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2-colored Rogers-Ramanujan partition identities

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Abstract: In this paper, we combined two types of partitions and introduced 2-colored Rogers-Ramanujan partitions. By finding some functional equations and using a constructive method, some identities have been found. Some overpartition identities coincide with our findings. A correspondence between colored partitions and overpartitions is provided.

Key words: Colored partitions, Rogers-Ramanujan, overpartition

1. Introduction

Agarwal and Andrews [1] gave some identities for Rogers-Ramanujan type partitions with n copies of n, which correspond to colored partitions. Sandon and Zanello [6] proved identities on the colored case of some partition types. This gave us the idea of combining colored partitions and Roger-Ramanujan type partitions and applying a constructive method to find identities on those partitions.

Rogers-Ramanujan identities can be interpreted using partitions in which the difference between every two consecutive parts is at least 2. We combined these types of partitions and colored ones to have the following partition type.

Definition 1.1 *A 2-colored Rogers-Ramanujan partition of n consists of two separated parts, each of the same color, and the difference between every two consecutive parts of the same color is at least two, moreover, parts in different colors do not intersect.*

As an example, all 2-colored Rogers-Ramanujan partitions of 6 are

$$6, \color{red}{6}, 5+1, \color{red}{5}+1, 5+\color{red}{1}, \color{red}{5}+\color{red}{1}, 4+2, \color{red}{4}+2, 4+\color{red}{2}, \color{red}{4}+\color{red}{2}, 3+\color{red}{2}+1, \color{red}{3}+2+\color{red}{1}$$

.

A constructive way from Kurşungöz's papers [4] and [5] was applied in the generating functions for these types of partitions to find identities on 2-colored Rogers-Ramanujan partitions.

For this purpose, functional equations relating to generating functions for given partition types have been found. By these functional equations, in two main steps, we can find the generating function as an infinite sum. Finally, by some transformation such as Jacobi's triple products [2], we can find the divisibility part of identities.

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Theorem 1.2 (Jacobi's triple product) *For $z \neq 0$, $|q| < 1$,*

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \prod_{n=0}^{\infty}(1-q^{2n+2})(1+zq^{2n+1})(1+z^{-1}q^{2n+1}).$$

By the method and the transformation mentioned above we reached the following identity.

Theorem 1.3 *Let $R(n)$ denote the number of 2-colored Rogers-Ramanujan partitions of n, then for $|q| < 1$*

$$\sum_{n\geq 0} r(n)q^n = \frac{(-q)_\infty (q^2, q^2, q^4; q^4)_\infty}{(q)_\infty}.$$

We observed that our results are identical to identities for special cases of overpartitions in [3]. An overpartition of a positive integer n is a nonincreasing sequence of positive integers such that the summation of them is n and the first occurrence of any part can be overlined. The next theorem is Rogers-Ramanujan-Gordon theorem for overpartitions discovered and proved by Chen, Sang and Shi.

Theorem 1.4 *For $k \geq a \geq 1$, let $D_{k,a}(n)$ denote the number of overpartitions of n of the form $d_1+d_2+\cdots+d_s$, such that 1 can occur as a nonoverlined part at most $a-1$ times, and $d_{j+k-1}-d_j \geq 1$ if d_j is overlined and $d_{j+k-1}-d_j \geq 2$ otherwise. For $k > i \geq 1$, let $C_{k,i}(n)$ denote the number of overpartitions of n whose nonoverlined parts are not congruent to $0, \pm i$ modulo $2k$ and let $C_{k,k}(n)$ denote the number of overpartitions of n with parts not divisible by k. Then $C_{k,i}(n) = D_{k,i}(n)$.*

In the first section, we will go over the 2-colored Rogers-Ramanujan partition type, accordingly, we will find two functional equations, and then constructively, mentioned above, we will find a partition identity. In the third section, a correspondence between our identities and one for overpartitions is given. Finally, possible future works are introduced in the last section.

2. Colored Rogers-Ramanujan partitions

Throughout this paper, for $|q| < 1$,

$$(a;q)_n = \prod_{i=0} n-1(1-aq^i),$$

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n,$$

$$(a_1, a_2, \cdots, a_k; q)_n := (a_1;q)_n(a_2;q)_n \cdots (a_k;q)_n.$$

According to 2-colored Rogers-Ramanujan partitions, the following definition is given.

Definition 2.1 *For $1 \leq j \leq 2$, let $R_j(x)$ be the generating function of 2-colored Rogers-Ramanujan partitions with smallest part greater than or equal to j.*

With respect to these definitions, one can find the following functional equations relating $R_1(x)$ and $R_2(x)$.

$$R_1(x) - R_2(x) = xqR_1(xq) + xqR_2(xq) \tag{2.1}$$

and

$$R_2(x) = R_1(xq)\ . \tag{2.2}$$

Equation (2.2) is clear, as shifting every part of R_1 by 1 unit it will change to R_2. The proof of equation (2.1) is as follows.

