Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 3276 – 3299
© TÜBİTAK
doi:10.55730/1300-0098.3333

From ordered semigroups to ordered Γ-hypersemigroups

Niovi KEHAYOPULU*

TBAE, The Scientific and Technological Research Council of Turkey, Turkey

Received: 22.07.2022 • **Accepted/Published Online:** 27.09.2022 • **Final Version:** 09.11.2022

Abstract: In an attempt to show the way we pass from ordered semigroups to ordered Γ-hypersemigroups, we examine the results of Semigroup Forum (1992; 46: 341-346) for an ordered Γ-hypersemigroup. It has been shown that the concept of semisimple ordered Γ-hypersemigroup S is identical with the concept "the ideals of S are idempotent" and the ideals of S are idempotent if and only if for all ideals A, B of S, we have $A \cap B = (A\Gamma B]$. The main results of the paper are the following: The ideals of an ordered Γ-hypersemigroup S are weakly prime if and only if they form a chain and S is semisimple. The ideals of S are prime if and only if they form a chain and S is intraregular. It should be finally mentioned that the concepts "prime ideal" and "both semiprime and weakly prime ideal" are the same; and that in commutative ordered Γ-hypersemigroups the prime and weakly prime ideals coincide. For an abstract formulation of the above statements we refer to Turk J Math (2016; 40: 310–316).

Key words: Ordered Γ-hypersemigroup, prime, weakly prime, semisimple, intraregular

1. Introduction

The concept of prime ideal has played an important role in the theory of commutative rings. Semigroups in which the ideals are prime or weakly prime have been considered by Szász [14]. Ordered semigroups in which the ideals are prime or weakly prime have been studied in 1992 in Semigroup Forum [1]. From the results on ordered semigroups given in [1], corresponding results on semigroups (without order) can be obtained as every semigroup endowed with the relation $\leq = \{(x, y) \mid x = y\}$ is an ordered semigroup. Later, the results of ordered semigroups have been studied for ordered Γ-semigroups [3]; and for ordered hypersemigroups [5]. Relationship between lattice ordered semigroups and ordered hypersemigroups has been given in [10]. An le-semigroup is a lattice ordered semigroup possessing a greatest element usually denoted by "e". An abstract formulation of the results for le-semigroups has been given in [4]. In this type of semigroups, the ideal elements (instead of ideals) play the essential role. The paper in [4] is in the same spirit with the abstract formulation of general topology (the so-called *topology without points*) initiated by Koutský [12], Nöbeling [13] and, even earlier, by Chittenden, Terasaka, Nakamura, Monteiro and Ribeiro (see [7]).

The present paper, as a continuation of the papers in [5, 10], shows the way we pass from the results on ordered semigroups considered in [1] to ordered Γ-hypersemigroups. It is shown that an ordered Γ-hypersemigroup is semisimple if and only if the ideals of S are idempotent. On the other hand the ideals of S are idempotent if and only if for all ideals A, B of S, we have $A \cap B = \{t \in S \mid t \leq a\gamma b$ for some $a \in A$, $b \in$

*Correspondence: niovi.kehayopulu-tbae@tubitak.gov.tr

2010 *AMS Mathematics Subject Classification:* 06F99, 06F05.

$B, \gamma \in \Gamma\}$. The main results are the following: (a) The ideals of an ordered Γ-hypersemigroup S are weakly prime if and only if they form a chain and S is semisimple and (b) The ideals of an ordered Γ-hypersemigroup S are prime if and only if they form a chain and S is intraregular. It should be mentioned here that the concepts "prime" and "both semiprime and weakly prime" ideals are the same and that in commutative ordered Γ-hypersemigroups the prime and weakly prime ideals coincide. The sets in the proofs show their pointless character and that they come from the *poe*-semigroups.

2. Prerequisites

Let S and Γ be two nonempty sets. The set S is called a Γ-hypergroupoid [6] if the following assertions are satisfied:

(1) if $a, b \in S$ and $\gamma \in \Gamma$, then $\emptyset \neq a\gamma b \subseteq S$ and

(2) if $a, b, c, d \in S$ and $\gamma, \mu \in \Gamma$ such that $a = c$, $\gamma = \mu$ and $b = d$, then $a\gamma b = c\mu d$.

If S is a Γ-hypergroupoid then, for every $\gamma \in \Gamma$, we denote by $\overline{\gamma}$ the operation on the set $\mathcal{P}^*(S)$ of all nonempty subsets of S defined by

$$A\overline{\gamma}B := \bigcup_{a\in A,\, b\in B} a\gamma b$$

and by Γ the operation on $\mathcal{P}^*(S)$ defined by

$$A\Gamma B := \bigcup_{\gamma\in\Gamma} A\overline{\gamma}B.$$

As one can easily see, $A\Gamma B = \bigcup\limits_{a\in A,\, b\in B,\, \gamma\in\Gamma} a\gamma b$. As a consequence, the following holds:

$$x \in A\Gamma B \text{ if and only if } x \in a\gamma b \text{ for some } a \in A, b \in B, \gamma \in \Gamma;$$

from which we have the following: if $a \in A$, $b \in B$ and $\gamma \in \Gamma$, then $a\gamma b \subseteq A\Gamma B$.

We write, for short, $a\Gamma b$ instead of $\{a\}\Gamma\{b\}$.

Lemma 2.1 *[6, Lemma 3.5] If S is a hypergroupoid then, for any $x, y \in S$ and any $\gamma \in \Gamma$, we have*

$$\{x\}\overline{\gamma}\{y\} = x\gamma y.$$

Lemma 2.2 *[6, Lemmas 3.6 and 3.8] If S is a Γ-hypergroupoid, A, B, C, D nonempty subsets of S, $\gamma \in \Gamma$, $A \subseteq B$ and $C \subseteq D$, then $A\Gamma C \subseteq B\Gamma D$ and $A\overline{\gamma}C \subseteq B\overline{\gamma}D$.*

A Γ-hypergroupoid S is called a Γ-hypersemigroup [6, Definition 3.14] if, for any $a, b, c \in S$ and any $\gamma, \mu \in \Gamma$, we have

$$\{a\}\overline{\gamma}(b\mu c) = (a\gamma b)\overline{\mu}\{c\}.$$

Lemma 2.3 *[6, Proposition 3.17] If S is a Γ-hypersemigroup then, for any nonempty subsets A, B, C of S, we have $(A\Gamma B)\Gamma C = A\Gamma(B\Gamma C)$.*

Lemma 2.4 *[8, Lemma 2] If S is a Γ-hypersemigroup then, for any nonempty subsets A, B, C of S and any $\gamma, \mu \in \Gamma$, we have $(A\overline{\gamma}B)\overline{\mu}C = A\overline{\gamma}(B\overline{\mu}C)$.*

Lemma 2.5 *(see also [6, Proposition 3.13]) If S is a Γ-hypergroupoid then, for any nonempty subsets A, B of S, we have*

(1) $(A \cup B)\Gamma C = A\Gamma C \cup B\Gamma C$ *and*

(2) $A\Gamma(B \cup C) = A\Gamma B \cup A\Gamma C$.

Lemma 2.6 *If S is a Γ-hypergroupoid then, for any nonempty subsets A, B of S, we have*

(1) $(A \cap B)\Gamma C \subseteq A\Gamma C \cap B\Gamma C$ *and*

(2) $A\Gamma(B \cap C) \subseteq A\Gamma B \cap A\Gamma C$.

Proof (1) Let $x \in (A \cap B)\Gamma C$. Then $x \in d\gamma c$ for some $d \in A \cap B$, $\gamma \in \Gamma$, $c \in C$. Since $d \in A$, $\gamma \in \Gamma$, $c \in C$, we have $d\gamma c = \{d\}\overline{\gamma}\{c\} \subseteq A\Gamma C$. Since $d \in B$, $\gamma \in \Gamma$, $c \in C$, we have $d\gamma b = \{d\}\overline{\gamma}\{b\} \subseteq B\Gamma C$. Thus we have $x \in A\Gamma C \cap B\Gamma C$. The proof of (2) is similar. □

Lemma 2.7 *[6, Proposition 3.12] If S is a Γ-hypergroupoid, A is a right ideal of S and B is a left ideal of S, then $A \cap B \neq \emptyset$.*

A Γ-hypergroupoid S is called commutative if $A\Gamma B = B\Gamma A$ for any nonempty subsets A, B of S.

A Γ-hypergroupoid S is called an ordered Γ-hypergroupoid [8] if there exists an order relation $\leq$ of S such that $a \leq b$ implies $a\gamma c \preceq b\gamma c$ and $c\gamma a \preceq c\gamma b$ for every $c \in S$ and every $\gamma \in \Gamma$, in the sense that for every $u \in a\gamma c$ there exists $v \in b\gamma c$ such that $u \leq v$ and for every $u \in c\gamma a$ there exists $v \in c\gamma b$ such that $u \leq v$.

Lemma 2.8 *[9, Lemma 2.2] If S is an ordered Γ-hypergoupoid, $a \leq b$, $c \leq d$ and $\gamma \in \Gamma$, then $a\gamma c \preceq b\gamma d$.*

For an ordered Γ-hypersemigroup S and a nonempty subset A of S, denote by $(A]$ the subset of S defined by:

$$(A] := \{t \in S \text{ such that } t \leq a \text{ for some } a \in A\}$$

and, for any nonempty subsets A, B of S, we have $S = (S]$; $A \subseteq (A]$; if $A \subseteq B$, then $(A] \subseteq (B]$; $\big((A]\big] = (A]$; if A is a right (or left) ideal of S, then $(A] = A$ (see also [1] -as the order does not play any role in these properties).

3. Main results

Definition 3.1 *[9, Definition 2.5] Let S be an ordered Γ-hypergroupoid. A nonempty subset A of S is called an ideal of S if*

(1) $S\Gamma A \subseteq A$, $A\Gamma S \subseteq A$; *that is if $x \in s\gamma a$ for some $s \in S$, $\gamma \in \Gamma$, $a \in A$, then $x \in A$ and if $x \in a\gamma s$ for some $a \in A$, $\gamma \in \Gamma$, $s \in S$, then $x \in A$.*

(2) *if $a \in A$ and $S \ni b \leq a$, then $b \in A$; that is if $(A] = A$.*

Definition 3.2 *Let S be a Γ–hypergroupoid or an ordered Γ–hypergroupoid. A nonempty subset T of S is called prime if, for any nonempty subsets A, B of S such that $A\Gamma B \subseteq T$, we have $A \subseteq T$ or $B \subseteq T$.*

For $A = \{a\}$ and $B = \{b\}$ we write, for short, $a\Gamma b$ instead of $\{a\}\Gamma\{b\}$.

Proposition 3.3 *Let S be a Γ-hypergroupoid or an ordered Γ-hypergroupoid and T a nonempty subset of S. The following are equivalent:*

(1) *T is prime.*

(2) *For any $a, b \in S$ such that $a\Gamma b \subseteq T$, we have $a \in T$ or $b \in T$.*

Proof (1) $\Longrightarrow$ (2). Let $a, b \in S$ such that $a\Gamma b \subseteq T$. Since $\{a\}, \{b\}$ are nonempty subsets of S, $a\Gamma b := \{a\}\Gamma\{b\} \subseteq T$ and T is prime, by (1), we have $\{a\} \subseteq T$ or $\{b\} \subseteq T$; that is $a \in T$ or $b \in T$.
(2) $\Longrightarrow$ (1). Let A, B be nonempty subsets of S such that $A\Gamma B \subseteq T$. Suppose that $A \nsubseteq T$ and let $b \in B$. We will prove that $b \in T$. For this purpose, take an element $a \in A$ such that $a \notin T$. By Lemma 2.2, we have $\{a\}\Gamma\{b\} \subseteq A\Gamma B \subseteq T$ i.e. $a\Gamma b \subseteq T$. Since $a, b \in S$ such that $a\Gamma b \subseteq T$, by (2), we have $a \in T$ or $b \in T$. Since $a \notin T$, we have $b \in T$. Thus we get $B \subseteq T$ and T is prime. □

Definition 3.4 *Let S be a Γ–hypergroupoid or an ordered Γ-hypergroupoid. A nonempty subset T of S is called weakly prime if, for any ideals A, B of S such that $A\Gamma B \subseteq T$, we have $A \subseteq T$ or $B \subseteq T$.*

Definition 3.5 *Let S be a Γ–hypergroupoid or an ordered Γ-hypergroupoid. A nonempty subset T of S is called semiprime if, for any nonempty subset A of S such that $A\Gamma A \subseteq T$, we have $A \subseteq T$.*

