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Oscillation of third-order neutral differential equations with oscillatory operator

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Abstract: A third-order damped neutral sublinear differential equation for which its differential operator is oscillatory is studied. Sufficient conditions are given under which every solution is either oscillatory or the derivative of its neutral term is oscillatory (or it tends to zero).

Key words: Third order, neutral, delay differential equation, oscillation

1. Introduction

Consider the third-order nonlinear differential equation

$$z''' + q(t)z' + r(t)|x(\sigma(t)|^{\lambda} \operatorname{sgn} x(\sigma(t)) = 0\,, \quad t \geq 0 \tag{1.1}$$

with

$$z(t) = x(t) + a(t)x\big(\tau(t)\big)\,, \tag{1.2}$$

where $\lambda \in (0,1]$, $q \in C(\mathbb{R}_+)$, $r \in C(\mathbb{R}_+)$, $a \in C(\mathbb{R}_+)$, $\sigma \in C(\mathbb{R})$, $\tau \in C(\mathbb{R})$, $\mathbb{R}_+ = [0,\infty)$, $\mathbb{R} = (\infty,\infty)$, $q(t) \geq 0$, $r(t) > 0$, $a(t) \geq 0$ for $t \in \mathbb{R}_+$ and $\lim_{t\to\infty} \sigma(t) = \lim_{t\to\infty} \tau(t) = \infty$.

Throughout the paper the following hypotheses are assumed.

(H_1) $\sigma \in C^1(\mathbb{R})$, $\tau \in C^1(\mathbb{R})$, $\sigma(t) \leq \tau(t) \leq t$ for $t \in \mathbb{R}$ and constants σ_1 and τ_0 exist such that $0 < \sigma'(t) \leq \sigma_1$ and $0 < \tau_0 \leq \tau'(t)$ for $t \in \mathbb{R}$;

(H_2) the associated linear equation $h'' + q(t)h = 0$, $t \geq 0$ is oscillatory.

Sometimes, the following hypothesis is assumed:

(H_3) numbers a_0 and a_1 exist such that $0 < a_0 \leq a(t) \leq a_1$ holds for $t \in \mathbb{R}_+$.

Definition 1.1 *Let $T \in \mathbb{R}_+$ and $T_0 = \sigma(T)$. Then a function x is said to be a solution of (1.1) on $[T,\infty)$ if x is defined and continuous on $[T_0,\infty)$, $z \in C^3[T,\infty)$ and (1.1) is satisfied on $[T,\infty)$, where z is given by (1.2). A solution x is said to be nonoscillatory if $x(t) \neq 0$ for large t; otherwise it is said to be oscillatory.*

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A continuous function $v \in C[T, \infty)$ is said to be oscillatory if there exists an increasing sequence of its zeros tending to the infinity.

We need to define some types of (1.1) which are studied below.

Definition 1.2 *Equation (1.1) is said to have Property A if every solution is either oscillatory or $z(t)z'(t) < 0$ for large t and $\lim_{t\to\infty} z(t) = 0$. It is said to have Property A^0 if every solution x is either oscillatory or $\lim_{t\to\infty} z(t) = 0$ or z' oscillates. It is said to have Property A^* if every solution x is either oscillatory or z' oscillates.*

In all the paper, if we study a solution x of (1.1), z is given by (1.2) without mentioned it.

The oscillation and the asymptotic behavior of solutions of third order differential equations are still of an intensive investigation. Some equations have applications in the mathematical modelling in biology and physics, see e.g., [6], the study of entry-flow phenomenon [10], the regulation of a stream turbine [16], the propagation of electrical pulses in the nerve of a squid [14] and the feedback nuclear reactor problem [18].

Most of oscillation results, connected with (1.1) and its special cases, are written on the equations taken the form either

$$x'''(t) + q(t)x'(t) + r(t)\big|x\big(\sigma(t)\big)\big|^{\lambda} \operatorname{sgn} x\big(\sigma(t)\big) = 0 \tag{1.3}$$

or, more generally,

$$\big(r_2(t)(r_1(t)x'(t))'\big)' + Q(t)x'(t) + R(t)\big|x\big(\sigma(t)\big)\big|^{\lambda} x\big(\sigma(t)\big) = 0 \tag{1.4}$$

with $r_i \in C(\mathbb{R}_+)$, $Q \in C(\mathbb{R}_+)$, $R \in C(\mathbb{R}_+)$, $r_i(t) > 0$, $Q(t) \geq 0$, $R(t) > 0$ for $t \in \mathbb{R}_+$ and $i = 1, 2$. The most results are devoted to equations for which

$$h'' + q(t)h = 0 \quad \big((r_2(t)(r_1(t)h)')' + Q(t)h = 0\big) \tag{1.5}$$

is nonoscillatory in case of (1.3) (of (1.4)). Mainly, sufficient conditions have been studied under which equation (1.3) ((1.4)) has Property A, see e.g., [1], [2], [5], [6] and the references therein.

In the recent years, the above given results are generalized for the neutral equation of the form either (1.1) (see e.g., [3]) or

$$\big(r_2(t)(r_1(t)z')'\big)' + R(t)\big|x(\sigma(t))\big|^{\lambda} \operatorname{sgn} x\big(\sigma(t)\big) = 0\,, \tag{1.6}$$

see e.g., [7], [8], [13], [15], [17] and the references therein.

