier spectral method

For the given data u_j^n, where $1 \leq j \leq N_x$, the discrete Fourier transform is defined as:

$$\hat{u}_m^n = \sum_{j=1}^{N_x} u_j^n e^{-ik_m x_j}, \quad -\frac{N_x}{2} + 1 \leq m \leq \frac{N_x}{2}, \tag{3.6}$$

where $k_m = 2\pi m/L_x$. The inverse discrete Fourier transform is:

$$u_j^n = \frac{1}{N_x} \sum_{m=1}^{N_x} \hat{u}_m^n e^{ik_m x_j}. \tag{3.7}$$

Note that we can obtain spectral derivatives as if we perform an analytic differentiation in the Fourier space. We assume that $u(x,t)$ is sufficiently smooth and extended to continuous version of the numerical approximation u_j^n. The following shows step-by-step description of how the differentiation works in the Fourier transform with finite basis.

$$\frac{\partial}{\partial x} u(x,t) = \frac{1}{N_x} \sum_{m=1}^{N_x} (ik_m) \hat{u}(k_m, t) e^{ik_m x}.$$

This process enables one to derive spectral derivatives in the Fourier space easily, not differentiate directly in the physical space. Therefore, we can represent the second derivative in the Fourier space as follows:

$$\frac{\partial^2}{\partial x^2} u(x,t) = \frac{1}{N_x} \sum_{m=1}^{N_x} (-k_m^2) \hat{u}(k_m, t) e^{ik_m x}.$$

Now we present the numerical solutions of Eq. (2.1). Firstly, we derive the numerical solution for $u(x,t)$, which starts with the Fourier transform of both sides of Eq. (2.1).

$$\frac{\partial \{u\}_m}{\partial t} = \left\{ \frac{\partial^2 u}{\partial x^2} \right\}_m - q(t)\{u\}_m, \tag{3.8}$$

where $\{\cdot\}_m$ is the Fourier transform of the quantity inside the bracket and m is the coefficient of the m-Fourier mode. Then, in the Fourier space, Eq. (3.8) becomes:

$$\frac{\partial \hat{u}_m}{\partial t} = -k_m^2 \hat{u}_m - q(t)\hat{u}_m.$$

Taking the forward difference in time derivative yields:

$$\frac{\hat{u}_m^{n+1} - \hat{u}_m^n}{\Delta t} = -k_m^2 \hat{u}_m^{n+1} - q^n \hat{u}_m^n$$

in which Δt is the time between the time steps $n+1$ and n. Therefore, with forward Euler time marching, we obtain the following discrete Fourier transform:

$$\hat{u}_m^{n+1} = \hat{u}_m^n \Big(\frac{1 - \Delta t q^n}{1 + \Delta t k_m^2} \Big). \tag{3.9}$$

Then, the updated numerical solution u_j^{n+1} can be computed using Eq. (3.7):

$$u_j^{n+1} = \frac{1}{N_x} \sum_{m=1}^{N_x} \hat{u}_m^{n+1} e^{ik_m x_j}. \tag{3.10}$$

Next, we employ the overdetermination condition (2.3) to obtain the unknown coefficient q^n by taking the spectral derivative in the Fourier space, Eq. (2.3) can be rewritten as follows:

$$\begin{aligned} \frac{\partial u(x)}{\partial x}\Big|_{x=0}^{n+1} &= \Big\{ \frac{1}{N_x} \sum_{m=1}^{N_x} (ik_m) \hat{u}_m^{n+1} e^{ik_m x} \Big\}_{x=0} \\ &= \frac{1}{N_x} \sum_{m=1}^{N_x} (ik_m) \hat{u}_m^{n+1} = h^{n+1}. \end{aligned} \tag{3.11}$$

By inserting (3.10) in (3.11), we obtain the following:

$$\frac{1}{N_x} \sum_{m=1}^{N_x} \Big[(ik_m) \hat{u}_m^n \Big(\frac{1 - \Delta t q^n}{1 + \Delta t k_m^2} \Big) \Big] = \frac{1 - \Delta t q^n}{N_x} \sum_{m=1}^{N_x} \frac{ik_m \hat{u}_m^n}{1 + \Delta t k_m^2} = h^{n+1}. \tag{3.12}$$

Now, the unknown coefficient q^n can be easily determined from (3.12):

$$q^n = \Big(1 - \frac{N_x h^{n+1}}{A_m^n} \Big) \frac{1}{\Delta t}, \quad A_m^n = \sum_{m=1}^{N_x} \frac{ik_m \hat{u}_m^n}{1 + \Delta t k_m^2}. \tag{3.13}$$

Thus, for (2.1), we now have the following solution steps with the Fourier spectral method:

1) perform the discrete Fourier transform (3.6) of u_j^n;

2) compute q^n using (3.13);

2) evaluate the updated numerical solution in the Fourier space using (3.9);

3) perform the inverse discrete Fourier transform (3.7) of u_j^{n+1}.