Proposition 3.6 *Let S be a Γ–hypergroupoid or an ordered Γ–hypergroupoid and T a nonempty subset of S. The following are equivalent:*

(1) *T is semiprime.*

(2) *For any $a \in S$ such that $a\Gamma a \subseteq T$, we have $a \in T$.*

Proof (1) $\Longrightarrow$ (2). Let $a \in S$ such that $a\Gamma a \subseteq T$. Since $\{a\}\Gamma\{a\} \subseteq T$ and T is semiprime, we have $\{a\} \subseteq T$ and so $a \in T$.
(2) $\Longrightarrow$ (1). Let A be a nonempty subset of S such that $A\Gamma A \subseteq T$ and let $a \in A$. By Lemma 2.2, we have $a\Gamma a \subseteq A\Gamma A \subseteq T$. Since $a \in S$ and $a\Gamma a \subseteq T$, by (2), we have $a \in T$. Thus we have $A \subseteq T$ and so T is semiprime. □

Definition 3.7 *Let S be a Γ–hypergroupoid or an ordered Γ-hypergroupoid. A nonempty subset T of S is called weakly semiprime if, for every ideal A of S such that $A\Gamma A \subseteq T$, we have $A \subseteq T$.*

Lemma 3.8 *[9, Lemma 2.6] If S is an ordered Γ-hypergroupoid then, for any nonempty subsets A, B of S, we have*

$$(A]\Gamma(B] \subseteq (A\Gamma B].$$

Lemma 3.9 *[9, lemma 2.8] If S is an ordered hypergroupoid then, for any nonempty subsets A, B of S, we have*

$$(A\Gamma B] = \Big(A\Gamma(B]\Big] = \Big((A]\Gamma B\Big] = \Big((A]\Gamma(B]\Big].$$

Lemma 3.10 *[9, Lemma 2.17] If S is an ordered Γ-hypersemigroup then, for any subsets A, B, C of S, we have*

$$\Big(A\Gamma(B]\Gamma C\Big] = (A\Gamma B\Gamma C].$$

Lemma 3.11 *Let $(S,\Gamma,\leq)$ be an ordered Γ-hypergroupoid and A, B nonempty subsets of S. Then we have the following:*

(1) *If A and B are ideals of S, then the sets $A\cap B$ and $A\cup B$ are also ideals of S.*

In particular, if S is an ordered Γ-hypersemigroup, then we have the following:

(2) *If A is a left ideal of S and B is a right ideal of S, then $(A\Gamma B]$ is an ideal of S.*

(3) *$(S\Gamma A\Gamma S]$ is an ideal of S for every nonempty subset A of S.*

Proof (1) Let A, B be ideals of S. By Lemma 2.7, $A\cap B$ is a nonempty set. Moreover we have $(A\cap B)\Gamma S\subseteq A\Gamma S\cap B\Gamma S\subseteq A\cap B$ and $S\Gamma(A\cap B)\subseteq S\Gamma A\cap S\Gamma B\subseteq A\cap B$. If $x\in A\cap B$ and $S\ni y\leq x$, then $y\in A\cap B$. Thus $A\cap B$ is an ideal of S. Similarly, $A\cup B$ is an ideal of S.

(2) Let A be a left and B a right ideal of S. Then $(A\Gamma B]$ is a nonempty subset of S and we have

$$\begin{aligned}(A\Gamma B]\Gamma S &= (A\Gamma B]\Gamma(S]\subseteq \Big((A\Gamma B)\Gamma S\Big] \text{ (by Lemma 3.8)}\\ &= \Big(A\Gamma(B\Gamma S)\Big] \text{ (by Lemma 2.3)}\\ &\subseteq (A\Gamma B],\end{aligned}$$

$$S\Gamma(A\Gamma B]=(S]\Gamma(A\Gamma B]\subseteq\Big(S\Gamma(A\Gamma B)\Big]=\Big((S\Gamma A)\Gamma B\Big]\subseteq(A\Gamma B].$$

Let $x\in(A\Gamma B]$ and $S\ni y\leq x$. Since $x\in(A\Gamma B]$, we have $x\leq t$ for some $t\in A\Gamma B$. Then we have $y\leq t\in A\Gamma B$ and so $y\in(A\Gamma B]$. Thus $(A\Gamma B]$ is an ideal of S.

(3) Let A be a nonempty subset of S. Then we have

$$\begin{aligned}(S\Gamma A\Gamma S]\Gamma S &= (S\Gamma A\Gamma S]\Gamma(S]\subseteq\Big((S\Gamma A\Gamma S)\Gamma S\Big] \text{ (by Lemma 3.8)}\\ &\subseteq \Big(S\Gamma A\Gamma(S\Gamma S)\Big] \text{ (by Lemma 2.3)}\\ &\subseteq (S\Gamma A\Gamma S] \text{ (since } S\Gamma S\subseteq S),\end{aligned}$$

$$S\Gamma(S\Gamma A\Gamma S]=(S]\Gamma(S\Gamma A\Gamma S]\subseteq\Big(S\Gamma(S\Gamma A\Gamma S)\Big]=\Big((S\Gamma S)\Gamma A\Gamma S\Big]\subseteq(S\Gamma A\Gamma S].$$

Let $x\in(S\Gamma A\Gamma S]$ and $S\ni y\leq x$. Since $x\in(S\Gamma A\Gamma S]$, we have $x\leq t$ for some $t\in S\Gamma A\Gamma S$. We have $y\leq t\in S\Gamma A\Gamma S$ and so $y\in(S\Gamma A\Gamma S]$. Thus $(S\Gamma A\Gamma S]$ is an ideal of S. □

Proposition 3.12 *Let S be an ordered Γ-hypergroupoid. An ideal T of S is weakly prime if and only if for every ideals A, B of S such that $(A\Gamma B]\cap(B\Gamma A]\subseteq T$, we have $A\subseteq T$ or $B\subseteq T$.*

Proof $\Longrightarrow$. Let A, B be ideals of S such that $(A\Gamma B]\cap(B\Gamma A]\subseteq T$. By Lemma 3.11(2), the sets $(A\Gamma B]$ and $(B\Gamma A]$ are ideals of S. Thus we have

$$(A\Gamma B]\Gamma(B\Gamma A]\subseteq(A\Gamma B]\Gamma S\cap S\Gamma(B\Gamma A]\subseteq(A\Gamma B]\cap(B\Gamma A]\subseteq T.$$

Since $(A\Gamma B]$, $(B\Gamma A]$ are ideals of S, $(A\Gamma B]\Gamma(B\Gamma A]\subseteq T$ and T is weakly prime, we have $(A\Gamma B]\subseteq T$ or $(B\Gamma A]\subseteq T$, then $A\Gamma B\subseteq T$ or $B\Gamma A\subseteq T$. If $A\Gamma B\subseteq T$ then, since A,B are ideals of S and T is weakly prime, we have $A\subseteq T$ or $B\subseteq T$. If $B\Gamma A\subseteq T$, in a similar way, we get $A\subseteq T$ or $B\subseteq T$.

$\Longleftarrow$. Let A, B be ideals of S such that $A\Gamma B \subseteq T$. Then $(A\Gamma B]\cap(B\Gamma A] \subseteq (A\Gamma B] \subseteq (T] = T$. By hypothesis, we have $A \subseteq T$ or $B \subseteq T$ and so T is weakly prime. $\square$

For a nonempty subset A of an ordered hypersemigroup S, we denote by $I(A)$ the ideal of S generated by A and we have $I(A) = (A \cup S\Gamma A \cup A\Gamma S \cup S\Gamma A\Gamma S]$ (see [9, Lemma 2.7]). When is convenient we write, for short, A^3, A^4 instead of $A\Gamma A\Gamma A$, $A\Gamma A\Gamma A\Gamma A$ and so on.

Lemma 3.13 *If S is an ordered Γ-hypersemigroup and A, B nonempty subsets of S, then we have*

$$I(A)\Gamma I(B) \subseteq (S\Gamma B \cup S\Gamma B\Gamma S] \tag{3.1}$$

$$I(A)^3 \subseteq (S\Gamma A\Gamma S] \tag{3.2}$$

If S is commutative, then

$$I(A)\Gamma I(B) \subseteq (A\Gamma B \cup S\Gamma A\Gamma B] \tag{3.3}$$

Proof

$$\begin{aligned} I(A)\Gamma I(B) &= (A \cup S\Gamma A \cup A\Gamma S \cup S\Gamma A\Gamma S]\Gamma(B \cup S\Gamma B \cup B\Gamma S \cup S\Gamma B\Gamma S] \\ &\subseteq \Big((A \cup S\Gamma A \cup A\Gamma S \cup S\Gamma A\Gamma S)\Gamma(B \cup S\Gamma B \cup B\Gamma S \cup S\Gamma B\Gamma S)\Big]. \end{aligned}$$

On the other hand,

$$\begin{aligned} & (A \cup S\Gamma A \cup A\Gamma S \cup S\Gamma A\Gamma S)\Gamma(B \cup S\Gamma B \cup B\Gamma S \cup S\Gamma B\Gamma S) \\ = \; & A\Gamma B \cup S\Gamma A\Gamma B \cup A\Gamma S\Gamma B \cup S\Gamma A\Gamma S\Gamma B \cup A\Gamma S\Gamma B \cup S\Gamma A\Gamma S\Gamma B \cup A\Gamma S\Gamma S\Gamma B \cup S\Gamma A\Gamma S\Gamma S\Gamma B \\ & \cup A\Gamma B\Gamma S \cup S\Gamma A\Gamma B\Gamma S \cup A\Gamma S\Gamma B\Gamma S \cup S\Gamma A\Gamma S\Gamma B\Gamma S \cup A\Gamma S\Gamma B\Gamma S \cup S\Gamma A\Gamma S\Gamma B\Gamma S \\ & \cup A\Gamma S\Gamma S\Gamma B\Gamma S \cup S\Gamma A\Gamma S\Gamma S\Gamma B\Gamma S \\ & \subseteq S\Gamma B \cup S\Gamma B\Gamma S. \end{aligned}$$

This is because

$A\Gamma B \subseteq S\Gamma B$; $S\Gamma A\Gamma B = (S\Gamma A)\Gamma B \subseteq S\Gamma B$; $A\Gamma S\Gamma B = (A\Gamma S)\Gamma B \subseteq S\Gamma B$;

$S\Gamma A\Gamma S\Gamma B = (S\Gamma A\Gamma S)\Gamma B \subseteq S\Gamma B$ as $S\Gamma A\Gamma S = (S\Gamma A)\Gamma S \subseteq S\Gamma S \subseteq S$;

$A\Gamma S\Gamma S\Gamma B = (A\Gamma S\Gamma S)\Gamma B \subseteq S\Gamma B$, as $A\Gamma S\Gamma S = A\Gamma(S\Gamma S) \subseteq A\Gamma S \subseteq S$;

$S\Gamma A\Gamma S\Gamma S\Gamma B = (S\Gamma A\Gamma S\Gamma S)\Gamma B \subseteq S\Gamma B$ as $S\Gamma A\Gamma S\Gamma S = S\Gamma A\Gamma(S\Gamma S) \subseteq S\Gamma A\Gamma S$;

$A\Gamma B\Gamma S \subseteq S\Gamma B\Gamma S$; $S\Gamma A\Gamma B\Gamma S = (S\Gamma A)\Gamma B\Gamma S \subseteq S\Gamma B\Gamma S$; $A\Gamma S\Gamma B\Gamma S = (A\Gamma S)\Gamma B\Gamma S \subseteq S\Gamma B\Gamma S$;

$S\Gamma A\Gamma S\Gamma B\Gamma S = (S\Gamma A\Gamma S)\Gamma B\Gamma S \subseteq S\Gamma B\Gamma S$ as $S\Gamma A\Gamma S = S\Gamma(A\Gamma S) \subseteq S\Gamma S \subseteq S$;

$A\Gamma S\Gamma B\Gamma S = (A\Gamma S)\Gamma B\Gamma S \subseteq S\Gamma B\Gamma S$;

$S\Gamma A\Gamma S\Gamma B\Gamma S = (S\Gamma A\Gamma S)\Gamma B\Gamma S \subseteq S\Gamma B\Gamma S$ as $S\Gamma A\Gamma S = (S\Gamma A)\Gamma S \subseteq S\Gamma S \subseteq S$;

$A\Gamma S\Gamma S\Gamma B\Gamma S = (A\Gamma S\Gamma S)\Gamma B\Gamma S \subseteq S\Gamma B\Gamma S$ as $A\Gamma S\Gamma S = A\Gamma(S\Gamma S) \subseteq A\Gamma S \subseteq S$;

$S\Gamma A\Gamma S\Gamma S\Gamma B\Gamma S = (S\Gamma A\Gamma S\Gamma S)\Gamma B\Gamma S \subseteq S\Gamma B\Gamma S$ as $S\Gamma A\Gamma S\Gamma S = (S\Gamma A)\Gamma(S\Gamma S) \subseteq S\Gamma S \subseteq S$.