Let

$$R_i(x) = \sum_{m\geq 0}\sum_{n\geq 0} r_i(m,n)x^m q^n \ \ ; \ \ i = 1,2$$

be the generating function for the types that have been mentioned above, where m is referring to the number of parts in partitions.

Let π be a 2-colored Rogers-Ramanujan partition of n with m parts, all possible ways such that the smallest part ≥ 1 will be counted by $r_1(m,n)$, if all partitions that the smallest part is ≥ 2 $(r_2(m,n))$ have been removed, then all remaining partitions are of 2-colored Rogers-Ramanujan type with the smallest part that is exactly 1 $(r_1(m,n) - r_2(m,n))$.

Now, let us count all partitions with the smallest part 1 in another way, if 1 is removed from all partitions, then there will be two cases:

(i) The smallest part is ≥ 2 with different color than 1, so one can subtract 1 from each part, the enumeration of these partitions is by $r_1(m-1,n-m)$.

(ii) The smallest part is ≥ 3 with the same color as 1, if 1 is subtracted from each part, the enumeration of these partitions is $r_2(m-1,m-n)$, note that part 2 is not possible here.

Therefore,

$$r_1(m,n) - r_2(m,n) = r_1(m-1,n-m) + r_2(m-1,n-m).$$

Multiplying all terms by $x^m q^n$ and taking the summation over m and n for all terms, $m,n \geq 0$ and both integers, we will have

$$\sum_{m\geq 0}\sum_{n\geq 0} r_1(m,n)x^m q^n - \sum_{m\geq 0}\sum_{n\geq 0} r_2(m,n)x^m q^n =$$
$$\sum_{m\geq 0}\sum_{n\geq 0} r_1(m-1,n-m)x^m q^n + \sum_{m\geq 0}\sum_{n\geq 0} r_2(m-1,n-m)x^m q^n.$$

By changing $m-1$ to m and $n-m$ to $n-m+1$ on the right hand side of this equation, we will have

$$\sum_{m\geq 0}\sum_{n\geq 0} r_1(m,n)x^m q^n - \sum_{m\geq 0}\sum_{n\geq 0} r_2(m,n)x^m q^n =$$
$$\sum_{m\geq 0}\sum_{n\geq 0} r_1(m,n)x^{m+1} q^{n+m+1} + \sum_{m\geq 0}\sum_{n\geq 0} r_2(m,n)x^{m+1} q^{n+m+1}\ .$$

Therefore, one will get the functional equation as follows:

$$R_1(x) - R_2(x) = xqR_1(xq) + xqR_2(xq)$$

.

We will follow the steps described in [4] and [5]. We assume that each of them has the form

$$R_i(x) = \sum_{n\geq 0} \alpha_n(x) q^{nA_i} + \beta_n(x) x^{B_i} q^{C_i} q^{nD_i} \quad ; \quad i = 1, 2 \tag{2.3}$$

with the initial condition that $R_i(0) = 1$ (for the empty partition of 0).

Our first goal is to find exponents of x and q in (2.3). For this end, we will construct the functional equations using Equation (2.1) in the following form

$$\sum_{n\geq 0} \alpha_n(x)(q^{nA_1} - q^{nA_2}) + \beta_n(x)(x^{B_1} q^{C_1} q^{nD_1} - x^{B_2} q^{C_2} q^{nD_2})$$

$$= \sum_{n\geq 0} \alpha_n(xq) xq(q^{nA_1} + q^{nA_2}) + \beta_n(xq) xq(x^{B_1} q^{B_1+C_1} q^{nD_1} + x^{B_2} q^{B_2+C_2} q^{nD_2})$$

in the last term, we substitute n by $n-1$, then we use another assumption that

$$\alpha_n(x)(q^{nA_1} - q^{nA_2}) = \beta_{n-1}(xq) xq(x^{B_1} q^{B_1+C_1} q^{(n-1)D_1} + x^{B_2} q^{B_2+C_2} q^{(n-1)D_2})$$

and

$$\beta_n(x)(x^{B_1} q^{C_1} q^{nD_1} - x^{B_2} q^{C_2} q^{nD_2}) = \alpha_n(xq) xq(q^{nA_1} + q^{nA_2}).$$

Now, we find α_n and β_n in terms of α_0.

