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$3D$-flows generated by the curl of a vector potential & Maurer-Cartan equations

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Abstract: We examine $3D$ flows $\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x})$ admitting vector identity $M\mathbf{v} = \nabla \times \mathbf{A}$ for a multiplier M and a potential field $\mathbf{A}$. It is established that, for those systems, one can complete the vector field $\mathbf{v}$ into a basis fitting an $\mathfrak{sl}(2)$-algebra. Accordingly, in terms of covariant quantities, the structure equations determine a set of equations in Maurer-Cartan form. This realization permits one to obtain the potential field as well as to investigate the (bi-)Hamiltonian character of the system. The latter occurs if the system has a time-independent first integral. In order to exhibit the theoretical results on some concrete cases, three examples are provided, namely the Gulliot system, a system with a nonintegrable potential, and the Darboux-Halphen system in symmetric polynomials.

Key words: $3D$-flows, vector potential, bi-Hamiltonian systems, Maurer-Cartan equations

1. Introduction

In a recent work [6], we have presented (bi)Hamiltonian analysis of $3D$ dynamical systems $\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x})$ where the velocity field is the curl of a vector potential that is $\mathbf{v} = \nabla \times \mathbf{A}$ for some $\mathbf{A}$. The analysis was divided into two cases according to the (Frobenius) integrability of the potential vector field $\mathbf{A}$. Two examples have been provided; a bi-Hamiltonian system admitting a nonintegrable potential and a non-Hamiltonian system admitting an integrable potential. If a system possesses bi-Hamiltonian character then the flow is the line of intersection of two surfaces determined by the Hamiltonian functions [1, 2, 21]. This geometric realization is a particular instance of superintegrability [7].

In the present paper, we are addressing the same problem from an algebraic point of view. The goal is to determine a potential field $\mathbf{A}$ for a given system $\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x})$ satisfying $\mathbf{v} = \nabla \times \mathbf{A}$. Evidently, this holds for volume-preserving flows. For the other case, the curl identity needs to be upgraded to $M\mathbf{v} = \nabla \times \mathbf{A}$ where M is a conformal factor called Jacobi last multiplier, see [3, 22]. More concretely, in this work, we shall argue that the existence of a vector potential is manifesting a representation of $\mathfrak{sl}(2)$ algebra of vector fields spanning the configuration space. Referring to the dualization between a vector field and a one-form section, we carry this algebra to the level of differential forms. This leads us to determine Maurer-Cartan type equations closing the exterior algebra of sections. We call this geometrization as Maurer-Cartan $\mathfrak{sl}(2)$ algebra of curl vector fields. Interestingly, in the Maurer-Cartan $\mathfrak{sl}(2)$ algebra, one of the one-form sections is not necessarily integrable. This

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permits us to claim that the present analysis is applicable even for nonintegrable cases. Further, referring to the conformal invariance of the algebra, we shall state that a perturbation is possible by taking a nonintegrable potential to an integrable one.

The paper is organized into 5 main sections. In the following one, we shall state some necessary background of $3D$ systems. Then, in Section 3, we shall focus on $3D$ systems admitting vector potentials. In Section 4, we shall provide Maurer-Cartan $\mathfrak{sl}(2)$ algebra of one-forms. Referring to the classical duality, in Section 5, it will be shown that $3D$ systems admitting vector potentials are inducing the $\mathfrak{sl}(2)$ algebra of vector fields. Three examples will be provided in Section 6 including the Gulliot system, a system with a nonintegrable potential, and the Darboux-Halphen system in symmetric polynomials.