Thus we have $I(A)\Gamma I(B) \subseteq (S\Gamma B \cup S\Gamma B\Gamma S]$ and property (3.1) holds.

By (3.1), we have

$$\begin{aligned}
I(A)^3 &\subseteq (S\Gamma A\cup S\Gamma A\Gamma S]\Gamma(A\cup S\Gamma A\cup A\Gamma S\cup S\Gamma A\Gamma S]\\
&\subseteq \Big((S\Gamma A\cup S\Gamma A\Gamma S)\Gamma(A\cup S\Gamma A\cup A\Gamma S\cup S\Gamma A\Gamma S)\Big]\\
&= \Big(S\Gamma A\Gamma A\cup S\Gamma A\Gamma S\Gamma A\cup S\Gamma A\Gamma A\Gamma S\cup S\Gamma A\Gamma S\Gamma A\Gamma S\cup S\Gamma A\Gamma S\Gamma A\cup S\Gamma A\Gamma S\Gamma S\Gamma A\\
&\qquad \cup S\Gamma A\Gamma S\Gamma A\Gamma S\cup S\Gamma A\Gamma S\Gamma S\Gamma A\Gamma S\Big]\\
&\subseteq (S\Gamma A\Gamma S]
\end{aligned}$$

and property (3.2) holds.

If S is commutative, then $I(A)\Gamma I(B)\subseteq(A\Gamma B\cup S\Gamma A\Gamma B]$ (and property (3.3) holds).
This is because

(a) $A\Gamma S\Gamma B=(A\Gamma S)\Gamma B=(S\Gamma A)\Gamma B=S\Gamma A\Gamma B$ (since S is commutative).

(b) $S\Gamma A\Gamma S\Gamma B=S\Gamma(A\Gamma S)\Gamma B=S\Gamma(S\Gamma A)\Gamma B=(S\Gamma S)\Gamma A\Gamma B\subseteq S\Gamma A\Gamma B$.

(c) $A\Gamma S\Gamma S\Gamma B=A\Gamma(S\Gamma S)\Gamma B\subseteq A\Gamma S\Gamma B\subseteq S\Gamma A\Gamma B$ (by (a)).

(d) $S\Gamma A\Gamma S\Gamma S\Gamma B=S\Gamma A\Gamma(S\Gamma S)\Gamma B\subseteq S\Gamma A\Gamma S\Gamma B\subseteq S\Gamma A\Gamma B$ (by (b)).

(e) $A\Gamma B\Gamma S=A\Gamma(B\Gamma S)=A\Gamma(S\Gamma B)=(A\Gamma S)\Gamma B=(S\Gamma A)\Gamma B=S\Gamma A\Gamma B$.

(f) $S\Gamma A\Gamma B\Gamma S=S\Gamma A\Gamma(B\Gamma S)=S\Gamma A\Gamma(S\Gamma B)=S\Gamma A\Gamma S\Gamma B\subseteq S\Gamma A\Gamma B$ (by (b)).

(g) $A\Gamma S\Gamma B\Gamma S=(A\Gamma S)\Gamma B\Gamma S=(S\Gamma A)\Gamma B\Gamma S=S\Gamma A\Gamma B\Gamma S\subseteq S\Gamma A\Gamma B$ (by (b)).

(h) $S\Gamma A\Gamma S\Gamma B\Gamma S=S\Gamma A\Gamma S\Gamma(B\Gamma S)=S\Gamma A\Gamma S\Gamma(S\Gamma B)=S\Gamma A\Gamma(S\Gamma S)\Gamma B\subseteq S\Gamma A\Gamma S\Gamma B\subseteq S\Gamma A\Gamma B$ (by (b)).

(i) $A\Gamma S\Gamma B\Gamma S=(A\Gamma S)\Gamma(B\Gamma S)=(S\Gamma A)\Gamma(S\Gamma B)=S\Gamma(A\Gamma S)\Gamma B=S\Gamma A\Gamma S\Gamma B\subseteq S\Gamma A\Gamma B$ (by (b)).

(j) $S\Gamma A\Gamma S\Gamma B\Gamma S=S\Gamma(A\Gamma S)\Gamma(B\Gamma S)=S\Gamma(S\Gamma A)\Gamma(S\Gamma B)=(S\Gamma S)\Gamma A\Gamma S\Gamma B\subseteq S\Gamma A\Gamma S\Gamma B\subseteq S\Gamma A\Gamma B$ (by (b)).

(k) $A\Gamma S\Gamma S\Gamma B\Gamma S=A\Gamma(S\Gamma S)\Gamma B\Gamma S\subseteq A\Gamma S\Gamma B\Gamma S\subseteq S\Gamma A\Gamma B$ (by (i)).

(l) $S\Gamma A\Gamma S\Gamma S\Gamma B\Gamma S=S\Gamma A\Gamma(S\Gamma S)\Gamma B\Gamma S\subseteq S\Gamma A\Gamma S\Gamma B\Gamma S\subseteq S\Gamma A\Gamma B$ (by (j)).

□

Proposition 3.14 *An ideal T of an ordered Γ-hypersemigroup S is prime if and only if it is semiprime and weakly prime. In commutative ordered Γ-hypersemigroups the prime and weakly prime ideals coincide.*

Proof If T is prime, then clearly it is semiprime and weakly prime. For the converse statement, suppose T is an ideal of S both semiprime and weakly prime and let A, B be nonempty subsets of S such that $A\Gamma B\subseteq T$. Then

$$(B\Gamma S\Gamma A]\Gamma(B\Gamma S\Gamma A]\subseteq\Big((B\Gamma S)\Gamma(A\Gamma B)\Gamma(S\Gamma A)\Big]\subseteq\Big(S\Gamma(A\Gamma B)\Gamma S\Big]\subseteq(S\Gamma T\Gamma S]\subseteq(T]=T.$$

Since T is semiprime, we have $(B\Gamma S\Gamma A]\subseteq T$. Then we have

$$\begin{aligned}
(S\Gamma B\Gamma S]\Gamma(S\Gamma A\Gamma S] &\subseteq \Big(S\Gamma\Big(B\Gamma(S\Gamma S)\Gamma A\Big)\Gamma S\Big]\subseteq\Big(S\Gamma(B\Gamma S\Gamma A)\Gamma S\Big]\\
&= \Big(S\Gamma(B\Gamma S\Gamma A]\Gamma S\Big]\subseteq(S\Gamma T\Gamma S]\subseteq(T]=T.
\end{aligned}$$

Since $(S\Gamma B\Gamma S]$, $(S\Gamma A\Gamma S]$ are ideals of S and T is weakly prime, we have $(S\Gamma B\Gamma S]\subseteq T$ or $(S\Gamma A\Gamma S]\subseteq T$. Let $(S\Gamma A\Gamma S]\subseteq T$. Then we have

$$\begin{aligned}\Big(I(A)\Gamma I(A)\Big]\Gamma I(A) &= \Big(I(A)\Gamma I(A)\Big]\Gamma\Big(I(A)\Big]\subseteq\Big(I(A)\Gamma I(A)\Gamma I(A)\Big]\\ &\subseteq \Big((S\Gamma A\Gamma S]\Big] \text{ (by Lemma 3.13)}\\ &= (S\Gamma A\Gamma S]\subseteq T.\end{aligned}$$

As $I(A)$ is an ideal of S, $\Big(I(A)\Gamma I(A)\Big]$ is an ideal of S (by Lemma 3.11(2)), and T is weakly prime, we have $\Big(I(A)\Gamma I(A)\Big]\subseteq T$ or $I(A)\subseteq T$. If $I(A)\subseteq T$, then $A\subseteq T$. If $\Big(I(A)\Gamma I(A)\Big]\subseteq T$, then $I(A)\Gamma I(A)\subseteq T$ then, since T is weakly prime, we have $I(A)\subseteq T$ and so $A\subseteq T$. From $(S\Gamma B\Gamma S]\subseteq T$, in a similar way, we get $B\subseteq T$. Therefore T is prime.

Let now S be commutative, T a weakly prime ideal of S and A, B nonempty subsets of S such that $A\Gamma B\subseteq T$. Then, by (3.3), $I(A)\Gamma I(B)\subseteq (A\Gamma B\cup S\Gamma A\Gamma B]\subseteq (T\cup S\Gamma T]=(T]=T$. Since $I(A)$, $I(B)$ are ideals of S, $I(A)\Gamma I(B)\subseteq T$ and T is weakly prime, we have $I(A)\subseteq T$ or $I(B)\subseteq T$, then $A\subseteq T$ or $B\subseteq T$ and so T is prime. □

Definition 3.15 *Let S be an ordered Γ-hypergroupoid. A subset T of S is called meet-irreducible if for any ideals A,B of S such that $A\cap B=T$, we have $A=T$ or $B=T$.*

Proposition 3.16 *Let S be an ordered Γ-hypergroupoid. If a subset A of S is weakly prime, then it is meet-irreducible. If an ideal of S is weakly semiprime and meet-irreducible, then it is weakly prime.*

Proof $\Longrightarrow$. Let T be an weakly prime subset of S and A, B be ideals of S such that $A\cap B=T$. We have $A\Gamma B\subseteq A\Gamma S\subseteq A$ and $A\Gamma B\subseteq S\Gamma B\subseteq B$ and so $A\Gamma B\subseteq A\cap B=T$. Since A, B are ideals of S, $A\Gamma B\subseteq T$ and T is weakly prime, we have $A\subseteq T$ or $B\subseteq T$. Then we have $A=T$ or $B=T$ and T is meet-irreducible. $\Longleftarrow$. Let T be an weakly semiprime and meet-irreducible ideal of S and A, B be ideals of S such that $A\Gamma B\subseteq T$. By Lemma 3.11(1), $A\cap B$ is an ideal of T. On the other hand, $(A\cap B)\Gamma(A\cap B)\subseteq A\Gamma B\subseteq T$. Since T is weakly semiprime subset of S and $A\cap B$ is an ideal of S, we have $A\cap B\subseteq T$. Then we have $T=T\cup(A\cap B)=(T\cup A)\cap(T\cup B)$. Since T,A,B are ideals of S, by Lemma 3.11(1), $T\cup A$ and $T\cup B$ are ideals of S. Since T is meet-irreducible, we have $T\cup A=T$ or $T\cup B=T$. Then $A\subseteq T$ or $B\subseteq T$ and T is weakly semiprime. □

Definition 3.17 *[11, Definition 2.16] An ordered Γ-hypersemigroup S is said to be semisimple if for every $a\in S$ there exist $x,y,z,t\in S$ and $\gamma,\mu,\rho,\zeta\in\Gamma$ such that*

$$t\in(x\gamma a)\overline{\mu}(y\rho a)\overline{\zeta}\{z\} \text{ and } a\le t.$$

Proposition 3.18 *Let S be an ordered Γ-hypersemigroup. The following are equivalent:*

(1) *S is semisimple.*

(2) *For every $a\in S$ there exist $\gamma,\mu,\rho,\zeta\in\Gamma$ such that $a\in\Big(S\overline{\gamma}\{a\}\overline{\mu}S\overline{\rho}\{a\}\overline{\zeta}S\Big]$.*

(3) *For any nonempty subset A of S, we have* $A \subseteq (S\Gamma A\Gamma S\Gamma A\Gamma S]$.

(4) *For any* $a \in S$, *we have* $a \in (S\Gamma a\Gamma S\Gamma a\Gamma S]$.

Proof (1) $\Longrightarrow$ (2). Let $a \in S$. Since S is semiprime, there exist $x, y, z, t \in S$ and $\gamma, \mu, \rho, \zeta \in \Gamma$ such that $t \in (x\gamma a)\overline{\mu}(y\rho a)\overline{\zeta}\{z\}$ and $a \leq t$. Since $a \leq t \in (x\gamma a)\overline{\mu}(y\rho a)\overline{\zeta}\{z\}$, by Lemmas 2.1, 2.4 and 2.2, we have

$$a \in \Big((x\gamma a)\overline{\mu}(y\rho a)\overline{\zeta}\{z\}\Big] = \Big(\{x\}\overline{\gamma}\{a\}\overline{\mu}\{y\}\overline{\rho}\{a\}\overline{\zeta}\{z\}\Big] \subseteq \Big(S\overline{\gamma}\{a\}\overline{\mu}S\overline{\rho}\{a\}\overline{\zeta}S\Big]$$

and property (2) is satisfied.