$$\alpha_n(x) = \tilde{\alpha}_0(xq^{2n}) \frac{x^{2n} q^{n(2n+1)} q^{-nA_2} (-1; q^E)_n (-x^F q^G q^{F-H} q^{nH}; q^{2F-H})_\infty}{(q^E; q^E)_n (x^F q^G q^{F-H} q^{nH}; q^{2F-H})_\infty}$$

and

$$\beta_n(x) = -\tilde{\alpha}_0(xq^{2n+1}) \frac{x^{2n+1} q^{(n+1)(2n+1)} x^{-B_2} q^{-C_2} q^{-nD_2} (-1; q^E)_{n+1} (-x^F q^G q^{2F-H} q^{nH}; q^{2F-H})_\infty}{(q^E; q^E)_n (x^F q^G q^{nH}; q^{2F-H})_\infty}$$

, where

$$\tilde{\alpha}_0(xq^{2n}) = \alpha_0(xq^{2n}) \frac{((xq^{2n})^F q^G q^{F-H}; q^{2F-H})_\infty}{(-(xq^{2n})^F q^G q^{F-H}; q^{2F-H})_\infty},$$

$E = A_1 - A_2$, $F = B_1 - B_2$, $G = C_1 - C_2$ and $H = D_1 - D_2$.

Next, we use these α_n and β_n in Equation (2.2). We also need one more assumption to simplify the infinite products and make them into rational functions, then by crossmultiplying, we then obtain an identity between polynomials, and we can find E, F, G, and H. The assumption that we need here is $2F - H | F - H$; therefore, $F = \frac{1-t}{1-2t} H$ for some integer t, in this part we choose the smallest or the simplest solutions among infinitely many ones, if they do not work, we will choose the other ones. Note that after rearranging monomials, there should be the same number of positive monomials on each side. Considering $t = 0$, we have $F = H$, and identifying the exponents of x, q and q^n, we will have $F = G = H = 1$ and $E = -1$.

We wanted $\tilde{\alpha_0}$ to be constant with respect to x. Note that in (2.3), if we put $x=0$, in the right hand side, all terms will be eliminated except $\alpha_0(0)$ and in the left hand side, we have $R_i(0)$, the partitions of 0 which is 1 for the empty partition of 0, i.e. $\alpha_0(0)=1$, so $\tilde{\alpha}_0(xq^{2n})=1$.

Putting them in the generating functions for $R_i(x)$, then applying $x=1$, we will have

$$R_1(1)=\sum_{n\geq 0}\frac{(-1)^n q^{n(2n+1)}(-1;q)_n(-q^{n+1};q)_\infty}{(q)_n(q^{n+1};q)_\infty}-\frac{(-1)^n q^{(n+1)(2n+2)}(-1;q)_{n+1}(-q^{n+2};q)_\infty}{(q)_n(q^{n+1};q)_\infty}$$

and

$$R_2(1)=\sum_{n\geq 0}\frac{(-1)^n q^{n(2n+2)}(-1;q)_n(-q^{n+1};q)_\infty}{(q)_n(q^{n+1};q)_\infty}-\frac{(-1)^n q^{(n+1)(2n+1)}(-1;q)_{n+1}(-q^{n+2};q)_\infty}{(q)_n(q^{n+1};q)_\infty}.$$

They can be rewritten as follows:

$$R_1(1)=2\frac{(-q)_\infty}{(q)_\infty}\sum_{n\geq 0}(-1)^n q^{n(2n+1)}(\frac{1}{1+q^n}-\frac{q^{3n+2}}{1+q^{n+1}})$$

and

$$R_2(1)=2\frac{(-q)_\infty}{(q)_\infty}\sum_{n\geq 0}(-1)^n q^{n(2n+2)}(\frac{1}{1+q^n}-\frac{q^{n+1}}{1+q^{n+1}}).$$

Therefore,

$$\begin{aligned}R_1(1)&=2\frac{(-q)_\infty}{(q)_\infty}(\sum_{n\geq 0}\frac{(-1)^n q^{n(2n+1)}}{1+q^n}-\sum_{n\geq 0}\frac{(-1)^n q^{n(2n+1)}q^{3n+2}}{1+q^{n+1}})\\&=\frac{(-q)_\infty}{(q)_\infty}(1+2(\sum_{n\geq 1}\frac{(-1)^n q^{n(2n+1)}}{1+q^n}-\sum_{n\geq 1}\frac{(-1)^{n-1}q^{2n^2}}{1+q^n}))\\&=\frac{(-q)_\infty}{(q)_\infty}(1+2\sum_{n\geq 1}(-1)^n q^{2n^2})=\frac{(-q)_\infty}{(q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2}\ .\end{aligned}$$

By Theorem (1.2) for $z=-1$ and q^2, for 2-colored Rogers-Ramanujan defined in 2.1 the following identity holds

$$R_1(1)=\frac{(-q)_\infty(q^2,q^2,q^4;q^4)_\infty}{(q)_\infty}$$

.