2. Bi-Hamiltonian dynamics in $3D$

Let $(\mathcal{P},\{\bullet,\bullet\})$ be a 3-dimensional Poisson manifold equipped with a Poisson bracket $\{\bullet,\bullet\}$. Hamilton's equation generated by a Hamiltonian function H is defined to be

$$\dot{\mathbf{x}} = \{\mathbf{x}, H\}, \tag{2.1}$$

for local coordinates $(\mathbf{x})$ on $\mathcal{P}$. In 3-dimensions, we can replace the role of a Poisson bracket with a Poisson vector $\mathbf{J}$ [5, 12, 13]. In this case, the Jacobi identity turns out to be the following vector equation

$$\mathbf{J}\cdot(\nabla\times\mathbf{J}) = 0, \tag{2.2}$$

whereas Hamilton's equation (2.1) takes the particular form

$$\dot{\mathbf{x}} = \mathbf{J}\times\nabla H. \tag{2.3}$$

Here, ∇H is the gradient of H. The following theorem is exhibiting all possible solutions of the Jacobi identity (2.2) so it characterizes Poisson structures in 3-dimensions [14–16].

Theorem 2.1 *The general solution of the vector equation (2.2) is $\mathbf{J} = (1/M)\nabla F$ for arbitrary functions M and F.*

The existence of scalar multiple $1/M$ in the solution is a manifestation of conformal invariance of the identity (2.2). In the literature, M is called Jacobi's last multiplier [17, 18]. In this picture, a Hamiltonian system has the following generic form

$$\dot{\mathbf{x}} = \frac{1}{M}\nabla F\times\nabla H. \tag{2.4}$$

A dynamical system is bi-Hamiltonian if it admits two different Hamiltonian structures

$$\dot{\mathbf{x}} = \{\mathbf{x}, H_2\}_1 = \{\mathbf{x}, H_1\}_2, \tag{2.5}$$

with the requirement that the Poisson brackets $\{\bullet,\bullet\}_1$ and $\{\bullet,\bullet\}_2$ be compatible [8, 23]. That is any linear pencil $\{\bullet,\bullet\}_1 + c\{\bullet,\bullet\}_2$ must satisfy the Jacobi identity [19, 23]. In three dimensions, a bi-Hamiltonian system can be put into the form

$$M\dot{\mathbf{x}} = \mathbf{J}_1\times\nabla H_2 = \mathbf{J}_2\times\nabla H_1. \tag{2.6}$$

Referring to the system (2.4), we conclude that a Hamiltonian system in the form of (2.4) is bi-Hamiltonian

$$M\dot{\mathbf{x}} = \nabla H_1 \times \nabla H_2 = \mathbf{J}_1 \times \nabla H_1 = \mathbf{J}_2 \times \nabla H_2, \tag{2.7}$$

where, the first Poisson vector $\mathbf{J}_1 = -\nabla H_2$ whereas the second Poisson vector $\mathbf{J}_2 = \nabla H_1$. The following theorem determines the Hamiltonian picture of three-dimensional dynamical systems admitting an integral invariant. For the proof, we refer [5, 9].

Theorem 2.2 *A three-dimensional dynamical system* $\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x})$ *having a time-independent first integral is bi-Hamiltonian if and only if there exists a conformal factor, called Jacobi's last multiplier,* M *which makes* $M\mathbf{v}$ *divergence-free.*

3. Curl fields

We have depicted the generic form of the $3D$ dynamical systems in (2.4). In this section, for a given $3D$ dynamical system $\mathbf{v} = \dot{\mathbf{x}}$, we examine the existence of a potential vector field $\mathbf{A}$ determining the dynamics up to some conformal factor.

Assume that the dynamical field is in the bi-Hamiltonian form $\mathbf{v} = \nabla H_1 \times \nabla H_2$ where the multiplier M is unity. In this case, we can write $\mathbf{v}$ as a curl vector $\mathbf{v} = \nabla \times \mathbf{A}$ with $\mathbf{A} = H_1 \nabla H_2$ so that $\nabla \cdot \mathbf{v} = 0$. To see this realization in a covariant formulation, consider the standard coordinates (x, y, z) on the space, and define a two-form

$$\mathbf{v} \cdot d\mathbf{x} \wedge d\mathbf{x} := \iota_v \left(dx \wedge dy \wedge dz \right), \tag{3.1}$$

where ι_v is the interior product by the vector field $v = \mathbf{v} \cdot \nabla$. The volume preserving the character of the field $\mathbf{v}$ can be recorded in terms of the Lie derivative as