(2) $\Longrightarrow$ (3). Let A be a nonempty subset of S and $a \in A$. By (2), there exist $\gamma, \mu, \rho, \zeta \in \Gamma$ such that $a \in \Big(S\overline{\gamma}\{a\}\overline{\mu}S\overline{\rho}\{a\}\overline{\zeta}S\Big]$. We have $S\overline{\gamma}\{a\} \subseteq S\Gamma\{a\} \subseteq S\Gamma A$, $S\overline{\gamma}\{a\}\overline{\mu}S\overline{\rho}\{a\} \subseteq S\Gamma A\overline{\mu}S\overline{\rho}\{a\} \subseteq S\Gamma A\Gamma S\Gamma A$, $S\overline{\gamma}\{a\}\overline{\mu}S\overline{\rho}\{a\}\overline{\zeta}S \subseteq S\Gamma A\Gamma S\Gamma A\Gamma S$. Thus we have $a \in (S\Gamma A\Gamma S\Gamma A\Gamma S]$ and property (3) holds.
The implication (3) $\Rightarrow$ (4) is obvious.

(4) $\Longrightarrow$ (1). Let $a \in S$. By hypothesis, we have $a \in (S\Gamma a\Gamma S\Gamma a\Gamma S]$, thus $a \leq t$ for some $t \in S\Gamma a\Gamma S\Gamma a\Gamma S = (S\Gamma a\Gamma S)\Gamma(a\Gamma S)$ $(\subseteq S)$. Then we have

$t \in u\rho v$ for some $u \in (S\Gamma a)\Gamma S$, $\rho \in \Gamma$, $v \in a\Gamma S$,
$u \in w\mu y$ for some $w \in S\Gamma a$, $\mu \in \Gamma$, $y \in S$, $w \in x\gamma a$ for some $x \in S$, $\gamma \in \Gamma$ and
$v \in a\zeta z$ for some $\zeta \in \Gamma$, $z \in S$.

Hence we obtain

$$\begin{aligned} t \in u\rho v &= \{u\}\overline{\rho}\{v\} \subseteq (w\mu y)\overline{\rho}(a\zeta z) \text{ (by Lemmas 2.1 and 2.2)} \\ &= \Big(\{w\}\overline{\mu}\{y\}\Big)\overline{\rho}\Big(\{a\}\overline{\zeta}\{z\}\Big) \text{ (by Lemma 2.1)} \\ &= \{w\}\overline{\mu}\Big(\{y\}\overline{\rho}\{a\}\Big)\overline{\zeta}\{z\} \text{ (by Lemma 2.4)} \\ &\subseteq (x\gamma a)\overline{\mu}(y\rho a)\overline{\zeta}\{z\} \text{ (by Lemmas 2.1 and 2.2).} \end{aligned}$$

Then $t \in (x\gamma a)\overline{\mu}(y\rho a)\overline{\zeta}\{z\}$, where $x, y, z, t \in S$ and $a \leq t$ and so S is semisimple. □

Definition 3.19 *A nonempty subset A of an ordered Γ-hypergroupoid S is called idempotent if $A = (A\Gamma A]$. That is, for every $a \in A$, there exist $b, c \in A$, $\gamma \in \Gamma$ and $t \in b\gamma c$ such that $a \leq t$ and if $x \leq t$ and $t \in b\gamma c$ for some $b, c \in A$, $\gamma \in \Gamma$, then $x \in A$.*

Theorem 3.20 *Let S be an ordered Γ-hypersemigroup. The following are equivalent:*

(1) *S is semisimple.*

(2) *The ideals of S are idempotent.*

(3) $A \cap B = (A\Gamma B]$ *for any ideals A, B of S.*

(4) $I(A) \cap I(B) = \Big(I(A)\Gamma I(B)\Big]$ *for any nonempty subsets A, B of S.*

(5) $I(A) = \Big(I(A)\Gamma I(A)\Big]$ *for every nonempty subset A of S.*

Proof (1) $\Longrightarrow$ (2). Let A be an ideal of S. If $x \in (A\Gamma A]$, then $x \in (A\Gamma S] \subseteq (A] = A$ and so $x \in A$. Let now $x \in A$. Since S is semisimple, by Proposition 3.18, we have $x \in (S\Gamma x\Gamma S\Gamma x\Gamma S]$. Then $x \le t$ for some $t \in S\Gamma x\Gamma S\Gamma x\Gamma S = (S\Gamma x\Gamma S)\Gamma(x\Gamma S)$ (by Lemma 2.3) and $t \in a\gamma b$ for some $a \in S\Gamma x\Gamma S$, $\gamma \in \Gamma$, $b \in x\Gamma S$. Since $a \in S\Gamma x\Gamma S \subseteq S\Gamma A\Gamma S \subseteq A$, $b \in x\Gamma S \subseteq A\Gamma S \subseteq A$ and $\gamma \in \Gamma$, we have $a\gamma b \subseteq A\Gamma A$. We have $x \le t \in A\Gamma A$ and so $x \in (A\Gamma A]$.
(2) $\Longrightarrow$ (3). Let A, B be ideals of S. Then $(A\Gamma B] \subseteq (A\Gamma S] \subseteq (A] = A$ and $(A\Gamma B] \subseteq (S\Gamma B] \subseteq (B] = B$ and so $(A\Gamma B] \subseteq A \cap B$. By Lemma 3.11(1), $A \cap B$ is an ideal of S thus, by (2), we have

$$A \cap B = \Big((A \cap B)\Gamma(A \cap B)\Big] \subseteq (A\Gamma B].$$

Thus we get $A \cap B = (A\Gamma B]$ and property (3) holds.
The implications (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (5) are obvious.
(5) $\Longrightarrow$ (1). Let A be a nonempty subset of S. By hypothesis, we have $I(A) = \Big((I(A)\Gamma I(A)\Big]$. Then we have

$$I(A)\Gamma I(A) = \Big((I(A)\Gamma I(A)\Big]\Gamma I(A) = \Big((I(A)\Gamma I(A)\Big]\Gamma\Big(I(A)\Big] \subseteq \Big(I(A)\Gamma I(A)\Gamma I(A)\Big],$$
$$I(A)^3 \subseteq \Big(I(A)\Gamma I(A)\Gamma I(A)\Big]\Gamma I(A) = \Big(I(A)\Gamma I(A)\Gamma I(A)\Big]\Gamma\Big(I(A)\Big] \subseteq \Big(I(A)\Gamma I(A)\Gamma I(A)\Gamma I(A)\Big].$$

We continue at the same way, and we have

$$I(A)^4 \subseteq \Big(I(A)\Gamma I(A)\Gamma I(A)\Gamma I(A)\Gamma I(A)\Big].$$

Hence we have

$$\begin{aligned} I(A) &= \Big(I(A)\Gamma I(A)\Big] \subseteq \Big(\Big(I(A)\Gamma I(A)\Gamma I(A)\Big]\Big] = \Big(I(A)\Gamma I(A)\Gamma I(A)\Big] \subseteq \Big(\Big(I(A)\Gamma I(A)\Gamma I(A)\Gamma I(A)\Big]\Big] \\ &= \Big(I(A)\Gamma I(A)\Gamma I(A)\Gamma I(A)\Big] \subseteq \Big(\Big(I(A)\Gamma I(A)\Gamma I(A)\Gamma I(A)\Gamma I(A)\Big]\Big] \\ &= \Big(\Big(I(A)\Gamma I(A)\Gamma I(A)\Gamma I(A)\Big)\Gamma I(A)\Big] \subseteq \Big(S\Gamma I(A)\Big] \subseteq \Big(I(A)\Big] = I(A) \end{aligned}$$

(as $I(A)\Gamma I(A) \subseteq S\Gamma S \subseteq S$, $I(A)\Gamma I(A)\Gamma I(A) \subseteq S\Gamma I(A) \subseteq S\Gamma S \subseteq S$, $I(A)\Gamma I(A)\Gamma I(A)\Gamma I(A) \subseteq S\Gamma I(A) \subseteq S$).
Thus we have

$$I(A) = \Big(I(A)\Gamma I(A)\Gamma I(A)\Gamma I(A)\Gamma I(A)\Big] = \Big(I(A)^5\Big].$$

On the other hand, by Lemma 3.13, we have

$$I(A)\Gamma I(A)\Gamma I(A) \subseteq (S\Gamma A\Gamma S].$$

Then we have

$$\begin{aligned} I(A)^4 &\subseteq (S\Gamma A\Gamma S]\Gamma(A \cup S\Gamma A \cup A\Gamma S \cup S\Gamma A\Gamma S] \\ &\subseteq \Big((S\Gamma A\Gamma S)\Gamma(A \cup S\Gamma A \cup A\Gamma S \cup S\Gamma A\Gamma S)\Big] \\ &= \Big(S\Gamma A\Gamma S\Gamma A \cup S\Gamma A\Gamma(S\Gamma S)\Gamma A \cup S\Gamma A\Gamma S\Gamma A\Gamma S \cup S\Gamma A\Gamma(S\Gamma S)\Gamma A\Gamma S\Big] \\ &= (S\Gamma A\Gamma S\Gamma A \cup S\Gamma A\Gamma S\Gamma A\Gamma S]. \end{aligned}$$

$$\begin{aligned} I(A)^5 &\subseteq (S\Gamma A\Gamma S\Gamma A \cup S\Gamma A\Gamma S\Gamma A\Gamma S]\Gamma(A \cup S\Gamma A \cup A\Gamma S \cup S\Gamma A\Gamma S] \\ &\subseteq \Big((S\Gamma A\Gamma S\Gamma A \cup S\Gamma A\Gamma S\Gamma A\Gamma S)\Gamma(A \cup S\Gamma A \cup A\Gamma S \cup S\Gamma A\Gamma S)\Big] \\ &= (S\Gamma A\Gamma S\Gamma A\Gamma A \cup S\Gamma A\Gamma S\Gamma A\Gamma S\Gamma A \cup S\Gamma A\Gamma S\Gamma A\Gamma A\Gamma S \\ &\quad \cup S\Gamma A\Gamma S\Gamma A\Gamma S\Gamma A\Gamma S \cup S\Gamma A\Gamma S\Gamma A\Gamma S\Gamma A \cup S\Gamma A\Gamma S\Gamma A\Gamma S\Gamma S\Gamma A \\ &\quad \cup S\Gamma A\Gamma S\Gamma A\Gamma S\Gamma A\Gamma S \cup S\Gamma A\Gamma S\Gamma A\Gamma S\Gamma S\Gamma A\Gamma S] \\ &\subseteq (S\Gamma A\Gamma S\Gamma A\Gamma S]. \end{aligned}$$

Therefore, we have

$$A \subseteq I(A) = \Big(I(A)^5\Big] \subseteq \Big((S\Gamma A\Gamma S\Gamma A\Gamma S]\Big] = (S\Gamma A\Gamma S\Gamma A\Gamma S].$$

Hence we obtain $A \subseteq (S\Gamma A\Gamma S\Gamma A\Gamma S]$ for every $A \in \mathcal{P}^*(S)$ and, by Proposition 3.18 $(3) \Rightarrow (1)$, S is semisimple.