Notice that the investigation of (1.1) or (1.3) or (1.4) (if (1.5) holds) consists in the very known equivalent transformation into the two-term equation with quasiderivatives (1.6) or (1.4) (with $Q \equiv 0$) or (1.4) (with $Q \equiv 0$), respectively. Moreover, any nonoscillatory solution x of (1.6) satisfies

$$z(t)z'(t) \neq 0 \quad \text{for large } t. \tag{1.7}$$

If equation $h'' + q(t)h = 0$ is oscillatory, a transformation of (1.1) into (1.6) does not exist in a neighborhood of ∞ and (1.7) does not hold for (1.1) as a nonoscillatory solution x, satisfying $z(t) \neq 0$ for large t and z' oscillates, may exist, see e.g., [11] for equation (1.3). This is the reason why Property A^0 is studied instead of Property A, see e.g., [1] for equation (1.3) or [2] for (1.4).

If r is large with respect to q, then, under some hypotheses on λ, σ and τ, solutions satisfying (1.7) of the equation

$$z^{(4)} + q(t)z'' + r(t)\big|x(\sigma(t))\big|^{\lambda} x\big(\sigma(t)\big) = 0 \tag{1.8}$$

do not exist (z is given by (1.2)), see also [4]. Moreover, in [9] sufficient conditions are given for every nontrivial solution x of (1.8) to have z'' oscillating.

To the best of our knowledge, nothing is known regarding Property A* for (1.1). Hence, our goal is to study either Property A^0 or Property A* for (1.1) if (H_2) holds.

In all the paper, σ^{-1} (τ^{-1}) denotes the inverse function to σ (τ). For a sake of brevity, we define

$$h(t) = \sigma^{-1}(\tau(t)), \quad h_1(t) = \tau^{-1}\big(\sigma(t)\big)\,,$$

$$J(t) = [h_1(t), \sigma^{-1}(t)]\,, r^*(t) = \min\big(r(\sigma^{-1}(t)), r(h(t))\big)\,,$$

$$\bar{r}(t) = \max_{s \in J(t)} r(s),\ \bar{q}(t) = \max_{s \in J(t)} q(s)\,,$$

$$\Delta(t) = \sigma^{-1}(t) - h_1\big(\tau(t)\big)\,,\ t \in \mathbb{R}_+\,.$$

Notice, that according to (H_1), h and h_1 are increasing.

2. Preliminaries

To prove our main results, we first present some lemmas to be used in the process.

Lemma 2.1 ([12] Lemma 5.2) *Let* $[a,b] \subset \mathbb{R}$, $m \geq 2$ *be integer,* $u \in C^m[a,b]$ *and* $\varrho_i = \max\limits_{a \leq t \leq b} |u^{(i)}(t)|$, $i = 0, 1, \dots, m$. *Then*

$$\varrho_i \leq 2^{-1} m! m^m (b-a)^{-i} \varrho_0 + 2^{(i-1/m)(m-i)} (m!)^{(m-i)/m} m^{m-i} \times \varrho_0^{(m-i)/m} \varrho_m^{i/m}\,, \quad i = 0, 1, \dots, m\,.$$

Lemma 2.2 *Let* $[a,b] \subset \mathbb{R}$, $u \in C[a,b]$, $(-1)^i u^{(i)}(t) \geq 0$ *for* $t \in [a,b]$, $i = 0,1,2$ *and let* $u''(b) = \min\limits_{a \leq s \leq b} u''(s)$. *Then* $u(a) \geq \frac{1}{2}(b-a)^2 u''(b)$.

Proof As

$$-u'(s) \geq -u'(s) + u'(b) = \int_s^b u''(t)\,dt \geq u''(b)(b-s)$$

for $a \leq s \leq b$, the integration implies

$$u(a) \geq u(a) - u(b) = -\int_a^b u'(s)\,ds \geq \int_a^b u''(b)(b-a)\,ds = \frac{1}{2}(b-a)^2 u''(b)\,.$$

□

Lemma 2.3 *Let $\lambda = 1$, $0 \le a < b < \infty$, there exist a_0 such that $0 < a_0 \le a(t)$ for $t \in [h_1(a), h_1(b)]$ and let x be a solution of (1.1), defined on $[h_1(a), \infty)$, satisfying*

$$x\big(\sigma(t)\big) > 0 \quad \text{for } t \in [a, b], \;\; z'(t) < 0 \quad \text{for } t \in [h_1(a), b]\,. \tag{2.1}$$

Then

$$\big|z''(t)\big| \le K\ z\big(h_1(a)\big) \quad \text{for} \quad t \in [a, b]\,, \tag{2.2}$$

where $q_0 = \max\limits_{a \le t \le b} q(t)$, $r_0 = \max\limits_{a \le t \le b} r(t)$ and

$$K = \max\Big\{\frac{162}{(b-a)^2} + 72\Big[\Big(\frac{4q_0}{b-a}\Big)^{2/3} + \Big(\frac{r_0}{a_0}\Big)^{2/3}\Big], 3456\sqrt{2}q_0\Big\}\,. \tag{2.3}$$