$$\mathcal{L}_v (dx \wedge dy \wedge dz) = d\iota_v (dx \wedge dy \wedge dz) = (\nabla \cdot \mathbf{v})\, dx \wedge dy \wedge dz = 0, \tag{3.2}$$

where we have employed the Cartan's identity $\mathcal{L}_v = d\iota_v + \iota_v d$. If $\mathbf{v}$ is not divergence-free, that is $\nabla \cdot \mathbf{v} \neq 0$, then assume an invariant volume

$$*1 = M dx \wedge dy \wedge dz \tag{3.3}$$

involving a conformal factor M. Here, $*$ is the Hodge star operator. This is a manifestation of Theorem 2.2. In this case, one can recast the conservation of the invariant volume as

$$\mathcal{L}_v \left(*1 \right) = d\iota_v \left(*1 \right) = \left(\nabla \cdot M\mathbf{v} \right) dx \wedge dy \wedge dz. \tag{3.4}$$

Accordingly, a vector potential in form $\mathbf{A} = H_1 \nabla H_2$ satisfies the equation $M\mathbf{v} = \nabla \times \mathbf{A}$.

To switch to a covariant picture, we define the Poisson one-form

$$\mathbf{J} \longrightarrow J = J_i dx^i. \tag{3.5}$$

In this case, the Jacobi identity is given by the Frobenius integrability condition $J \wedge dJ = 0$. We have coordinate independent manifestation of the dynamics

$$\iota_v \left(*1 \right) = J^{(1)} \wedge J^{(2)} \tag{3.6}$$

where the left-hand side can be written as

$$\begin{aligned} \iota_v \left(*1 \right) &= (M\mathbf{v}) \cdot d\mathbf{x} \wedge d\mathbf{x} = \nabla \times \mathbf{A} \cdot d\mathbf{x} \wedge d\mathbf{x} \\ &= d \left(\mathbf{A} \cdot d\mathbf{x} \right) = d\gamma \end{aligned}$$

for a one-form (potential) $\gamma = \mathbf{A} \cdot d\mathbf{x}$ of $M\mathbf{v}$. Thus, casting $M\mathbf{v}$ into bi-Hamiltonian form is the same as writing a Maurer-Cartan like equation

$$d\gamma = J^{(1)} \wedge J^{(2)}. \tag{3.7}$$

4. Structure equations

Given a three dimensional dynamical system $\dot{\mathbf{x}} = \mathbf{v}$, our interest is investigating a potential one-form for $\mathbf{v}$. We first assume the existence of an integrable one-form potential. Then, referring to this one-form, we shall construct an algebra in the space of one-form sections on $\mathbb{R}^3$. Later, the case of nonintegrable potential one-forms will be examined in light of the present discussion.

Assume that γ represents an integrable potential one-form for $\mathbf{v}$. The integrability condition $\gamma \wedge d\gamma = 0$ implies that there exists a one-form α, mimicking the role of an integrating factor, such that

$$d\gamma = 2\alpha \wedge \gamma. \tag{4.1}$$

Taking the exterior derivative of (4.1), we arrive at the following

$$2d\alpha \wedge \gamma = 2\alpha \wedge 2\alpha \wedge \gamma = 0. \tag{4.2}$$

This identity determines two possibilities. First, α is a closed one-form, then we can integrate and we are done. Second, $d\alpha \neq 0$ and we have

$$d\alpha = \gamma \wedge \beta \tag{4.3}$$

for some one-form β. Since we have assumed that α is not closed, γ and β are linearly independent. We further assume that the set $\{\alpha, \beta, \gamma\}$ determines a basis for the one-form sections. An implication of this is $\alpha \wedge d\alpha \neq 0$ which says that α is not integrable. We shall comment on this after conformal invariance of the structure equations is obtained. From the exterior derivative of (4.3)

$$2\alpha \wedge \gamma \wedge \beta - \gamma \wedge d\beta = 0. \tag{4.4}$$

and linear independence of basis one-forms, we obtain

$$d\beta = -2\alpha \wedge \beta. \tag{4.5}$$

Thus, by starting with an integrable one-form γ, we obtained a linearly independent basis satisfying the structure equations (4.1), (4.3), and (4.5). We collect these Maurer-Cartan type equations in the following theorem while exhibiting $\mathfrak{sl}(2)$ algebra character of the system.