The equivalence $(2) \Leftrightarrow (3)$ holds in ordered Γ-hypergroupoids in general. □

Remark 3.21 *In the above theorem we tried to use sets (instead of elements) to show the pointless character of the results. However, conditions*

(1) $I(a) \cap I(b) = \big(I(a)\Gamma I(b)\big]$ *or*

(2) $I(a) = \big(I(a)\Gamma I(a)\big]$

for any $a, b \in S$, also characterize the semisimple Γ-hypersemigroups. From (2) one can prove that S is semisimple exactly as in $(5) \Rightarrow (1)$ *of Theorem 3.20.*

Remark 3.22 *We have seen in [10, Theorem 3.15] that for an le-semigroup S the following are equivalent: (1) S is semisimple [4, 10]. (2) $a \wedge b = ab$ for any ideal elements a, b of S. (3) $a^2 = a$ for any ideal element a of S. (4) Every ideal element of S is weakly semiprime. Therefore, the following theorem also holds.*

Theorem 3.23 *An ordered Γ-hypersemigroup S is semisimple if and only if every ideal of S is weakly semiprime.*

Proof Let T be an ideal of S and A be an ideal of S such that $A\Gamma A \subseteq T$. Since S is semisimple, by Theorem 3.20 $(1) \Rightarrow (5)$, we have $I(A) = \Big(I(A)\Gamma I(A)\Big]$. Since A is an ideal of S, we have $I(A) = A$. Then we have $A = (A\Gamma A] \subseteq (T] = T$, then $A \subseteq T$ and so T is weakly semiprime.
The "$\Leftarrow$-part can be proved by and easy modification of the proof of $(4) \Rightarrow (1)$ in [10, Theorem 3.18] (in fact by replacing the $*$ by Γ) as follows: Let A be a nonempty subset of S. Then we have $I(A)^8 \subseteq (S\Gamma A\Gamma S\Gamma A]$ and since $(S\Gamma A\Gamma S\Gamma A]$ is an ideal of S and S is weakly semiprime, we have $A \subseteq (S\Gamma A\Gamma S\Gamma A]$ and by Proposition 3.18 $(3) \Rightarrow (1)$, S is semiprime. □

Theorem 3.24 *Let S be an ordered Γ-hypersemigroup. The ideals of S are weakly prime if and only if they form a chain and one of the nine equivalent conditions given in Theorem 3.20, Remark 3.21 or Theorem 3.23 holds in S.*

Proof $\Longrightarrow$. Let A, B be ideals of S. Since $(A\Gamma B]$ is an ideal of S, by hypothesis, it is weakly prime. Since A, B are ideals of S, $A\Gamma B \subseteq (A\Gamma B]$ and $(A\Gamma B]$ is weakly prime, we have $A \subseteq (A\Gamma B]$ or $B \subseteq (A\Gamma B]$. Then we have $A \subseteq (S\Gamma B] \subseteq (B] = B$ or $B \subseteq (A\Gamma S] \subseteq (A] = A$ i.e. $A \subseteq B$ or $B \subseteq A$, and the ideals of S form a chain. For the rest of the $\Rightarrow$-part of the theorem, it is enough to prove that the ideals of S are idempotent. For this purpose, let A be an ideal of S. By Lemma 3.11(2), $(A\Gamma A]$ is an ideal of S. By hypothesis, $(A\Gamma A]$ is weakly prime. Since A is an ideal of S, $A\Gamma A \subseteq (A\Gamma A]$ and $(A\Gamma A]$ is weakly prime, we have $A \subseteq (A\Gamma A]$. If now $x \in (A\Gamma A]$, then $x \in (A\Gamma S] \subseteq (A] = A$ and so $x \in A$. Thus we have $A = (A\Gamma A]$.
$\Longleftarrow$. Suppose condition (3) of Theorem 3.20 holds in S and let A, B, T be ideals of S such that $A\Gamma B \subseteq T$. By hypothesis, we have $A \subseteq B$ or $B \subseteq A$. If $A \subseteq B$ then, we have $A = A \cap B = (A\Gamma B] \subseteq (T] = T$. If $B \subseteq A$, then $B = A \cap B = (A\Gamma B] \subseteq T$. Thus we have $A \subseteq T$ or $B \subseteq T$ and so T is weakly prime. □

As a consequence, the following theorem holds.

Theorem 3.25 *The ideals of an ordered Γ-hypersemigroup S are weakly prime if and only if they form a chain S is semisimple.*

Definition 3.26 *[9, Definition 2.10] An ordered Γ-hypersemigroup S is called intraregular if for every $a \in S$ there exist $x, y, t \in S$ and $\gamma, \mu, \rho \in \Gamma$ such that $t \in (x\gamma a)\overline{\mu}(a\rho y)$ and $a \le t$.*

This is equivalent to saying $A \subseteq (S\Gamma A\Gamma A\Gamma S]$ for any nonempty subset A of S [9, Proposition 2.11].

Proposition 3.27 *(For ordered semigroups see [2]). If S is an intraregular ordered Γ-hypersemigroup then, for any nonempty subsets A, B of S, we have*

$$(S\Gamma A\Gamma B\Gamma S] = (S\Gamma B\Gamma A\Gamma S].$$

Proof Let A, B be nonempty subsets of S. Since S is intraregular and $A\Gamma B$ is a nonempty subset of S, by [9, Proposition 2.11], we have

$$A\Gamma B \subseteq \Big(S\Gamma(A\Gamma B)\Gamma(A\Gamma B)\Gamma S\Big] = \Big((S\Gamma A)\Gamma B\Gamma A\Gamma(B\Gamma S)\Big] \subseteq (S\Gamma B\Gamma A\Gamma S].$$

Then

$$\begin{aligned} S\Gamma(A\Gamma B)\Gamma S &\subseteq S\Gamma(S\Gamma B\Gamma A\Gamma S]\Gamma S = (S]\Gamma(S\Gamma B\Gamma A\Gamma S]\Gamma(S] \\ &\subseteq \Big((S\Gamma S)\Gamma B\Gamma A\Gamma(S\Gamma S)\Big] \subseteq (S\Gamma B\Gamma A\Gamma S]. \end{aligned}$$

Thus we have $(S\Gamma A\Gamma B\Gamma S] \subseteq \Big((S\Gamma B\Gamma A\Gamma S]\Big] = (S\Gamma B\Gamma A\Gamma S]$. By symmetry, we have $(S\Gamma B\Gamma A\Gamma S] \subseteq (S\Gamma A\Gamma B\Gamma S]$ and equality holds. □

Proposition 3.28 *Let S be an ordered Γ-hypersemigroup. If S is intraregular, then it is semisimple. If S is commutative and semisimple, then S is intraregular. Therefore, a commutative ordered Γ-hypersemigroup is semisimple if and only if it is intraregular.*

Proof $\Longrightarrow$. Let A be a nonempty subset of S. Since S is intraregular, we have

$$\begin{aligned}
A &\subseteq (S\Gamma A\Gamma A\Gamma S] \subseteq \Big(S\Gamma(S\Gamma A\Gamma A\Gamma S]\Gamma A\Gamma S\Big] \text{ (by Lemma 2.2)}\\
&= \Big(S\Gamma(S\Gamma A\Gamma A\Gamma S)\Gamma A\Gamma S\Big] \text{ (by Lemma 3.10)}\\
&= \Big((S\Gamma S)\Gamma A\Gamma A\Gamma S\Gamma A\Gamma S\Big] \text{ (by Lemma 2.3)}\\
&\subseteq \Big(S\Gamma A\Gamma A\Gamma S\Gamma A\Gamma S\Big] \text{ (since } S\Gamma S\subseteq S)\\
&\subseteq \Big(S\Gamma A\Gamma(A\Gamma S)\Gamma A\Gamma S\Big] \text{ (by Lemma 2.3)}\\
&\subseteq (S\Gamma A\Gamma S\Gamma A\Gamma S] \text{ (since } A\Gamma S\subseteq S).
\end{aligned}$$

By Proposition 3.18 $(3)\Rightarrow(1)$, S is semisimple.

$\Longleftarrow$. Let A be a nonempty subset of S. Since S is semisimple, by Proposition 3.18 $(1)\Rightarrow(3)$, we have

$$\begin{aligned}
A &\subseteq (S\Gamma A\Gamma S\Gamma A\Gamma S] = \Big(S\Gamma A\Gamma(S\Gamma A)\Gamma S\Big] \text{ (by Lemma 2.3)}\\
&= \Big(S\Gamma A\Gamma(A\Gamma S)\Gamma S\Big] \text{ (since } S \text{ is commutative)}\\
&= \Big(S\Gamma(A\Gamma A)\Gamma(S\Gamma S)\Big] \text{ (by Lemma 2.3)}\\
&\subseteq \Big(S\Gamma(A\Gamma A)\Gamma S\Big] \text{ (since } S\Gamma S\subseteq S)\\
&= (S\Gamma A\Gamma A\Gamma S) \text{ (by Lemma 2.3)}.
\end{aligned}$$

By [9, Proposition 2.11], S is intraregular. □

Theorem 3.29 *Let S be an ordered Γ-hypersemigroup. The ideals of S are prime if and only if they form a chain and S is intraregular.*

Proof $\Longrightarrow$. If the ideals of S are prime, then they are weakly prime, so, by Theorem 3.24, they form a chain. To show that S is intraregular, let A be a nonempty subset of S. By [9, Proposition 2.11], it is enough to prove that $A\subseteq(S\Gamma A\Gamma A\Gamma S]$. By Lemma 3.11(3), the set $(S\Gamma A\Gamma A\Gamma S]$ is an ideal of S; by hypothesis, $(S\Gamma A\Gamma A\Gamma S]$ is prime, and so semiprime as well. Since $(A\Gamma A)\Gamma(A\Gamma A)\subseteq(S\Gamma A\Gamma A\Gamma S]$ and $(S\Gamma A\Gamma A\Gamma S]$ is semiprime, we have $A\Gamma A\subseteq(S\Gamma A\Gamma A\Gamma S]$ and $A\subseteq(S\Gamma A\Gamma A\Gamma S]$.

$\Longleftarrow$. Since S is intraregular, the ideals of S are semiprime. In fact, if T is an ideal of S and A a nonempty subset of S such that $A\Gamma A\subseteq T$ then, since S is is intraregular, by [9, Proposition 2.11], we have

$$A\subseteq\Big(S\Gamma(A\Gamma A)\Gamma S\Big]\subseteq(S\Gamma T\Gamma S]=\Big((S\Gamma T)\Gamma S\Big]\subseteq(T\Gamma S]\subseteq(T]=T,$$

then $A\subseteq T$ and so T is semiprime.

Since the ideals of S are semiprime we have (1) and (2) below:

(1) $I(A)=(S\Gamma A\Gamma S]$ for every nonempty subset A of S. Indeed:

$(A\Gamma A)\Gamma(A\Gamma A)\subseteq(S\Gamma A\Gamma S]$, where $(S\Gamma A\Gamma S]$ is an ideal of S. Since $(S\Gamma A\Gamma S]$ is semiprime, we have $A\Gamma A\subseteq$

$(S\Gamma A\Gamma S]$ and $A \subseteq (S\Gamma A\Gamma S]$. Then we have $I(A) \subseteq (S\Gamma A\Gamma S] \subseteq I(A)$, then $I(A) = (S\Gamma A\Gamma S]$ and condition (1) is satisfied.

(2) $I(x\Gamma y) = I(x) \cap I(y)$ for every $x, y \in S$. Indeed:
Let $x, y \in S$. We have $x\Gamma y \subseteq I(x)\Gamma S \subseteq I(x)$ and $x\Gamma y \subseteq S\Gamma I(y) \subseteq I(y)$, thus we have $I(x\Gamma y) \subseteq I(x) \cap I(y)$. Let now $t \in I(x) \cap I(y)$. By (1), we have $t \in I(x) = (S\Gamma x\Gamma S]$ and $t \in I(y) = (S\Gamma y\Gamma S]$. Then we have

$$t \le u \text{ for some } u \in S\Gamma x\Gamma S \text{ and } t \le w \text{ for some } w \in S\Gamma y\Gamma S.$$

Since $u \in (S\Gamma x)\Gamma S$, we have $u \in v\mu b$ for some $v \in S\Gamma x$, $\mu \in \Gamma$, $b \in S$. Since $v \in S\Gamma x$, we have $v \in a\zeta x$ for some $a \in S$, $\zeta \in \Gamma$. Then we have

$$u \in v\mu b = \{v\}\overline{\mu}\{b\} \subseteq (a\zeta x)\overline{\mu}\{b\} = \{a\}\overline{\zeta}\{x\}\overline{\mu}\{b\}.$$

Since $w \in S\Gamma(y\Gamma S)$, we have $w \in c\xi z$ for some $c \in S$, $\xi \in \Gamma$, $z \in y\Gamma S$. Since $z \in y\Gamma S$, we have $z \in y\omega d$ for some $\omega \in \Gamma$, $d \in S$. Then we have

$$w \in c\xi z = \{c\}\overline{\xi}\{z\} \subseteq \{c\}\overline{\xi}(y\omega d) = \{c\}\overline{\xi}\{y\}\overline{\omega}\{d\}.$$

Hence we obtain

$$t \le u, \text{ where } u \in \{a\}\overline{\zeta}\{x\}\overline{\mu}\{b\} \text{ for some } a, b \in S,\ \zeta, \mu \in \Gamma,$$

$$t \le w, \text{ where } w \in \{c\}\overline{\xi}\{y\}\overline{\omega}\{d\} \text{ for some } c, d \in S,\ \xi, \omega \in \Gamma.$$