Proof Let x be a solution of (1.1) (with $\lambda = 1$) satisfying (2.1). Put $c_0 = \frac{4}{b-a}$, $c_1 = 4$, $c_2 = \frac{81}{(b-a)^2}$, $c_3 = 4 \cdot 3^{4/3}$, $\varrho_i = \max\limits_{a \le t \le b} \big|z^{(i)}(t)\big|$, $i = 0, 1, 2, 3$ and $\varrho_* = z\big(h_1(a)\big)$. Hence, using (2.1), $\varrho_0 \le \varrho_*$. Applying Lemma 2.1 with $u = z$, $m = 2$, we get

$$\varrho_1 \le c_0 \varrho_0 + c_1 (\varrho_0 \varrho_2)^{1/2}\,. \tag{2.4}$$

Similarly, Lemma 2.1 with $u = z$, $m = 3$ implies

$$\varrho_2 \le c_2 \varrho_0 + c_3 \varrho_0^{1/3} \varrho_3^{2/3}\,. \tag{2.5}$$

Now we estimate ϱ_3 using (1.2) and (2.1):

$$x\big(\sigma(t)\big) \le \big(a(h_1(t))\big)^{-1} z\big(h_1(t)\big) \le \frac{1}{a_0} z\big(h_1(a)\big) = \frac{\varrho_*}{a_0}$$

for $t \in [a, b]$. From this and from (1.1)

$$\varrho_3 \le q_0 \varrho_1 + r_0 \max_{a \le t \le b} x\big(\sigma(t)\big) \le q_0 \varrho_1 + \frac{r_0}{a_0} \varrho_*\,. \tag{2.6}$$

Furthermore, (2.4), (2.5) and (2.6) imply

$$\begin{aligned}
\varrho_2 &\le c_2 \varrho_0 + c_3 \varrho_0^{1/3} \big[c_0 q_0 \varrho_0 + c_1 q_0 \varrho_0^{1/2} \varrho_2^{1/2} + \frac{r_0}{a_0} \varrho_*\big]^{2/3} \\
&\le c_2 \varrho_* + 3^{2/3} c_3 \varrho_*^{1/3} \Big[(c_0 q_0 \varrho_0)^{2/3} + (c_1 q_0^{2/3}) \varrho_*^{1/3} \varrho_2^{1/3} + \Big(\frac{r_0}{a_0}\Big)^{2/3} \varrho_*^{2/3}\Big] \\
&\le \varrho_*^{2/3} \big[c_4 \varrho_1^{1/3} + c_5 \varrho_2^{1/3}\big]
\end{aligned} \tag{2.7}$$

with $c_4 = c_2 + 3^{2/3} c_3 \big[(c_0 q_0)^{2/3} + \big(\frac{r_0}{a_0}\big)^{2/3}\big]$, $c_5 = 3^{2/3} c_3 (c_1 q_0)^{2/3}$. If $c_4 \varrho_*^{1/3} \ge c_5 \varrho_2^{1/3}$, then (2.7) implies $\varrho_2 \le 2c_4 \varrho_*$. If $c_4 \varrho_*^{1/3} < c_5 \varrho_2^{1/3}$, then (2.7) implies $\varrho_2 \le 2c_5 \varrho_*^{2/3} \varrho_2^{1/3}$; hence $\varrho_2 \le (2c_5)^{3/2} \varrho_*$. Thus, in both cases

$$\varrho_2 \le \max\big(2c_4, (2c_5)^{3/2}\big) \varrho_* = K\ z\big(h_1(a)\big)$$

and (2.2) holds. □

Lemma 2.4 *Suppose* (H_3) *with* $a_0 > 1$ *and* x *is a solution of* (1.1) *such that*

$$x\big(\sigma(t)\big) > 0\,,\ z'(t) < 0 \quad on \quad [T,\infty) \subset \mathbb{R}_+\,. \tag{2.8}$$

Then

$$x\big(\sigma(t)\big) \ge \frac{a_0 - 1}{a_0 a_1} z\big(h_1(t)\big) \quad for \quad t \ge h(T)\,. \tag{2.9}$$

Proof It follows from [5, Lemma 1] and $a(t) \le a_1$. □

Lemma 2.5 *Suppose* $\lambda = 1$, $q \in C^1(\mathbb{R}_+)$, (H_3) *with* $a_0 > 1$, x *is a solution of* (1.1) *satisfying* (2.8) *and*

$$\frac{a_0 a_1}{2(a_0 - 1)} q'(t) \le r(t) \tag{2.10}$$

for $t \ge T$. *Then*

$$F(t) := -2z''(t)z(t) - q(t)z^2(t) + \big(z'(t)\big)^2 \le 0 \tag{2.11}$$

for $t \ge T$ *and*

$$\int_T^\infty \Big[2\frac{a_0 - 1}{a_0 a_1} r(t) - q'(t)\Big] z^2\big(h_1(t)\big)\, dt < \infty\,. \tag{2.12}$$

Proof Let x be a solution of (1.1) with (2.8). Then

$$F'(t) = 2r(t)z(t)x\big(\sigma(t)\big) - q'(t)z^2(t)\,, \quad t \ge T\,. \tag{2.13}$$

According to (2.10), (2.13) and Lemma 2.4

$$\begin{aligned} F'(t) &\ge 2\frac{a_0 - 1}{a_0 a_1} r(t)z(t)z\big(h_1(t)\big) - q'(t)z^2(t) \\ &\ge \Big[2\frac{a_0 - 1}{a_0 a_1} r(t) - q(t)\Big] z^2\big(h_1(t)\big) \ge 0 \end{aligned} \tag{2.14}$$

for $t \ge h(T)$. Case $z''(t) \le 0$ for large t is impossible as, otherwise, z becomes negative due to (2.8). Hence, either

1° $z''(t) \ge 0$ for large t, or

2° z'' changes its sign infinitely many times.