Theorem 4.1 *An integrable one-form γ determines 3-dimensional basis satisfying*

$$d\beta = -2\alpha \wedge \beta, \qquad d\alpha = \gamma \wedge \beta, \qquad d\gamma = 2\alpha \wedge \gamma, \tag{4.6}$$

where $d\alpha \neq 0$,

The algebra given in (4.6) admits a symmetry by being invariant under some conformal transformations.

Theorem 4.2 *The Maurer-Cartan system (4.6) is invariant under conformal transformations*

$$\beta \mapsto \rho\beta, \qquad \alpha \mapsto \alpha - \frac{1}{2} d \ln \rho, \qquad \gamma \mapsto \frac{\gamma}{\rho}, \tag{4.7}$$

for a nonvanishing function ρ*.*

To prove this assertion, we start with $\beta \mapsto \rho\beta$ and compute

$$d\,(\rho\beta) = d\rho \wedge \beta + \rho \wedge d\beta = \frac{d\rho}{\rho} \wedge (\rho\beta) + \rho\,(-2\alpha \wedge \beta) = -\big(2\alpha - \frac{d\rho}{\rho}\big) \wedge (\rho\beta)$$

which defines $\alpha \mapsto \alpha - \frac{d\rho}{2\rho}$. Differentiating

$$d\big(\alpha - \frac{d\rho}{2\rho}\big) = d\alpha = \gamma \wedge \beta = \gamma \wedge \frac{\rho}{\rho}\beta = \frac{\gamma}{\rho} \wedge (\rho\beta)$$

gives the form of transformation $\gamma \mapsto \frac{\gamma}{\rho}$. Further differentiation closes the algebra

$$d\big(\frac{\gamma}{\rho}\big) = \frac{d\gamma}{\rho} - \frac{1}{\rho^2} d\rho \wedge \gamma = \frac{2\alpha \wedge \gamma}{\rho} - \frac{d\rho}{\rho} \wedge \frac{\gamma}{\rho} = 2\big(\alpha - \frac{1}{2}\frac{d\rho}{\rho}\big) \wedge \frac{\gamma}{\rho}.$$

Nonintegrable integrating factor. In the structure equations, the one-form α appears as an integrating factor for integrable one-forms β and γ. Yet α itself is nonintegrable, because

$$\alpha \wedge d\alpha = \alpha \wedge \gamma \wedge \beta \neq 0$$

for an orientable three-manifold. As an integrating factor, we can seek for functions f and g which will make α integrable in

$$\begin{aligned} d\beta &= -2(\alpha + f\beta) \wedge \beta = -2\alpha \wedge \beta \\ d\gamma &= 2(\alpha + g\gamma) \wedge \gamma = 2\alpha \wedge \gamma. \end{aligned} \tag{4.8}$$

that is we require f, g to satisfy

$$\begin{aligned} (\alpha + f\beta) \wedge d(\alpha + f\beta) &= 0 \\ (\alpha + g\gamma) \wedge d(\alpha + g\gamma) &= 0. \end{aligned} \tag{4.9}$$

These conditions imply linear first-order PDEs for the functions f and g

$$\begin{aligned} \alpha \wedge (\gamma + df) \wedge \beta &= 0 \\ \alpha \wedge (-\beta + dg) \wedge \gamma &= 0 \end{aligned} \tag{4.10}$$

which can always be locally solvable. This means that, in the decomposition of two-forms $d\beta$ and $d\gamma$ we can replace nonintegrable integrating factor α with integrable ones $\alpha + f\beta$ in $d\beta$ and $\alpha + g\gamma$ in $d\gamma$ to make them integrable and hence Poisson one-forms for vector fields corresponding to $d\beta$ and $d\gamma$.