Let now $\gamma \in \Gamma$. By Lemma 2.8, we have

$$t\gamma t \preceq w\gamma u = \{w\}\overline{\gamma}\{u\} \subseteq \Big(\{c\}\overline{\xi}\{y\}\overline{\omega}\{d\}\Big)\overline{\gamma}\Big(\{a\}\overline{\zeta}\{x\}\overline{\mu}\{b\}\Big) = \{c\}\overline{\xi}\Big(\{y\}\overline{\omega}\{d\}\overline{\gamma}\{a\}\overline{\zeta}\{x\}\Big)\overline{\mu}\{b\}.$$

On the other hand, $\{y\}\overline{\omega}\{d\}\overline{\gamma}\{a\}\overline{\zeta}\{x\} \subseteq I(x\Gamma y)$. Indeed: We have

$$\begin{aligned}\Big(\{y\}\overline{\omega}\{d\}\overline{\gamma}\{a\}\overline{\zeta}\{x\}\Big)\Gamma\Big(\{y\}\overline{\omega}\{d\}\overline{\gamma}\{a\}\overline{\zeta}\{x\}\Big) &\subseteq S\overline{\zeta}\{x\}\Gamma\{y\}\overline{\omega}S \\ &= S\overline{\zeta}(x\Gamma y)\overline{\omega}S \subseteq S\Gamma(x\Gamma y)\Gamma S \subseteq I(x\Gamma y).\end{aligned}$$

Since the ideal $I(x\Gamma y)$ is semiprime, we have

$$\{y\}\overline{\omega}\{d\}\overline{\gamma}\{a\}\overline{\zeta}\{x\} \subseteq I(x\Gamma y),$$

and then

$$\{c\}\overline{\xi}\Big(\{y\}\overline{\omega}\{d\}\overline{\gamma}\{a\}\overline{\zeta}\{x\}\Big)\overline{\mu}\{b\} \subseteq \{c\}\overline{\xi}I(x\Gamma y)\overline{\mu}\{b\} \subseteq S\Gamma I(x\Gamma y)\Gamma S \subseteq I(x\Gamma y).$$

Hence we obtain $t\gamma t \preceq w\gamma u \subseteq I(x\Gamma y)$ for every $\gamma \in \Gamma$. Then $t\Gamma t \subseteq I(x\Gamma y)$. Indeed: Let $z \in t\Gamma t$. Then $z \in t\rho t$ for some $\rho \in \Gamma$. Since $t\gamma t \preceq w\gamma u \subseteq I(x\Gamma y)$ for every $\gamma \in \Gamma$, we have $t\rho t \preceq w\rho u \subseteq I(x\Gamma y)$. Since $t\rho t \preceq w\rho u$ and $z \in t\rho t$, there exists $h \in w\rho u$ such that $z \le h$. We have $z \le h \in w\rho u \subseteq I(x\Gamma y)$ and so $z \in I(x\Gamma y)$. Therefore, $t\Gamma t \subseteq I(x\Gamma y)$. Since $t\Gamma t \subseteq I(x\Gamma y)$ and $I(x\Gamma y)$ is semiprime, we have $t \in I(x\Gamma y)$. Thus $I(x) \cap I(y) \subseteq I(x\Gamma y)$, and property (2) is satisfied.

Let now T be an ideal of S and $a, b \in S$ such that $a\Gamma b \subseteq T$. By hypothesis, $I(a) \subseteq I(b)$ or $I(b) \subseteq I(a)$. If $I(a) \subseteq I(b)$ then, by property (2), we have $a \in I(a) = I(a) \cap I(b) = I(a\Gamma b) \subseteq I(T) = T$ and so $a \in T$. If $I(b) \subseteq I(a)$, then $b \in I(b) = I(a) \cap I(b) = I(a\Gamma b) \subseteq I(T) = T$ and so $b \in T$. By Proposition 3.3, T is prime.
□

Corollary 3.30 *(For ordered semigroups see [2]). An ordered* Γ*-hypersemigroup* S *is intraregular and the ideals of* S *form a chain if and only if, for any nonempty subsets* A, B *of* S*, we have*

$$A \subseteq (S\Gamma A\Gamma B\Gamma S] \text{ or } B \subseteq (S\Gamma A\Gamma B\Gamma S].$$

Proof $\Longrightarrow$. If S is intraregular and the ideals of S form a chain, then, by Theorem 3.29, the ideals of S are prime. Since $(S\Gamma A\Gamma B\Gamma S]$ is an ideal of S and $(A\Gamma A)\Gamma(B\Gamma B) \subseteq (S\Gamma A\Gamma B\Gamma S]$, we have $A\Gamma A \subseteq (S\Gamma A\Gamma B\Gamma S]$ or $B\Gamma B \subseteq (S\Gamma A\Gamma B\Gamma S]$, then $A \subseteq (S\Gamma A\Gamma B\Gamma S]$ or $B \subseteq (S\Gamma A\Gamma B\Gamma S]$.
$\Longleftarrow$. Suppose that for any nonempty subsets A, B of S, we have $A \subseteq (S\Gamma A\Gamma B\Gamma S]$ or $B \subseteq (S\Gamma A\Gamma B\Gamma S]$. To prove that S is intraregular and the ideals of S form a chain, by Theorem 3.29, it is enough to prove that the ideals of S are prime. For this, let T be an ideal of S and A, B nonempty subsets of S such that $A\Gamma B \subseteq T$. By hypothesis, we have $A \subseteq (S\Gamma A\Gamma B\Gamma S]$ or $B \subseteq (S\Gamma A\Gamma B\Gamma S]$. If $A \subseteq (S\Gamma A\Gamma B\Gamma S]$, then we have $A \subseteq (S\Gamma T\Gamma S] \subseteq (STS] \subseteq (T] = T$. If $B \subseteq (S\Gamma A\Gamma B\Gamma S]$, then $B \subseteq (S\Gamma T\Gamma S] \subseteq T$. Thus we have $A \subseteq T$ or $B \subseteq T$ and T is prime. □

Remark 3.31 *According to the present paper, the results on ordered semigroups in [1] hold not only for elements but for sets as well that shows that the results on [1] are also based on le(poe)-semigroups.*

Remark 3.32 *A poe-semigroup* S *is called intraregular if for every* $a \in S$ *we have* $a \le ea^2e$*. It is called semisimple if* $a \le eaeae$ *for every* $a \in S$ *[4]. The abstract formulation of Proposition 3.27 is the following: If* S *is an intraregular poe-semigroup then, for any* $a, b \in S$*, we have* $eabe = ebae$*. In fact, if* $a, b \in S$ *then, since* S *is intraregular, we have* $ab \le e(ab)^2e = eababe \le ebae$*, then* $eabe \le e^2bae^2 \le ebae$*. By symmetry, we have* $ebae \le eabe$ *and equality holds. The abstract formulation of Proposition 3.28 is as follows. If* S *is intraregular and* $a \in S$*, then* $a \le ea^2e \le e(ea^2e)ae = (e^2a)aeae \le eaeae$ *and so* S *is semisimple. If* S *is commutative and semisimple, then* $a \le e(ae)(ae) = e(ea)(ae) = e^2a^2e \le ea^2e$ *and so* S *is intraregular.*

4. Examples

We apply the results of this paper to the following examples.

For the first example we have to give the following definitions:

An ordered Γ-hypersemigroup S is called regular if for every $a \in S$ there exist $x, t \in S$ and $\gamma, \mu \in \Gamma$ such that $t \in (a\gamma x)\overline{\mu}\{a\}$ and $a \le t$ [9, Definition 2.3]. An ordered Γ-hypersemigroup S is regular if and only if for any nonempty subset A of S we have $A \subseteq (A\Gamma S\Gamma A]$ [9, Proposition 2.4]. An ordered Γ-hypersemigroup S is called right regular if for every $a \in S$ there exist $x, t \in S$ and $\gamma, \mu \in \Gamma$ such that $t \in (a\gamma a)\overline{\mu}\{x\}$ and $a \le t$ [9, Definition 2.13]; equivalently if $A \subseteq (A\Gamma A\Gamma S]$ for every nonempty subset A of S [9, Proposition 2.14]. It is called left regular if for every $a \in S$ there exist $x, t \in S$ and $\gamma, \mu \in \Gamma$ such that $t \in \{x\}\overline{\gamma}(a\mu a)$ and $a \le t$ [9, Definition 2.13]; equivalently if $A \subseteq (S\Gamma A\Gamma A]$ for any nonempty subset A of S [9, Proposition 2.15].

Example 4.1 *Here we give some examples of ordered* Γ*-hypersemigroups in which the ideals are idempotent. In a regular ordered* Γ*-hypersemigroup the right (or left) ideals are idempotent. In fact, if* A *is a right ideal of* S*, then* $A \subseteq (A\Gamma S\Gamma A] = \Big((A\Gamma S)\Gamma A\Big] \subseteq (A\Gamma A] \subseteq (A\Gamma S] \subseteq (A] = A$ *and so* $(A\Gamma A] = A$*. Therefore, in a regular ordered* Γ*-hypersemigroup, the ideals are idempotent. If* S *is a semisimple ordered* Γ*-hypersemigroup and* A

is an ideal of S*, then* $A \subseteq (S\Gamma A\Gamma S\Gamma A\Gamma S] = \Big((S\Gamma A)\Gamma S\Gamma(A\Gamma S)\Big] \subseteq (A\Gamma S\Gamma A] = \Big((A\Gamma S)\Gamma A\Big] \subseteq (A\Gamma A] \subseteq (A\Gamma S] \subseteq (A] = A$ *and so* A *is idempotent. As a result, in an intraregular ordered* Γ*-hypersemigroup the ideals are idempotent. The right (or left) regular ordered* Γ*-hypersemigroups are intraregular. In fact, if* S *is right regular and* A *is a nonempty subset of* S*, then* $A \subseteq (A\Gamma A\Gamma S] \subseteq \Big(A\Gamma(A\Gamma A\Gamma S]\Gamma S\Big] = \Big(A\Gamma(A\Gamma A\Gamma S)\Gamma S\Big] = \Big(A\Gamma A\Gamma A\Gamma(S\Gamma S)\Big] \subseteq (S\Gamma A\Gamma A\Gamma S]$ *and so* S *is intraregular. Thus in right regular or left regular ordered* Γ*-hypersemigroups, the ideals are idempotent.*

Remark 4.2 *Regarding Example 4.1, clearly, the idea comes from the poe-semigroups. If* S *is a regular poe-semigroup and* a *is a right (resp. left) ideal element of* S*, then* $a \le aea = (ae)a \le a^2 \le ae \le a$ *(resp.* $a \le aea \le a(ea) \le a^2 \le ea \le a$*) and so* $a^2 = a$*. If* S *is an intraregular poe-semigroup and* a *is an ideal element of* S*, then* $a \le ea^2e = (ea)(ae) \le a^2 \le ae \le a$ *and so* $a^2 = a$*. If* S *is a semisimple poe-semigroup and* a *is an ideal element of* S*, then* $a \le eaeae \le (ea)e(ae) \le aea \le a^2 \le a$ *and so* $a^2 = a$*. If* S *is a right regular poe-semigroup, then* $a \le a^2e \le a(a^2e)e \le ea^2e$ *so* $a \le ea^2e$ *for every* $a \in S$ *and so* S *is intraregular. If* S *is a left regular poe-semigroup, then* $a \le ea^2 = e(ea^2)a \le ea^2e$ *for every* $a \in S$ *and so* S *is intraregular. For the necessary definitions see [4] and [6].*

Example 4.3 *We consider the ordered* Γ*-semigroup* $S = \{a, b, c, d\}$ *given by Tables 1 and 2 and Figure 1. Using the methodology described in [11], from this ordered* Γ*-semigroup the ordered* Γ*-hypersemigroup given by Tables 3 and 4 and the same figure (Figure 1) can be obtained. This is an intraregular ordered* Γ*-hypersemigroup as, for example,*

$a \in (a\gamma a)\overline{\mu}(a\gamma a) = \{a\}\overline{\mu}\{a\} = a\mu a = \{a, b, c\}$ *and* $a \le a$.