Let us choose an increasing sequence $\{t_k\}_{k=1}^\infty$ such that $t_1 \ge h(T)$, $\lim\limits_{k\to\infty} t_k = \infty$ and t_k is arbitrary in case 1° ($z''(t_k) = 0$ and z' has local maxima at t_k, $k = 1, 2, \ldots$ in case 2°). It is easy to see that $\lim\limits_{k\to\infty} z'(t_k) = 0$ as, otherwise, z becomes negative for large t and that contradicts (2.8). Hence, $\lim\limits_{k\to\infty} F(t_k) \le 0$ and (2.11) follows from this and from (2.14). Finally, (2.11) and (2.14) yield (2.12). □

3. Property A^0

The following theorems show the possible types of nonoscillatory solutions of (1.1).

Theorem 3.1 *Any nonoscillatory solution* x *of* (1.1) *either satisfies*

$$z(t)z'(t) < 0 \quad \textit{for large } t \tag{3.1}$$

or z' *oscillates.*

Proof It is similar as the one of Theorem 2.1 in [1] given for (1.3). □

Remark 3.2 *Notice that Theorem 3.1 is valid without hypotheses* (H_1) *and* $q \geq 0$.

Theorem 3.3 *Suppose* $q \in C^1(\mathbb{R}_+)$, (H_3) *holds with* $a_0 > 1$ *and*

$$\int_0^\infty \Big[2\frac{a_0-1}{a_0 a_1} r(t) - q'(t)\Big]\, dt = \infty \qquad \textit{in case } \lambda = 1, \tag{3.2}$$

and for any $K > 0$

$$\int_0^\infty \Big[K\, r(t) - q'(t)\Big]\, dt = \infty \quad \textit{in case } \lambda < 1. \tag{3.3}$$

Equation (1.1) *has Property* A^0.

Proof Let x be a nonoscillatory solution of (1.1) which is positive for large t (case $x < 0$ can be studied similarly). Suppose that (1.1) has not Property A^0. Then Theorem 3.1 implies the existence of $T_0 \geq 0$ and $c > 0$ such that $x\big(\sigma(t)\big) > 0$, $z(t) > 0$, $z'(t) < 0$ for $t \geq T_0$ and

$$\lim_{t\to\infty} z(t) = c. \tag{3.4}$$

At first, suppose $\lambda = 1$. There exists $T \geq T_0$ such that (3.2) and $z(t) \leq 2c$ hold for $t \geq T$. From this, (2.10) is valid, and Lemma 2.5, (2.12) and (3.4) imply

$$\int_T^\infty \Big[2\frac{a_0-1}{a_0 a_1} r(t) - q'(t)\Big]\, dt < \infty.$$

This contradicts (3.2).
Now, let $0 < \lambda < 1$. Then x is the solution of the equation

$$z''' + q(t)z' + R(t)x\big(\sigma(t)\big) = 0 \tag{3.5}$$

on $[T_0, \infty)$ with $R(t) = x^{\lambda-1}(\sigma(t))r(t)$. According to (3.4), $T \geq T_0$ exists such that $z(t) \leq 2c$ for $t \geq T$. From Lemma 2.4 and (2.9)

$$R(t) \geq \Big(\frac{a_0-1}{a_0 a_1}\Big)^{\lambda-1} z^{\lambda-1}\big(h_1(t)\big) r(t) \geq \Big(2c\frac{a_0-1}{a_0 a_1}\Big)^{\lambda-1} r(t)$$

for $t \geq T_1 = h(t) \geq T$. Hence, using (3.3),

$$\int_T^\infty \Big[2\frac{a_0-1}{a_0 a_1} R(t) - q'(t)\Big]\, dt \geq \int_T^\infty \Big[2^\lambda \Big(\frac{a_0-1}{a_0 a_1}\Big)^\lambda c^\lambda r(t) - q'(t)\Big]\, dt = \infty.$$

Thus, (3.2) is valid for (3.5) and according to proved part of the theorem for $\lambda = 1$, $\lim_{t\to\infty} z(t) = 0$. This contradicts (3.4). □

Theorem 3.4 *Suppose $q \in C^1(\mathbb{R}_+)$, $q'(t) \le 0$ for large t and $\int\limits_0^\infty r(t)dt = \infty$. Then (1.1) has Property A^0 and for any nonoscillatory solution x of (1.1)*

$$\liminf_{t\to\infty} x(t) = 0\,. \tag{3.6}$$

Proof Let x be a nonoscillatory solution of (1.1) such that $x(\sigma(t)) > 0$ and $q'(t) \le 0$ for $t \ge T \ge 0$ (case $x(\sigma(t)) < 0$ can be studied similarly). With respect to Theorem 3.1 it is necessary to prove (3.6). Suppose, contrarily, that $c > 0$ exists such that $x(\sigma(t)) \ge c$ for $t \ge T$. If F is given by (2.11), then

$$F'(t) = 2r(t)z(t)x^\lambda(\sigma(t)) - q'(t)z^2(t) \ge 2c^{\lambda+1}r(t) > 0$$

for $t \ge T$; hence,

$$\lim_{t\to\infty} F(t) = F(t) + 2c^{\lambda+1}\int_T^\infty r(t)\,dt = \infty\,. \tag{3.7}$$

Suppose (3.1) is valid. Then z is bounded and $\limsup\limits_{t\to\infty} z'(t) = 0$ (otherwise z becomes negative). Case $z'' < 0$ for large t is impossible as, otherwise, z becomes negative. Let $\{t_k\}_{k=1}^\infty$ be an increasing sequence such that $t_1 \ge T$, $\lim\limits_{k\to\infty} t_k = \infty$ and $z''(t_k) = 0$ in case z'' oscillates (t_k is arbitrary if $z'' > 0$ for large t). From this, $\{F(t_k)\}_{k=1}^\infty$ is bounded which contradicts (3.7).