4. Numerical results and discussion

Numerical results obtained from both methods are presented for the test example for the inverse problem (2.1)–(2.3), in which we obtain the numerical solution for the coefficient of heat capacity and temperature, respectively. In this example, we take, for simplicity, $l = 2\pi$ and $T = 1$. The computational details have already been given in Section 3. We have also calculated the relative error to analyse the error between the exact and estimated solutions, defined as:

$$\eta(u) = max_{1 \le i \le N_x} |u_i^{numerical} - u_i^{exact}|$$

$$\eta(q) = max_{1 \le i \le N_t} |q_i^{numerical} - q_i^{exact}|$$

We solve this inverse problem (2.1)–(2.3) with following input data:

$$\varphi(x) = sin(x), \quad h(t) = 1 + t$$

for $x \in (0, l = 2\pi)$ and $t \in (0, T = 1)$. The exact solution is given by:

$$u(x,t) = (1+t)sin(x), \quad q(t) = \frac{-2-t}{1+t}.$$

Table 1 gives the numerical coefficients, obtained by FDM and FSM for $q(t)$ using $N_x \in \{32, 64, 128\}$ in comparison with the exact ones. In Figure 2, we present the plots of numerical and analytical results for both methods, as it can be seen, the comparison is relatively good for both methods. Table 2 illustrates the absolute errors for $u(x,t)$ for different number of grid points N_x. Figure 3 shows the numerical solutions of $u(x,t)$ obtained using the proposed numerical approaches at time $t = 1$.

Table 1. The exact and numerical coefficients of $q(t)$ for $N_x \in \{32, 64, 128\}$ done by FDM and FSM.

t	0.1	0.2	0.3	...	0.8	0.9	1	N_x
FDM	−1.9069	−1.8310	−1.7668	...	−1.5528	−1.5235	−1.4972	32
	−1.9092	−1.8333	−1.7691	...	−1.5552	−1.5258	−1.4995	64
	−1.9098	−1.8339	−1.7697	...	−1.5557	−1.5264	−1.5001	128
	−1.9091	−1.8333	−1.7692	...	−1.5555	−1.5263	−1.5000	exact
FSM	−1.91082	−1.83486	−1.77059	...	−1.55642	−1.52712	−1.50075	32, 64, 128
	−1.9091	−1.8333	−1.7692	...	−1.5555	−1.5263	−1.5000	exact

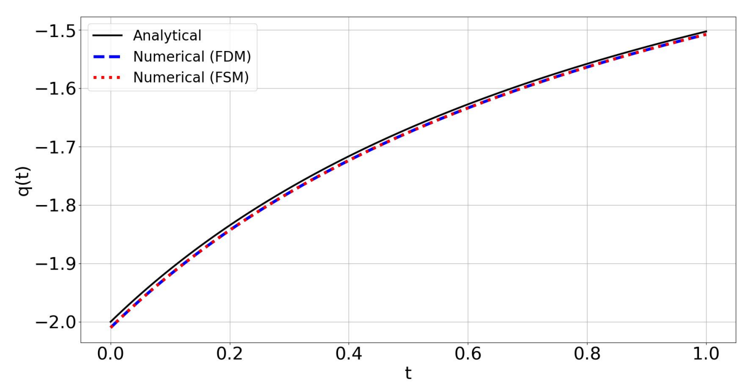

Figure 2. Comparison of the analytical and numerical solutions of $q(t)$, FDM, and FSM.

Table 2. The absolute errors for $u(x,t)$ obtained by FDM and FSM for $N_x \in \{32, 64, 128\}$.

t	0.1	0.2	0.3	...	0.8	0.9	1	N_x
FDM	6.229e-03	6.821e-03	7.417e-03	...	1.044e-02	1.106e-02	1.168e-02	32
$\eta(u)$	7.877e-04	8.711e-04	9.563e-04	...	1.412e-03	1.509e-03	1.608e-03	64
	5.966e-04	6.451e-04	6.926e-04	...	9.156e-04	9.572e-04	9.979e-04	128
FSM	4.774e-15	6.328e-15	4.775e-15	...	1.554e-14	1.376e-14	1.443e-14	32
$\eta(u)$	5.773e-15	5.662e-15	4.441e-15	...	1.112e-14	1.398e-14	2.132e-14	64
	2.886e-15	5.218e-15	9.104e-15	...	1.311e-14	1.176e-14	1.421e-14	128

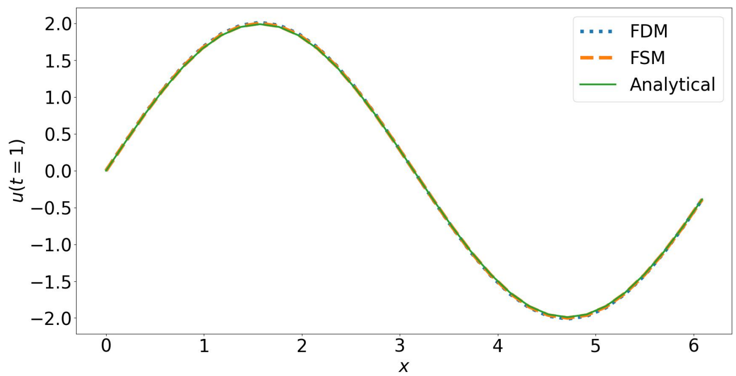

Figure 3. Comparison of the numerical solutions (FDM, FSM) with the analytical solution of $u(x,t=1)$.

5. Conclusion

This paper has presented the finite difference and Fourier spectral numerical approaches to identify simultaneously the time-dependent coefficient in the one-dimensional parabolic heat equation. The resulting inverse problem have been reformulated as constrained regularized minimization problem which was solved using MATLAB Optimization Toolbox routines. The numerically obtained results show that by increasing the number of grid points, which requires higher computation time, in the finite difference method, the absolute errors $\eta(u)$ and $\eta(q)$ are decreasing indeed. Nevertheless, the Fourier spectral method has the highest accuracy in $u(x,t)$ compared to the finite difference method. Moreover, one has to mention that changing the number of discretizated points in the Fourier spectral method has almost no influence in $\eta(u)$ and $\eta(q)$.

In the future, it is important to study the application possibilities of these numerical algorithms for the problem of determining the coefficients in fractional diffusion equations [6].

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