$b \in (a\gamma b)\overline{\mu}(b\gamma a) = \{b\}\overline{\mu}\{a, b, c\} = \bigcup\limits_{x\in\{a,b,c\}} b\mu x = b\mu a \cup b\mu b \cup b\mu c = \{a, b, c\}$ *and* $b \le b$.

$$\begin{aligned} c \in (c\gamma c)\overline{\mu}(c\gamma c) &= \{a, b, c\}\overline{\mu}\{a, b, c\} = \bigcup_{x\in\{a,b,c\}, y\in\{a,b,c\}} x\mu y \\ &= a\mu a \cup a\mu b \cup a\mu c \cup b\mu a \cup b\mu b \cup b\mu c \cup c\mu a \cup c\mu b \cup c\mu c = \{a, b, c\} \text{ and } c \le c. \end{aligned}$$

$d \in (d\gamma d)\overline{\gamma}(d\gamma d) = \{d\}\overline{\gamma}\{d\} = d\gamma d = \{d\}$ *and* $d \le d$.

The set of all nonempty subsets of S *is the set*

$$\Big\{\{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, S\Big\}.$$

The only ideals of S *are the sets* $\{a, b, c\}$ *and* S *and they form a chain. Indeed, we have*

$S\Gamma\{a\} = S\Gamma\{b\} = S\Gamma\{c\} = \{a, b, c\} \nsubseteq \{a\}, \{b\}, \{c\}$ *that means that* $\{a\}$*,* $\{b\}$*,* $\{c\}$ *cannot be ideals.*

$S\Gamma\{d\} = \{a, b, c, d\} \nsubseteq \{d\}$ *that means that* $\{d\}$ *cannot be ideal.*

$S\Gamma\{a, b\} = S\Gamma\{a, c\} = S\Gamma\{b, c\} = \{a, b, c\} \nsubseteq \{a, b\}, \{a, c\}, \{b, c\}$ *and so* $\{a, b\}$*,* $\{a, c\}$*,* $\{b, c\}$ *are not ideals.*

$S\Gamma\{a,d\} = S\Gamma\{b,d\} = S\Gamma\{c,d\} = S\Gamma\{a,b,d\} = S\Gamma\{a,c,d\} = S\Gamma\{b,c,d\} = \{a,b,c,d\}$
$\nsubseteq \{a,d\},\{b,d\},\{c,d\},\{a,b,d\},\{a,c,d\},\{b,c,d\}$
and so $\{a,d\},\{b,d\},\{c,d\},\{a,b,d\},\{a,c,d\},\{b,c,d\}$ *cannot be ideals.*

On the other hand,

$S\Gamma\{a,b,c\} = \{a,b,c\} \subseteq \{a,b,c\}$.

$\{a,b,c\}\Gamma S = \{a,b,c\}\{\gamma,\mu\}\{a,b,c,d\} = \{a,b,c\} \subseteq \{a,b,c\}$.

Let $x \in \{a,b,c\}$ *and* $\{a,b,c,d\} \ni y \le x$. *We have the following cases:*

(a) $x = a$ *and* $\{a,b,c,d\} \ni y \le a$. *Then* $y = a \in \{a,b,c\}$.

(b) $x = b$ *and* $\{a,b,c,d\} \ni y \le b$. *Then* $y = b \in \{a,b,c\}$.

(c) $x = c$ *and* $\{a,b,c,d\} \ni y \le c$. *Then* $y = a$ *or* $y = b$ *or* $y = c$ *and so* $y \in \{a,b,c\}$.

Therefore, the set $\{a,b,c\}$ *is an ideal of* S.

Theorem 3.29 can be applied and the set $\{a,b,c\}$ *is a prime ideal of* S. *Independently, one can write a computer program to see that* $\{a,b,c\}$ *is the only proper ideal of* S. *As* S *is intraregular, by Proposition 3.28, it is semisimple as well and so Theorem 3.20 can be also applied. As a consequence, the ideals of* S *are idempotent and for any ideals* A,B *of* S, *we have* $A \cap B = (A\Gamma B]$. *Independently,*

$$\begin{aligned}\Big(\{a,b,c\}\Gamma\{a,b,c\}\Big] &= \Big(\{a,b,c\}\{\gamma,\mu\}\{a,b,c\}\Big]\\ &= (a\gamma a \cup a\gamma b \cup a\gamma c \cup b\gamma a \cup b\gamma b \cup b\gamma c \cup c\gamma a \cup c\gamma b \cup c\gamma c\\ &\quad \cup a\mu a \cup a\mu b \cup a\mu c \cup b\mu a \cup b\mu b \cup b\mu c \cup c\mu a \cup c\mu b \cup c\mu c]\\ &= \Big(\{a,b,c\}\Big] = \{a,b,c\} \ (as\ \{a,b,c\}\ is\ an\ ideal\ of\ S)\\ &= \{a,b,c\} \cap \{a,b,c\}.\end{aligned}$$

$\Big(\{a,b,c\}\Gamma\{a,b,c,d\}\Big] = \Big(\{a,b,c\}\{\gamma,\mu\}\{a,b,c,d\}\Big] = \Big(\{a,b,c\}\Big] = \{a,b,c\} = \{a,b,c\} \cap \{a,b,c,d\}$.

$\Big(\{a,b,c,d\}\Gamma\{a,b,c\}\Big] = \Big(\{a,b,c,d\}\{\gamma,\mu\}\{a,b,c\}\Big] = \Big(\{a,b,c\}\Big] = \{a,b,c\} = \{a,b,c,d\} \cap \{a,b,c\}$.

$\Big(\{a,b,c,d\}\Gamma\{a,b,c,d\}\Big] = \{a,b,c,d\} = \{a,b,c,d\} \cap \{a,b,c,d\}$.

Table 1. The ω-operation of Example 4.3.

ω	a	b	c	d
a	a	b	c	a
b	c	c	c	b
c	c	c	c	c
d	a	b	c	d

Example 4.4 *We consider the ordered* Γ*-semigroup* $S = \{a,b,c,d,e\}$ *given by Tables 5 and 6 and Figure 2. From this ordered* Γ*-semigroup the ordered* Γ*-hypersemigroup given by Tables 7 and 8 and the same figure (Figure 2) can be obtained. This is an intraregular ordered* Γ*-hypersemigroup as, for example,*

Table 2. The ξ-operation of Example 4.3.

ξ	a	b	c	d
a	c	c	c	b
b	c	c	c	c
c	c	c	c	c
d	c	c	c	b

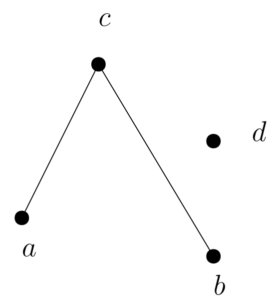

Figure 1. The order of Example 4.3.

Table 3. The γ-hyperoperation of Example 4.3.

γ	a	b	c	d
a	$\{a\}$	$\{b\}$	$\{a,b,c\}$	$\{a\}$
b	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{b\}$
c	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$
d	$\{a\}$	$\{b\}$	$\{a,b,c\}$	$\{d\}$

Table 4. The μ-hyperoperation of Example 4.3.

μ	a	b	c	d
a	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{b\}$
b	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$
c	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$
d	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{b\}$

$$a \in (a\gamma a)\overline{\mu}(a\gamma a) = \{a\}\overline{\mu}\{a\} = a\mu a = \{a\} \ \textit{and}\ a \le a\,,$$

$$\begin{aligned} b \in (b\gamma b)\overline{\mu}(b\gamma b) &= \{a,b,e\}\overline{\mu}\{a,b,e\} = \bigcup_{x,y\in\{a,b,e\}} x\mu y \\ &= a\mu a \cup a\mu b \cup a\mu e \cup b\mu a \cup b\mu b \cup b\mu e \cup e\mu a \cup e\mu b \cup e\mu e = \{a,b,e\} \ \textit{and}\ b \le b, \end{aligned}$$

$$c \in (c\gamma c)\overline{\gamma}(c\gamma c) = \{c\}\overline{\gamma}\{c\} = c\gamma c = \{c\} \ \textit{and}\ c \le c\,,$$
$$d \in (d\gamma d)\overline{\gamma}(d\gamma d) = \{c,d\}\overline{\gamma}\{c,d\} = c\gamma c \cup c\gamma d \cup d\gamma c \cup d\gamma d = \{c,d\} \ \textit{and}\ d \le d\,,$$
$$e \in (e\gamma e)\overline{\mu}(e\gamma e) = \{e\}\overline{\mu}\{e\} = e\mu e = \{e\} \ \textit{and}\ e \le e\,.$$

The set of all nonempty subsets of S *is the set*

$$\Big\{\{a\},\{b\},\{c\},\{d\},\{e\},\{a,b\},\{a,c\},\{a,d\},\{a,e\},\{b,c\},\{b,d\},\{b,e\},\{c,d\},\{c,e\},\{d,e\},$$
$$\{a,b,c\},\{a,b,d\},\{a,b,e\},\{a,c,d\},\{a,c,e\},\{a,d,e\},\{b,c,d\},\{b,c,e\},\{b,d,e\},\{c,d,e\},$$
$$\{a,b,c,d\},\{a,b,c,e\},\{a,b,d,e\},\{a,c,d,e\},\{b,c,d,e\},S\Big\}.$$

As in the previous example, one can check that the sets $\{a,b,e\}$ *and* S *are ideals of* S *and there is no any other ideal of* S*. As the ideals* $\{a,b,e\}$ *and* S *clearly form a chain, according to Theorem 3.29, they are prime ideals. As this is an example of semisimple ordered* Γ*-hypersemigroup as well, the results of the paper concerning the semisimple ordered* Γ*-hypersemigroup can be also applied.*

Table 5. The ω-operation of Example 4.4.

ω	a	b	c	d	e
a	a	b	a	b	a
b	a	b	b	b	b
c	a	b	c	d	e
d	a	b	d	d	b
e	a	b	e	b	e

Table 6. The ζ-operation of Example 4.4.

ζ	a	b	c	d	e
a	a	b	a	b	a
b	a	b	b	b	b
c	a	b	e	b	e
d	a	b	b	b	b
e	a	b	e	b	e

Table 7. The γ-hyperoperation of Example 4.4.

γ	a	b	c	d	e
a	$\{a\}$	$\{a,b,e\}$	$\{a\}$	$\{a,b,e\}$	$\{a\}$
b	$\{a\}$	$\{a,b,e\}$	$\{a,b,e\}$	$\{a,b,e\}$	$\{a,b,e\}$
c	$\{a\}$	$\{a,b,e\}$	$\{c\}$	$\{c,d\}$	$\{e\}$
d	$\{a\}$	$\{a,b,e\}$	$\{c,d\}$	$\{c,d\}$	$\{a,b,e\}$
e	$\{a\}$	$\{a,b,e\}$	$\{e\}$	$\{a,b,e\}$	$\{e\}$

Example 4.5 *We consider the ordered* Γ*-semigroup* $S=\{a,b,c,d,e\}$ *given by Tables 9 and 10 and Figure 3. From this ordered* Γ*-semigroup the ordered* Γ*-hypersemigroup given by Tables 11 and 12 and the same figure (Figure 3) can be obtained. This is an intraregular ordered* Γ*-hypersemigroup as, for example,*

Table 8. The μ-hyperoperation of Example 4.4.

μ	a	b	c	d	e
a	$\{a\}$	$\{a,b,e\}$	$\{a\}$	$\{a,b,e\}$	$\{a\}$
b	$\{a\}$	$\{a,b,e\}$	$\{a,b,e\}$	$\{a,b,e\}$	$\{a,b,e\}$
c	$\{a\}$	$\{a,b,e\}$	$\{e\}$	$\{a,b,e\}$	$\{e\}$
d	$\{a\}$	$\{a,b,e\}$	$\{a,b,e\}$	$\{a,b,e\}$	$\{a,b,e\}$
e	$\{a\}$	$\{a,b,e\}$	$\{e\}$	$\{a,b,e\}$	$\{e\}$

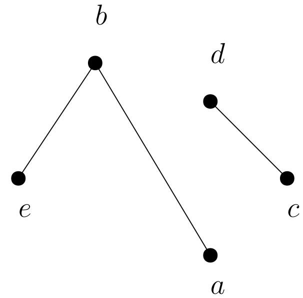

Figure 2. The order of Example 4.4.

$$a \in (a\gamma a)\overline{\mu}(a\gamma a) = \{a\} \text{ and } a \le a, \quad b \in (b\gamma b)\overline{\mu}(b\gamma b) = \{a,b\} \text{ and } b \le b,$$
$$c \in (c\gamma c)\overline{\gamma}(c\gamma c) = \{c\} \text{ and } c \le c, \quad d \in (d\gamma d)\overline{\gamma}(d\gamma d) = \{d\} \text{ and } d \le d,$$
$$e \in (e\gamma e)\overline{\gamma}(e\gamma e) = \{e\} \text{ and } e \le e.$$

The sets $\{a,b\}$, $\{a,b,d\}$, $\{a,b,e\}$, $\{a,b,d,e\}$ *and* S *are the only ideals of* S *and they do not form a chain. However, the ideals of* S *could be prime, but they are not. The ideal* $\{a,b\}$ *is not weakly prime as* $\{a,b,d\}\{\gamma,\mu\}\{a,b,e\} = \{a,b\}$ *but* $\{a,b,d\} \nsubseteq \{a,b\}$ *and* $\{a,b,e\} \nsubseteq \{a,b\}$.