Suppose z' oscillates. Then an increasing sequence $\{t_k\}_{k=1}^\infty$ exists such that $t_1 \ge T$, $\lim\limits_{k\to\infty} t_k = \infty$ and $z''(t_k) \ge 0$, $k = 1, 2, \ldots$. Then (2.11) implies $\{F(t_k)\}_{k=1}^\infty$ is bounded which contradicts (3.7). □

Remark 3.5 *(i) Notice that Theorem 3.4 is valid without the hypothesis (H_1).*

(ii) In [1, Theorem 4.1 and Corollary 4.3] equation (1.3) is investigated under the hypotheses $q \in C^1(\mathbb{R}_+)$, $q(t) \ge q_0 > 0$, $q'(t) \le 0$ for large t. It is proved that $\int\limits_0^\infty r(t)\,dt = \infty$ is necessary and sufficient condition for (3.6). Hence, Theorem 3.4 generalizes this result even for (1.3) and hypothesis $\int\limits_0^\infty r(t)\,dt = \infty$ cannot be weaken.

4. Property A*

In this paragraph, Property A* is studied for (1.1). Notice that nothing is known even for its special case (1.3). At first, two results are formulated for the linear equation.

Theorem 4.1 *Suppose $\lambda = 1$, (H_3), $\tau^{-1}(\sigma(t)) < \tau(t) < t$,*

$$\Big(M\int_{\tau(t)}^t r^*(s)\,ds - \bar q(t)\Big)\big(h(\tau(t)) - t\big)^2 > 1 \tag{4.1}$$

and

$$M\int_{\tau(t)}^t r^*(s)\,ds - 72\Big(\frac{\bar r(t)}{a_0}\Big)^{2/3} > H(t) + \bar q(t) \tag{4.2}$$

for large t, *where* $M = \min\left(\frac{1}{\sigma_1}, \frac{\tau_0}{a_1\sigma_1}\right)$ *and*

$$H(t) = \max\left\{\frac{162}{\Delta^2(t)} + 72\Big(\frac{4}{\Delta(t)}\bar{q}(t)\Big)^{2/3}, 3456\sqrt{2}\bar{q}(t)\right\}. \tag{4.3}$$

Then (1.1) *has Property* A^*.

Proof According to Theorem 3.1, it is sufficient to prove that a solution x of (1.1), satisfying (3.1), does not exist. Hence, suppose, contrarily, that there is a solution x of (1.1) such that the hypothesies of Theorem 4.1 hold for $t \ge T$ and

$$x\big(\sigma(t)\big) > 0\,,\ z(t) > 0\,,\ z'(t) < 0 \quad \text{for } t \ge T \ge 0 \tag{4.4}$$

(case $x(\sigma(t)) < 0$ can be studied similarly). Put $v(t) = z''(t)$ and notice that $' = \frac{d}{dt}$. Then (1.1) implies

$$\begin{aligned}
&\big(v(h(t))\big)' + q\big(h(t)\big)\big(z(h(t))\big)' + \frac{\tau_0}{\sigma_1 a_1} r\big(h(t)\big)a(t)x\big(\tau(t)\big)\\
&\le h'(t)\Big[\frac{(v(h(t)))'}{h'(t)} + q\big(h(t)\big)\frac{(z(h(ut)))'}{h'(t)}\\
&+ r\big(h(t)\big)x\big(\tau(t)\big)\Big] = 0
\end{aligned} \tag{4.5}$$

for $t \ge T$. Similarly,

$$\big(v(\sigma^{-1}(t))\big)' + q\big(\sigma^{-1}(t)\big)\big(z(\sigma^{-1}(t)\big)' + \frac{1}{\sigma_1} r\big(\sigma^{-1}(t)\big)x(t) \le 0 \tag{4.6}$$

for $t \ge T$. Now, (4.5) and (4.6) imply

$$\begin{aligned}
&-\big(v(\sigma^{-1}(t) + v(h(t))\big)' \ge q\big(\sigma^{-1}(t)\big)\big(z(\sigma^{-1}(t))\big)\\
&+ q(h(t))\big(z(h(t))\big)' + Mr^*(t)z(t)
\end{aligned} \tag{4.7}$$

for $t \ge T$. Notice that (H_1) yields

$$t < h\big(\sigma(t)\big) < h(t) < \sigma^{-1}(t)\,, \quad t \ge T\,. \tag{4.8}$$