Non-integrable potential one-form. Here, we assume that $\mathbf{v}$ admits a nonintegrable potential vector. In this case, we simply can identify the potential one-form to be α which already presents in $\mathfrak{sl}(2)$-structure. Then, the locally bi-Hamiltonian form of $\mathbf{v}$ is one of the Maurer-Cartan equations

$$\iota_v(dx \wedge dy \wedge dz) = d\alpha = \gamma \wedge \beta$$

with the potential α being nonintegrable

$$\alpha \wedge d\alpha = \alpha \wedge \gamma \wedge \beta \neq 0.$$

5. Dynamical system

Suppose the system $\dot{\mathbf{x}} = \mathbf{v}$ comes along with $\mathbf{u}$ and $\mathbf{w}$ constituting an $\mathfrak{sl}(2)$ algebra

$$[u, v] = 2v, \qquad [u, w] = -2w, \qquad [v, w] = u \tag{5.1}$$

where $u = \mathbf{u} \cdot \nabla$, $v = \mathbf{v} \cdot \nabla$ and $w = \mathbf{w} \cdot \nabla$. Here, the brackets are the Jacobi-Lie bracket of vector fields. In this realization, the invariant volume density is

$$\frac{1}{M} = (\mathbf{v} \times \mathbf{u}) \cdot \mathbf{w}. \tag{5.2}$$

We define the dual one-forms

$$\alpha = M(\mathbf{w} \times \mathbf{v}) \cdot d\mathbf{x}, \qquad \beta = M(\mathbf{u} \times \mathbf{w}) \cdot d\mathbf{x}, \qquad \gamma = M(\mathbf{v} \times \mathbf{u}) \cdot d\mathbf{x} \tag{5.3}$$

which can easily be shown to satisfy

$$\iota_v \beta = \iota_u \alpha = \iota_w \gamma = 1, \tag{5.4}$$

and all the other possible couplings are identically zero. In order to see the correspondence between $\mathfrak{sl}(2)$ algebra in (5.1) and the one exhibited in (4.6), it is enough to consider the very definition of the exterior derivative. For a one-form ω, it is given by

$$d\omega(u, v) = u(\iota_v \omega) - v(\iota_u \omega) - \iota_{[u,v]}\omega. \tag{5.5}$$

To show that $\mathfrak{sl}(2)$ is the natural structure regarding $\mathbf{v}$ as a curl vector field, we consider the one-form γ in (5.3). To show that, $d\gamma$ is a curl expression for $\mathbf{v}$, we compute

$$\begin{aligned} \nabla \times (M\mathbf{v} \times \mathbf{u}) &= \nabla M \times (\mathbf{v} \times \mathbf{u}) + M\nabla \times (\mathbf{v} \times \mathbf{u}) \\ &= (\mathbf{u} \cdot \nabla M)\mathbf{v} - (\mathbf{v} \cdot \nabla M)\mathbf{u} + M\,(\nabla \cdot \mathbf{u})\,\mathbf{v} - M\,(\nabla \cdot \mathbf{v})\,\mathbf{u} + M\overrightarrow{[u, v]} \\ &= \nabla \cdot (M\mathbf{u})\,\mathbf{v} - \nabla \cdot (M\mathbf{v})\,\mathbf{u} + 2M\mathbf{v} = 2M\mathbf{v} \end{aligned} \tag{5.6}$$

due to the invariance of M. Similarly, one can compute that

$$\nabla \times (M\mathbf{u} \times \mathbf{w}) = 2M\mathbf{w}, \qquad \nabla \times (M\mathbf{v} \times \mathbf{w}) = -M\mathbf{u}. \tag{5.7}$$