Moreover, the coefficients in the Taylor series of $R_2(1)$ coincides with the number of partitions for 2-colored Rogers-Ramanujan partitions with parts more than 1.

$$1+2q^2+2q^3+2q^4+4q^5+6q^6+8q^7+10q^8+14q^9+18q^{10}+\cdots$$

We will come back to $R_2(n)$ later.

3. Correspondence with overpartitions

There is a one-to-one correspondence between 2-colored Rogers-Ramanujan type partitions, $R(n)$ and previously defined overpartitions $D_{2,2}(n)$.

Let $\pi = (y_1, \cdots, y_m)$ be an arbitrary 2-colored Rogers-Ramanujan type partition of n, and $\overline{\pi} = (z_1, \cdots, z_m)$ be an arbitrary overpartition of n, both into m parts. Note that for both π and *overline*π all parts are distinct.

If the difference between parts in π is more than 1, it is clear that each part can get any of two colors, in this case, replacing all parts of one color with the same overlined parts give us an overpartition $\overline{\pi}$ satisfying in the conditions of $D_{2,2}(n)$. It is the same from an overpartition to a 2-colored Rogers-Ramanujan type partition.

Now, consider the case that there is one set of t consecutive parts in the partition. For more than one set of consecutive parts, it will be the same. In a 2-colored Rogers-Ramanujan type partition π, there are only two possibilities for this t parts, they should be alternatively one of two colors, e.g., for three consecutive parts i, $i-1$ and $i-2$, two cases are

$$j, i, {\color{red}i-1}, i-2, k \;\text{ and }\; j, {\color{red}i}, i-1, {\color{red}i-2}, k$$

, where $j > i+1$ and $k < i-3$. For the overpartition case, with respect to the condition on parts for $D_{2,2}(n)$, the first $t-1$ parts should be overlined and there are two possibilities for the last one; therefore, the number of overpartition for the t parts is 2, same as 2-colored case, e.g., for three consecutive parts i, $i-1$, and $i-2$, two cases are

$$j, \bar{i}, \overline{i-1}, i-2, k \;\text{ and }\; j, \bar{i}, \overline{i-1}, \overline{i-2}, k$$

, where $j > i+1$ and $k < i-3$.

Therefore, enumeration of 2-colored Rogers-Ramanujan type partitions and overpartitions satisfying in $d_{2,2}(n)$ is the same for any integer n.

Moreover, by Equations (2.1) and (2.2)

$$R_1(x) - R_1(xq) = xqR_1(xq) + xqR_1(xq^2).$$

Hence, by setting $R_1(x) = \sum_{n\geq 0} r_n x^n$, the q-difference equation will lead us to the induction

$$r_{n+1} = \frac{q^{n+1}(1+q^n)}{(1-q^{n+1})} r_n$$

, so

$$r_n = r_0 \frac{q^{n(n+1)/2}(-1;q)_n}{(q;q)_n}.$$

With $r_0 = 1$, it is clear that $R_1(x)$ is the generating function of the overpartitions enumerated by $D_{2,2}(n)$.

It is not hard to see another correspondence between $D_{2,1}(n)$ and the following partition type.

Definition 3.1 *Let $R_3(n)$ denote the number of 2-colored Rogers-Ramanujan partitions which do not allow to have a red 1 in the partition.*

This gives us the following identity.

Theorem 3.2 *For Definition 3.1 and* $|q| < 1$*,*

$$\sum_{n\geq 0} r_3(n)q^n = \sum_{q\geq 0} d_{2,1}(n)q^n = \frac{(-q)_\infty (q^1, q^3, q^4; q^4)_\infty}{(q)_\infty}$$

Note that in the definition of $R_3(n)$, we can choose any of two colors. This lead us to

$$R_2(x) = 2R_3(x) - R_1(x).$$

Therefore, we have the following identity for $R_2(n)$.

Theorem 3.3 *For* $|q| < 1$*,*

$$\sum_{n\geq 0} r_2(n)q^n = 2\frac{(-q)_\infty (q^1, q^3, q^4; q^4)_\infty}{(q)_\infty} - \frac{(-q)_\infty (q^2, q^2, q^4; q^4)_\infty}{(q)_\infty}.$$

4. Future studies

There are two options for further study on this topic, first one is to extend the number of colors to more than two, ideally for arbitrary number of colors. Our second suggestion is to extend it to 2-colored Gordon-Rogers-Ramanujan type partitions, i.e. for 2-colored partition $y_1 + \cdots + y_m$, and for $k \geq 1$, $y_j - y_{j+k-1} \geq 1$ if y_j is black and $y_j - y_{j+k-1} \geq 2$ otherwise.

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