The ideal $\{a,b,d,e\}$ *is weakly prime as*

$$\{a,b\}\Gamma\{a,b\} = \{a,b\}, \ \{a,b\}\Gamma\{a,b,d\} = \{a,b\}, \ \{a,b\}\Gamma\{a,b,e\} = \{a,b\},$$
$$\{a,b\}\Gamma\{a,b,d,e\} = \{a,b\}, \ \{a,b\}\Gamma\{a,b,c,d,e\} = \{a,b\},$$
$$\{a,b,d\}\Gamma\{a,b\} = \{a,b\}, \ \{a,b,d\}\Gamma\{a,b,d\} = \{a,b,d\}, \ \{a,b,d\}\Gamma\{a,b,e\} = \{a,b\},$$
$$\{a,b,d\}\Gamma\{a,b,d,e\} = \{a,b,d\}, \ \{a,b,d\}\Gamma\{a,b,c,d,e\} = \{a,b,d\},$$
$$\{a,b,e\}\Gamma\{a,b\} = \{a,b\}, \ \{a,b,e\}\Gamma\{a,b,d\} = \{a,b\}, \ \{a,b,e\}\Gamma\{a,b,e\} = \{a,b,e\},$$
$$\{a,b,e\}\Gamma\{a,b,d,e\} = \{a,b,e\}, \ \{a,b,e\}\Gamma\{a,b,c,d,e\} = \{a,b,e\},$$
$$\{a,b,d,e\}\Gamma\{a,b\} = \{a,b\}, \ \{a,b,d,e\}\Gamma\{a,b,d\} = \{a,b,d\}, \ \{a,b,d,e\}\Gamma\{a,b,e\} = \{a,b,e\},$$
$$\{a,b,d,e\}\Gamma\{a,b,d,e\} = \{a,b,d,e\}, \ \{a,b,d,e\}\Gamma\{a,b,c,d,e\} = \{a,b,d,e\},$$
$$\{a,b,c,d,e\}\Gamma\{a,b\} = \{a,b\}, \ \{a,b,c,d,e\}\Gamma\{a,b,d\} = \{a,b,d\}, \ \{a,b,c,d,e\}\Gamma\{a,b,e\} = \{a,b,e\},$$
$$\{a,b,c,d,e\}\Gamma\{a,b,d,e\} = \{a,b,d,e\}.$$

The ideals $\{a,b,d\}$ *and* $\{a,b,e\}$ *are weakly prime as well.*

If we take the ordered Γ*-semigroup given by Tables 9 and 10 and change Figure 3 to Figure 4, then we obtain the ordered* Γ*-hypersemigroup given by Tables 13 and 14.*

This is an intraregular Γ*-hypersemigroup as, for example,*

Table 9. The ρ-operation of Example 4.5.

ρ	a	b	c	d	e
a	a	b	a	b	a
b	a	b	b	b	b
c	a	b	c	d	e
d	a	b	d	d	b
e	a	b	e	b	e

Table 10. The ζ-operation of Example 4.5.

ζ	a	b	c	d	e
a	a	b	a	b	a
b	a	b	a	b	a
c	a	b	a	b	a
d	a	b	a	b	a
e	a	b	a	b	a

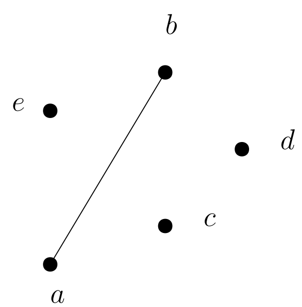

Figure 3. The order of Example 4.5.

Table 11. The γ-hyperoperation of Example 4.5.

γ	a	b	c	d	e
a	$\{a\}$	$\{a,b\}$	$\{a\}$	$\{a,b\}$	$\{a\}$
b	$\{a\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
c	$\{a\}$	$\{a,b\}$	$\{c\}$	$\{d\}$	$\{e\}$
d	$\{a\}$	$\{a,b\}$	$\{d\}$	$\{d\}$	$\{a,b\}$
e	$\{a\}$	$\{a,b\}$	$\{e\}$	$\{a,b\}$	$\{e\}$

$a \in (a\gamma a)\overline{\mu}(a\gamma a) = \{a\}$ *and* $a \le a$, $b \in (b\gamma b)\overline{\mu}(b\gamma b) = S$ *and* $b \le b$,
$c \in (c\gamma c)\overline{\gamma}(c\gamma c) = \{c\}$ *and* $c \le c$, $d \in (d\gamma d)\overline{\gamma}(d\gamma d) = \{d\}$ *and* $d \le d$,
$e \in (e\gamma e)\overline{\gamma}(e\gamma e) = \{e\}$ *and* $e \le e$ *(and so semisimple as well)*
having the S *as its only ideal that clearly is prime.*

Let us finally give an example of an intraregular *poe*-semigroup to apply the results mentioned in Remark 3.32.

Table 12. The μ-hyperoperation of Example 4.5.

μ	a	b	c	d	e
a	$\{a\}$	$\{a,b\}$	$\{a\}$	$\{a,b\}$	$\{a\}$
b	$\{a\}$	$\{a,b\}$	$\{a\}$	$\{a,b\}$	$\{a\}$
c	$\{a\}$	$\{a,b\}$	$\{a\}$	$\{a,b\}$	$\{a\}$
d	$\{a\}$	$\{a,b\}$	$\{a\}$	$\{a,b\}$	$\{a\}$
e	$\{a\}$	$\{a,b\}$	$\{a\}$	$\{a,b\}$	$\{a\}$

Table 13. The γ-hyperoperation of Example 4.5 related to Figure 4.

γ	a	b	c	d	e
a	$\{a\}$	S	$\{a\}$	S	$\{a\}$
b	$\{a\}$	S	S	S	S
c	$\{a\}$	S	$\{c\}$	$\{d\}$	$\{e\}$
d	$\{a\}$	S	$\{d\}$	$\{d\}$	S
e	$\{a\}$	S	$\{e\}$	S	$\{e\}$

Table 14. The μ-hyperoperation of Example 4.5 related to Figure 4.

μ	a	b	c	d	e
a	$\{a\}$	S	$\{a\}$	S	$\{a\}$
b	$\{a\}$	S	$\{a\}$	S	$\{a\}$
c	$\{a\}$	S	$\{a\}$	S	$\{a\}$
d	$\{a\}$	S	$\{a\}$	S	$\{a\}$
e	$\{a\}$	S	$\{a\}$	S	$\{a\}$

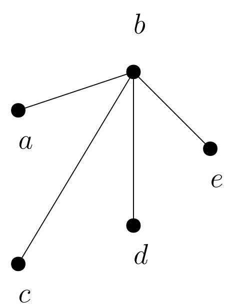

Figure 4. The (second) order of Example 4.5.

Example 4.6 *The poe-semigroup* $S = \{a,b,c,d,e\}$ *defined by Table 15 and Figure 5 is intraregular and so semisimple as well. From the fact that* S *is commutative and semisimple, we also get that* S *is intraregular. Also, for any* $a,b \in S$, *we have* $eabe = ebae$.

The poe-semigroup $S = \{a,b,c,d,e\}$ *given by Table 16 and Figure 6 is intraregular and not commutative. However, it is semisimple.*

Note: There is mistake in the proof of the implication (4) $\Rightarrow$ (1) in [10, Theorem 3.15] that affects the

Table 15. The multiplication of Example 4.6 related to Figure 5.

$\cdot$	a	b	c	d	e
a	c	b	a	b	e
b	b	b	b	b	b
c	a	b	c	b	e
d	b	b	b	b	b
e	e	b	e	b	e

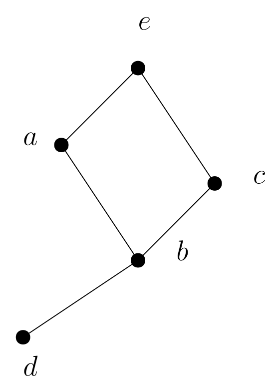

Figure 5. The order of Example 4.6 related to Table 15.

Table 16. The multiplication of Example 4.6 related to Figure 6.

$\cdot$	a	b	c	d	e
a	e	b	a	d	e
b	b	b	b	b	b
c	e	b	c	d	e
d	d	b	d	d	d
e	e	b	e	d	e

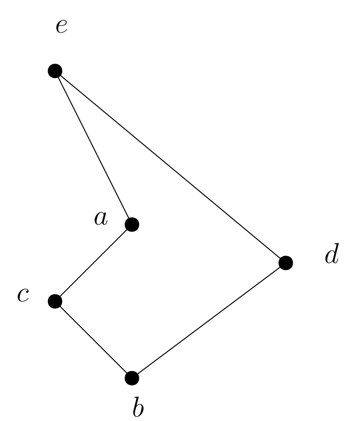

Figure 6. The order of Example 4.6 related to Table 16.

proof of (4) ⇒ (1) of [10, Theorem 3.18] as well corrected in Theorem 3.23 above. In Theorem 3.15, after the $\big(r(l(a)\big)^5 \le eaeae$, we should write $\big(r(l(a)\big)^8 \le eaeae$ and, since $eaeae$ is an ideal element of S and S is weakly semiprime, we have $a \le \big(r(l(a)\big) \le eaeae$ and so S is semiprime.

I would like to thank the two anonymous referees for their time to read the paper carefully and their prompt reply.

References

[1] Kehayopulu N. On prime, weakly prime ideals in ordered semigroups. Semigroup Forum 1992; 46: 341-346. doi:10.1007/BF02574353

[2] Kehayopulu N. On intra-regular ordered semigroups. Semigroup Forum 1993; 46 (3): 271-278. doi:10.1007/BF02573571

[3] Kehayopulu N. On prime, weakly prime ideals in po-Γ-semigroups. Lobachevskii Journal of Mathematics 2009; 30 (4): 257-262.

[4] Kehayopulu N. On le-semigroups. Turkish Journal of Mathematics 2016: 40: 310-316. doi:10.3906/mat-1503-74

[5] Kehayopulu N. On ordered hypersemigroups with idempotent ideals, prime or weakly prime ideals. European Journal of Pure and Applied Mathematics 2018; 11 (1): 10-22. doi:10.29020/nybg.ejpam.v11i1.3085

[6] Kehayopulu N. Lattice ordered semigroups and Γ-hypersemigroups. Turkish Journal of Mathematics 2020; 44 (5): 1835-1851. doi:10.3906/mat-1909-83

[7] Kehayopulu N. Bi-interior ideal elements in $\wedge e$-semigroups. European Journal of Pure and Applied Mathematics 2021; 14 (1): 43-52. doi:10.29020/nybg.ejpam.v14i1.3931

[8] Kehayopulu N. On ordered Γ-hypersemigroups, minimal bi-ideals, and minimal left ideals. Turkish Journal of Mathematics 2021; 45 (2): 909-918. doi:10.3906/mat-2011-35

[9] Kehayopulu N. On ordered Γ-hypersemigroups and their relation to lattice ordered semigroups. Turkish Journal of Mathematics 2021; 45 (3): 1120-1132. doi:10.3906/mat-2101-23

[10] Kehayopulu N. Relationship between lattice ordered semigroups and ordered hypersemigroups. Turkish Journal of Mathematics 2021; 45 (4): 1724-1737. doi:10.3906/mat-2103-98

[11] Kehayopulu N. Finite ordered Γ-hypersemigroups constructed by ordered Γ-semigroups. Turkish Journal of Mathematics 2022; 46 (1): 323-337. doi:10.3906/mat-2111-33

[12] Koutský K. Théorie the lattices topologiques. Publications de la Faculte des Sciences de l'Universite Masaryk (Brno). Spisy Vydavane Prirodovedeckou Facultou Masarykovy University 1952 (1952), 133-171.

[13] Nöbeling G. Grundlagen der analytischen Topologie. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Band LXXII. Springer-Verlag, Berlin-Göttingen-Heidelberg 1954. x+221 pp.

[14] G. Szász G. Eine Charakteristik der Primidealhalbgruppen. Publicationes Mathematicae Debrecen 1970; 17: 209-213.