Recall that for a one-form $\mathbf{A} \cdot d\mathbf{x}$ we have $d(\mathbf{A} \cdot d\mathbf{x}) = (\nabla \times \mathbf{A}) \cdot d\mathbf{x} \wedge d\mathbf{x}$. So that, the left-hand sides of the Maurer-Cartan system in (4.6) are the curls whereas the right-hand sides decompose these curls into two potentials that are Poisson vectors. (5.6) and (5.7) are coefficients of $d\mathbf{x} \wedge d\mathbf{x}$ in $d\gamma$, $d\beta$ and $d\alpha$, respectively. To verify the right-hand sides of Maurer-Cartan equations, we take as an example $d\gamma = 2\alpha \wedge \gamma$ and compute

$$\begin{aligned} 2M d\mathbf{x} \wedge d\mathbf{x} &= 2M(\mathbf{w} \times \mathbf{v}) \cdot d\mathbf{x} \wedge M(\mathbf{v} \times \mathbf{u}) \cdot d\mathbf{x} \\ &= 2M^2(\mathbf{w} \times \mathbf{v}) \times (\mathbf{v} \times \mathbf{u}) \cdot d\mathbf{x} \wedge d\mathbf{x} \\ &= 2M^2\mathbf{v}(\mathbf{u} \cdot \mathbf{w} \times \mathbf{v}) \cdot d\mathbf{x} \wedge d\mathbf{x} = 2M\mathbf{v} \cdot d\mathbf{x} \wedge d\mathbf{x} \end{aligned} \tag{5.8}$$

by definition of the multiplier M.

Heisenberg algebra. We consider a basis $\{\omega^1, \omega^2, \omega^3\}$ for the space of one-form sections and assume that the following structure equations hold

$$d\omega^1 = d\omega^3 = 0, \qquad d\omega^2 = \omega^3 \wedge \omega^1. \tag{5.9}$$

Referring to the standard coordinates (x, y, z), we can write the local realizations

$$\omega^1 = dx, \qquad \omega^2 = dy - xdz, \qquad \omega^3 = dz. \tag{5.10}$$

For each one form $\omega = \boldsymbol{\omega} \cdot d\mathbf{x}$, we associate a covector $\boldsymbol{\omega}$. Then referring to (5.10), we arrive at the following set

$$\boldsymbol{\omega}^1 = (1, 0, 0), \qquad \boldsymbol{\omega}^2 = (0, 1, -x), \qquad \boldsymbol{\omega}^3 = (0, 0, 1) \tag{5.11}$$

satisfying $M = \boldsymbol{\omega}^1 \times \boldsymbol{\omega}^2 \cdot \boldsymbol{\omega}^3 = 1$. Accordingly, we define the following basis

$$u = \boldsymbol{\omega}^1 \times \boldsymbol{\omega}^2 \cdot \nabla = x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \qquad v = \boldsymbol{\omega}^3 \times \boldsymbol{\omega}^1 \cdot \nabla = \frac{\partial}{\partial y}, \qquad w = \boldsymbol{\omega}^2 \times \boldsymbol{\omega}^3 \cdot \nabla = \frac{\partial}{\partial x} \tag{5.12}$$

satisfying

$$[v, w] = [v, u] = 0, \qquad [w, u] = v. \tag{5.13}$$

If $\mathbf{v}$ is the given dynamical system, these equations characterize $\mathbf{v}$ as having two symmetries whose commutator produces the dynamics. Note the dualities

$$\iota_w \omega^1 = \iota_v \omega^2 = \iota_u \omega^3 = 1.$$

It follows from (5.9) that $\boldsymbol{\omega}^1$ and $\boldsymbol{\omega}^3$ are conserved covariants and $\mathbf{v}$, expressed as a curl, is bi-Hamiltonian with these invariants. This shows that well-known bi-Hamiltonian systems with two integrals of motion can in fact be manifested in Heisenberg algebra (5.9). These local coordinates are indeed the case of final quadrature after having conserved quantities, say H_1 and H_2, and reducing the system by eliminating two coordinates. That is

$$\mathbf{x} = (H_1, y, H_2), \qquad \dot{\mathbf{x}} = (0, \dot{y}(y, H_1, H_2), 0).$$

6. Examples

6.1. Guillot system

As a first example, we consider the Guillot system [11]

$$\dot{x} = x^2 + y^4, \qquad \dot{y} = xy, \qquad \dot{z} = 2y^2 z - xz. \tag{6.1}$$