The integration of (4.7) from $\tau(t)$ to t, (4.4) and (4.8) imply

$$\begin{aligned}
&-\big(v(\sigma^{-1}(t) + v\big(h(\tau(t))\big) \ge \bar{q}(t)\big(z(\sigma^{-1}(t)) - z\big(h(t)\big)\big)'\\
&+ \bar{q}(t)\big(z(h(t)) - z(h(\tau(t)))\big) + M\int_{\tau(t)}^{t} r^*(s)z(s)\,ds\\
&\ge z(t)\Big[M\int_{\tau(t)}^{t} r^*(s)\,ds - \bar{q}(t)\Big]
\end{aligned} \tag{4.9}$$

for $t \ge T$. To estimate $v(\sigma^{-1}(t))$, we apply Lemma 2.3 with $a = h(t)$ and $b = \sigma^{-1}(t)$. From this, (2.3) and from (4.3)

$$\big|v(\sigma^{-1}(t))\big| \le H_1(t)z(t) \tag{4.10}$$

for $t \geq T$, where $H_1(t) = H(t) + 72\big(\frac{\bar{r}(t)}{a_0}\big)^{2/3}$. To estimate $v(h(\tau(t)))$, two possibilities are considered:

1° There exists an increasing sequence $\{t_k\}_{k=1}^{\infty}$ such that $T \leq t_1$, $\lim\limits_{k\to\infty} t_k = \infty$ and

$$v\big(h(\tau(t_k))\big) \leq 0 \quad k = 1, 2, \dots \tag{4.11}$$

2° $\qquad v(t) > 0 \quad$ for $\quad t \geq T^* \geq T$.

Case 1°**.** Due to (4.9), (4.10) and (4.11)

$$H_1(t_k)z(t_k) \geq \Big[M \int_{\tau(t_k)}^{t_k} r^*(s)\,ds - \bar{q}(t_k)\Big] z(t_k)\,,$$

$k = 1, 2, \dots$ that contradicts (4.2) for $k \to \infty$.

Case 2°**.** We have $\liminf\limits_{t\to\infty} v(t) = 0$ as, otherwise, z' becomes positive; it contradicts (4.4). Hence, an increasing sequence $\{t_k\}_{k=1}^{\infty}$ exists such that $t_1 \geq T^*$, $\lim\limits_{k\to\infty} t_k = \infty$ and

$$v\big(h(\tau(t_k))\big) = \min_{t_k \leq s \leq h(\tau(t_k))} v(s)\,, \quad k = 1, 2, \dots$$

Hence, Lemma 2.2 (with $a = t_k$, $b = h(\tau(t_k))$, $u = z$) implies

$$z(t_k) \geq \frac{1}{2}\big(h(\tau(t_k)) - t_k\big)^2 v\big(h(\tau(t_k))\big)\,.$$

From this and from (4.9)

$$2\big(h(\tau(t_k)) - t_k\big)^{-2} z(t_k) \geq z(t_k)\Big[M \int_{\tau(t_k)}^{t_k} r^*(s)\,ds - \bar{q}(t_k)\Big]$$

that contradicts (4.2) for large k. □

Theorem 4.2 *Suppose $\lambda = 1$, $q \in C^1(\mathbb{R}_+)$, (H_3) with $a_0 > 1$, $\eta \in C(\mathbb{R}_+)$ exists such that*

$$h_1(t) < \eta(t) < t\,, \tag{4.12}$$

$$\Big(\frac{a_0 - 1}{a_0 a_1} \int_{\eta(t)}^{t} r(s)\,ds - q^*(t)\Big)\big(\eta(t) - h_1(t)\big)^2 > 2 \tag{4.13}$$

and

$$r(t) \geq \frac{a_0 a_1}{a_0 - 1} \max\Big(q^{3/2}(t), \frac{q'(t)}{2}\Big) \tag{4.14}$$

for large t, where $q^*(t) = \max\limits_{\eta(t) \leq s \leq t} q(s)$. Then (1.1) has Property A^*.

Proof Due to Theorem 3.1, it is sufficient to prove the nonexistence of a solution x of (1.1) satisfying (3.1). Hence, let, contrarily, there exist $T \geq 0$ and a solution x of (1.1) such that (4.12), (4.13), (4.14) and

$$x\big(\sigma(t)\big) > 0\,,\ z(t) > 0\,,\ z'(t) < 0 \tag{4.15}$$

hold for $t \geq T$. Put $c = \frac{a_0-1}{a_0 a_1} > 0$. There are three possibilities:

(i) $z'' \leq 0$ for large t;

(ii) $z'' \geq 0$ for large t;

(iii) z'' changes its sign infinitely many times.

Case (i). It is impossible as, otherwise, z becomes negative for large t, see (4.15).

Case (ii). Equation (1.1), (4.15) and Lemma 2.4 imply

$$\begin{aligned} z''' + q(t)z'(t) + cr(t)z\big(h_1(t)\big) &\leq z''' + q(t)z'(t) \\ &+ r(t)x\big(\sigma(t)\big) = 0 \end{aligned}$$

for $t \geq T_2 = h(T_1)$. From this, from (4.15) and by the integration from $\eta(t)$ to t, $t \geq T_2$

$$\begin{aligned} -z''\big(\eta(t)\big) + q^*(t)\big(z(t) - z(\eta(t))\big) + cz\big(h_1(t)\big)\int_{\eta(t)}^{t} r(s)\,ds \\ \leq z''(t) - z''\big(\eta(t)\big) + \int_{\eta(t)}^{t} q(s)z'(s)\,ds + c\int_{\eta(t)}^{t} r(s)z\big(h_1(s)\big)\,ds \leq 0\,; \end{aligned}$$

hence,

$$\begin{aligned} z''\big(\eta(t)\big) &\geq -q^*(t)z\big(\eta(t)\big) + cz\big(h_1(t)\big)\int_{\eta(t)}^{t} r(s)\,ds \\ &\geq \left[c\int_{\eta(t)}^{t} r(s)\,ds - q^*(t)\right] z\big(h_1(t)\big)\,. \end{aligned} \tag{4.16}$$