In order to investigate a potential vector field for this system, the first step is to determine the vector field v generating the system (6.1) and then complete it to a 3 dimensional basis satisfying structure equations for $\mathfrak{sl}(2)$ that is (5.1) which has already been done by Guillot in search of vector fields of Darboux-Halphen type

$$\begin{aligned} v &= (x^2 + y^4)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + (2y^2 z - xz)\frac{\partial}{\partial z}, \\ u &= 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}, \\ w &= -\frac{\partial}{\partial x}. \end{aligned} \tag{6.2}$$

The reciprocal of the multiplier is given in (5.2). For the present system, it is computed to be

$$M = \frac{1}{\mathbf{v} \times \mathbf{u} \cdot \mathbf{w}} = \frac{1}{2zy^3} \tag{6.3}$$

provided that $2zy^3$ never vanishes. It is now straightforward to check that Mv is a divergence-free vector field. Referring to the identifications given in (5.3), we compute the following one-form sections

$$\begin{aligned}
\alpha &= \frac{2y^2 - x}{2y^3} dy - \frac{x}{2zy^2} dz, \\
\beta &= \frac{1}{2y^3} dy + \frac{1}{2zy^2} dz, \\
\gamma &= -dx + \frac{x^2 - y^4 - 4xy^2}{2y^3} dy + \frac{y^4 - x^2 y}{2zy^2} dz,
\end{aligned} \tag{6.4}$$

respectively. It is now the matter of a direct calculation to verify that the forms in (6.4) are satisfying the $\mathfrak{sl}(2)$ Maurer-Cartan equations (4.6) stated in Theorem 4.1. See also that if we write $\gamma = \mathbf{A} \cdot d\mathbf{x}$ then it is possible now to establish that $d\gamma = M\mathbf{v} \cdot d\mathbf{x} \wedge d\mathbf{x}$ which reads the vector $M\mathbf{v}$ as the curl of the potential field

$$\mathbf{A} = (-1, \frac{x^2 - y^4 - 4xy^2}{2y^3}, \frac{y^4 - x^2 y}{2zy^2}). \tag{6.5}$$

It is immediate to see that the following function

$$H_1 = \frac{x^2}{y^2} - y^2 \tag{6.6}$$

is a first integral of the Guillot system (6.1) for $y \neq 0$. Along with the existence of the multiplier M in (6.3), in the light of Theorem 2.2, we argue that the Guillot system (6.1) admits two conserved quantities so that one can recast it in the bi-Hamiltonian form. This implies that one may examine the Guillot system (6.1) in the framework of Heisenberg algebra (5.13) as well. To have this, we compute another first integral

$$H_2 = \epsilon \log \big(\frac{\epsilon x + y^2}{y(yz)^{\epsilon/2}} \big) \tag{6.7}$$

where log is the natural logarithm and ϵ stands for ± 1. A direct computation exhibits bi-Hamiltonian character of the dynamics

$$2\iota_v (dx \wedge dy \wedge dz) = \frac{1}{M} dH_2 \wedge dH_1 \tag{6.8}$$

where M is Jacobi's last multiplier in (6.3). Here the factor 2 on the right-hand side of (6.8) is a manifestation of the multiple 2 on the right-hand side of the first bracket in the $\mathfrak{sl}(2)$ algebra (5.1). The flow is the line of intersection of two surfaces determined by the Hamiltonian functions H_1 in (6.6) and H_2 in (6.7).

6.2. Nonintegrable potential

As a simple example, we take the vector field $\mathbf{u}$ in Guillot system. Dual one-form is α which is nonintegrable. This means that

$$-M\mathbf{u} \cdot d\mathbf{x} \wedge d\mathbf{x} = d\alpha = \gamma \wedge \beta \tag{6.9}$$

admits a potential one-form

$$\alpha = \frac{2y^2 - x}{2y^3} dy - \frac{x}{2zy^2} dz = \boldsymbol{\alpha} \cdot d\mathbf{x} \tag{6.10}$$

which is not integrable $\alpha \wedge d\alpha \neq 0$. Indeed, we have $M\mathbf{u} = \nabla \times \boldsymbol{\alpha}$ but

$$\boldsymbol{\alpha} \cdot \nabla \times \boldsymbol{\alpha} = \frac{1}{2zy^3} = M \tag{6.11}$$

as expected. Nonintegrable potential leads to the emergence of the cohomological element called Godbillon-Vey class. This is the three-form

$$\alpha \wedge d\alpha = (\boldsymbol{\alpha} \cdot \nabla \times \boldsymbol{\alpha})\, dx \wedge dy \wedge dz = *1 \tag{6.12}$$

which is obviously closed (we are in dimension three) but not exact, i.e. there is no two-form whose derivative is $*1$. In other words, there is no vector field $\mathbf{B}$ that solves the equation $\nabla \cdot \mathbf{B} = \boldsymbol{\alpha} \cdot \nabla \times \boldsymbol{\alpha}$.

6.3. Darboux-Halphen system in symmetric polynomials

We start with Darboux-Halphen system [4] given by

$$\begin{aligned} \dot{t}_1 &= t_2 t_3 - t_1 t_2 - t_1 t_3, \\ \dot{t}_2 &= t_1 t_3 - t_3 t_2 - t_1 t_2, \\ \dot{t}_3 &= t_1 t_2 - t_3 t_1 - t_3 t_2. \end{aligned} \tag{6.13}$$

See, for example, [20] for more recent discussions on this system. We introduce a new set of dependent variables

$$x := -2\,(t_1 + t_2 + t_3)\,, \qquad y := 4\,(t_1 t_2 + t_2 t_3 + t_1 t_3)\,, \qquad z := -8 t_1 t_2 t_3 \tag{6.14}$$

which take Darboux-Halphen system (6.13) into the following form

$$\dot{x} = \frac{1}{2} y, \qquad \dot{y} = 3z, \qquad \dot{z} = 2xz - \frac{1}{2} y^2. \tag{6.15}$$

We denote the dynamics governed by the equations (6.15) by a vector field v. We complete the vector field v into a basis (v, u, w) satisfying the $\mathfrak{sl}(2)$ algebra exhibited in (5.1). A direct computation gives this basis as

$$\begin{aligned} v &= \frac{1}{2} y \frac{\partial}{\partial x} + 3z \frac{\partial}{\partial y} + \left(2xz - \frac{1}{2} y^2\right) \frac{\partial}{\partial z}, \\ u &= 2x \frac{\partial}{\partial x} + 4y \frac{\partial}{\partial y} + 6z \frac{\partial}{\partial z}, \\ w &= -6 \frac{\partial}{\partial x} - 4x \frac{\partial}{\partial y} - 2y \frac{\partial}{\partial z}. \end{aligned} \tag{6.16}$$

Referring to the identity (5.2), the volume is computed to be

$$\frac{1}{M} = 72xyz - 16y^3 + 4x^2 y^2 - 16x^3 z - 108z^2. \tag{6.17}$$

According to (5.3), we get the one form sections as

$$\begin{aligned} \alpha &= M\big((2xy^2+6yz-8x^2z)dx+(12xz-4y^2)dy+(2xy-18z)dz\big), \\ \beta &= 4M\big((6xz-2y^2)dx+(xy-9z)dy+(6y-2x^2)dz\big), \\ \gamma &= M\big((18z^2-8xy+2y^3)dx+(4x^2z-xy^2-3yz)dy+(2y^2-6xz)dz\big), \end{aligned} \tag{6.18}$$

where M is the multiplier in (6.17). It is now straightforward to check that these forms are satisfying the equations (4.6) presented in Theorem 4.1. Polynomial character of the multiplier M permits us to assign dimensions $[x]=1$, $[y]=2$, and $[z]=3$. Thus $[M]=-6$. Accordingly, one computes the dimensions of the one-form sections in (6.18) as $[\alpha]=0$, $[\beta]=-1$ and $[\gamma]=1$, respectively. This is obeying $\mathfrak{sl}(2)$ algebra realization of the one-form sections. The exterior derivative $d\gamma$ will define the curl potential for the dynamics v. Since Darboux-Halphen is known not to admit any polynomial integrals, this structure shows only that bi-Hamiltonian structure exists.

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