As $\liminf\limits_{t\to\infty} z''(t) = 0$ (otherwise z' becomes positive for large t, see (4.15)), an increasing sequence $\{t_k\}_{k=1}^{\infty}$ exists such that $t_1 \geq T_2$, $\lim\limits_{k\to\infty} t_k = \infty$ and $z''(t_k) = \min\limits_{h_1(t_k)\leq s\leq t_k} z''(s)$, $k = 1, 2, \ldots$. From this and from Lemma 2.2 (with $a = h_1(t_k)$, $b = \eta(t_k)$) we get

$$z\big(h(t_k)\big) \geq \frac{1}{2}\big(\eta(t_k) - h_1(t_k)\big)^2 z''\big(\eta(t_k)\big)\,, \quad k = 1, 2, \ldots$$

The application to (4.16) for $t = t_k$ implies

$$2 \geq \left[c\int_{\eta(t_k)}^{t} r(s)\,ds - q^*(t_k)\right]\big(\eta(t_k) - h_1(t_k)\big)^2\,, \quad k = 1, 2, \ldots$$

which contradicts (4.13) for large k.

Case (iii). If $F_1(t) = -z''(t)z'(t)$, then

$$F_1'(t) = q(t)\big(z'(t)\big)^2 + r(t)z'(t)x\big(\sigma(t)\big) - \big(z''(t)\big)^2, \quad t \geq T.$$

As $z'(t) < 0$, F_1 changes its sign infinitely many times and an increasing sequence $\{t_k\}_{k=1}^{\infty}$ exists such that $t_1 \geq T$, $\lim\limits_{k\to\infty} t_k = \infty$, $z''(t_k) = 0$ and $F_1(t) < 0$ in a left neighborhood of t_k, $k = 1, 2, \ldots$. From this and from $F_1(t_k) = 0$

$$0 \leq F_1'(t_k) = q(t_k)\big(z'(t_k)\big)^2 + r(t_k)z'(t_k)x\big(\sigma(t_k)\big), \tag{4.17}$$

$k = 1, 2, \ldots$. Furthermore, (4.17) and Lemma 2.4 yield

$$\big|z'(t_k)\big| \geq \frac{r(t_k)}{q(t_k)}x\big(\sigma(t_k)\big) \geq c\frac{r(t_k)}{q(t_k)}z\big(h_1(t_k)\big) > c\frac{r(t_k)}{q(t_k)}z(t_k), \tag{4.18}$$

$t_k \geq h(T)$. On the other side, Lemma 2.5 implies

$$q(t_k)z^2(t_k) + \big(z'(t_k)\big)^2 \leq 0, \quad k = 1, 2, \ldots$$

Hence, from this and using (4.18)

$$c\frac{r(t_k)}{q(t_k)}z(t_k) < q^{1/2}(t_k)z(t_k)$$

that contradicts (4.14) for large k. □

The next theorem is devoted to the sublinear case.

Theorem 4.3 *Suppose that $\lambda < 1$, $\varepsilon \in (0,2)$, $\varepsilon_1 > 0$, $q \in C^1(\mathbb{R}_+)$, (H_3) with $a_0 > 1$, $\eta \in C^0(\mathbb{R}_+)$ are such that for large t*

$$h_1(t) < \eta(t) < t, \tag{4.19}$$

$$\Big(\frac{a_0 - 1}{a_0 a_1}\int_{\eta(t)}^{t} r(s)\,ds - q^*(t)\Big)\big(\eta(t) - h_1(t)\big)^2 > \varepsilon, \tag{4.20}$$

$$r(t) \geq \varepsilon_1 q^{3/2}(t) \tag{4.21}$$

and for any $K > 0$

$$\int_0^{\infty}\big[Kr(t) - q'(t)\big]\,dt = \infty, \tag{4.22}$$

where $q^*(t) = \max\limits_{\eta(t)\leq s\leq t} q(s)$. Then (1.1) has Property A^*.

Proof Similarly to the proof of Theorem 4.2, suppose, contrarily, that there exist $T \geq 0$ and a solution x (1.1) such that (4.19), (4.20), (4.21) and

$$x\big(\sigma(t)\big) > 0, \; z(t) > 0, \; z'(t) < 0 \tag{4.23}$$

hold for $t \ge T$. Then Theorem 3.3 and (4.22) imply

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = 0\,. \tag{4.24}$$

Furthermore, x is the solution of the equation

$$z^{'''} + q(t)z' + R(t)x\big(\sigma(t)\big) = 0 \tag{4.25}$$

for $t \ge T$ with

$$R(t) = r(t)x^{\lambda-1}\big(\sigma(t)\big)\,. \tag{4.26}$$

Put $a^* = \frac{a_0 a_1}{a_0 - 1} > 0$. In virtue of (4.24), there exists $T_1 \ge T$ such that

$$x^{\lambda-1}\big(\sigma(t)\big) \ge \max\Big(\frac{2}{\varepsilon}, \frac{a^*}{\varepsilon_1}\Big)\,. \tag{4.27}$$

Now, we apply Theorem 4.2 to equation (4.25). Then (4.20), (4.26) and (4.27) imply

$$\Big(\frac{1}{a^*}\int_{\eta(t)}^{t} R(s)\,ds - q^*(t)\Big)\big(\eta(t) - h_1(t)\big)^2 \ge \Big(\frac{1}{a^*}\int_{\eta(t)}^{t} \frac{2}{\varepsilon} r(s)\,ds$$
$$- \frac{2}{\varepsilon}q^*(t)\Big)\Big(\eta(t) - \tau^{-1}\big(\sigma(t)\big)\Big)^2 > 2\,;$$

hence, (4.13) is valid for (4.25). Furthermore, using (4.21), (4.26) and (4.27) we get

$$R(t) = r(t)x^{\lambda-1}\big(\sigma(t)\big) \ge \frac{a^*}{\varepsilon_1}r(t) \ge a^* q^{3/2}(t)\,. \tag{4.28}$$

Moreover, according to (4.22), $T_2 \ge T_1$ exists such that

$$q'(t) \le q'(t) + \frac{K}{t} \le \frac{2}{\varepsilon_1}r(t)\,, \quad t \ge T_2\,.$$

From here and from the first inequality in (4.28)

$$R(t) \ge \frac{a^*}{\varepsilon_1}r(t) \ge a^* q'(t)/2\,. \tag{4.29}$$

Thus, (4.28) and (4.29) imply the validity of (4.14) for (4.25). As all hypotheses of Theorem 4.2, applied to (4.25), are satisfied, z' oscillates and it contradicts (4.23). $\square$

5. Special case

Remark 5.1 *Theorems 4.2 and 4.3 can be applied to Equation (1.3) as it is equivalent with*

$$z^{'''} + q(t)z' + (1+a)r(t)\big|x^{\lambda}(\sigma(t))\big|^{\lambda}\operatorname{sgn} x\big(\sigma(t)\big) = 0\,,$$

where $\tau(t) \equiv t$, $a(t) \equiv a > 0$ *and* $z(t) = (1+a)x(t)$.

Consider a special type of (1.1) with constant delays,

$$z''' + q(t)z' + r(t)\big|x(t-c_1)\big|^{\lambda} \operatorname{sgn} x(t-c_1) = 0 \tag{5.1}$$

with $\lambda \in (0,1]$, $a > 0$, $0 \le c_0 < c_1$ and $z(t) = x(t) + ax(t-c_0)$.

Corollary 5.2 *Let $a > 1$, $q \in C^1(\mathbb{R}_+)$, $v \ge 0$, $w \in \mathbb{R}$, either $s > -2$ or $s = -2$ and $q_0 > \frac{1}{4}$, $r_0 > 0$, $q_0 > 0$, $q_0' \in \mathbb{R}$ and*

$$r(t) \ge r_0 t^v\,,\ q(t) \le q_0 t^s\,,\ q'(t) \le q_0' t^w \quad \textit{for large } t\,,$$

where $v \ge 0$, $v > \frac{3}{2}s$, $v > w$, and $r_0 > \frac{27a^2}{2(a-1)(c_1-c_0)^3}$ in case $\lambda = 1$ and $v = 0$. Then (5.1) has Property A^*.

Proof Let $\eta(t) = t - c_2$, where $c_2 \in (0, c_1 - c_0)$. The proof follows from Theorems 4.2 and 4.3. As $\eta(t) - \tau^{-1}(\sigma(t)) = c_1 - c_0 - c_2 > 0$ is constant, a necessary condition for validity of either (4.13) or (4.20) is $\liminf\limits_{t\to\infty} \int\limits_{\eta(t)}^{t} r(s)\,ds > 0$, i.e. $v \ge 0$. Similarly, necessary conditions for either (4.14) or (4.21) to be valid are $v \ge \frac{3}{2}s$ and $v \ge w$. In case $\lambda = 1$ and $v = 0$ we obtain the condition $r_0 > \frac{2a^2}{(a-1)c_2(c_1-c_0-c_2)^2}$ and the optimal value of c_2 is $c_2 = \frac{c_1-c_0}{3}$. We use $\varepsilon = \frac{1}{2}\big(\frac{a-1}{a^2} r_0 c_2 - q_0\big)(c_1 - c_0 - c_2)^2$ and $\varepsilon_1 = r_0 q_0^{-3/2}$ when Theorem 4.3 is applied. □

Corollary 5.3 *Let $\lambda = 1$, $a \le 1$, $0 < 2c_0 < c_1$, $r_0 > 0$, $r_1 > 0$, $q_0 > 0$, $\varepsilon \in (0,1)$ either $s > -2$ or $s = -2$ and $q_0 > \frac{1}{4}$, $v \ge 0$, $v > s$, $r_0 > \frac{1}{c_0}\big[\frac{162}{(c_1-c_0)^2} + 72\big(\frac{r_1}{a}\big)^{2/3}\big]$ in case $v = 0$ and*

$$r_0 t^v \le r(t) \le r_1 t^{(3-\varepsilon)v/2}\,,\ q(t) \le q_0 t^s \quad \textit{for large } t\,.$$

Then (5.1) has Property A^*.

Proof It follows from Theorem 4.1. □

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