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Notes on the quadraticity of linear combinations of a cubic matrix and a quadratic matrix that commute

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Abstract: Let A_1 and A_2 be an $\{\alpha_1, \beta_1, \gamma_1\}$-cubic matrix and an $\{\alpha_2, \beta_2\}$-quadratic matrix, respectively, with $\alpha_1 \neq \beta_1$, $\beta_1 \neq \gamma_1$, $\alpha_1 \neq \gamma_1$ and $\alpha_2 \neq \beta_2$. In this work, we characterize all situations in which the linear combination $A_3 = a_1A_1 + a_2A_2$ with the assumption $A_1A_2 = A_2A_1$ is an $\{\alpha_3, \beta_3\}$-quadratic matrix, where a_1 and a_2 are unknown nonzero complex numbers.

Key words: Quadratic matrix, cubic matrix, linear combination, diagonalization

1. Introduction

Let $\mathbb{C}$ be the field of all complex numbers and $\mathbb{C}^*$ be the set of all nonzero complex numbers. The symbols $\mathbb{C}_n$, I_n, and $\mathbf{0}$ will denote the set of all $n \times n$ complex matrices, identity matrix (of size n), and zero matrix of suitable size, respectively. When we do not want to emphasize the size of the identity matrix, we'll use the symbol I to indicate it. Moreover, the rank of a matrix A will be symbolized by $\mathrm{rk}(A)$. On the other hand, similarity and direct sum of two matrices A and B will be denoted by $A \sim B$ and $A \oplus B$, respectively, where two square matrices A and B are similar if there exists a nonsingular matrix S such that $S^{-1}AS = B$.

We say that a matrix A is an $\{\alpha, \beta, \gamma\}$-cubic matrix if the equality

$$(A - \alpha I_n)(A - \beta I_n)(A - \gamma I_n) = \mathbf{0} \tag{1.1}$$

holds with $\alpha, \beta, \gamma \in \mathbb{C}$. It is easily seen that if $\{\alpha, \beta, \gamma\} = \{1, -1, 0\}$ is taken, then the matrix A in (1.1) becomes a tripotent matrix, i.e. a matrix satisfying the equality $A^3 = A$.

Recall that a matrix A is called a generalized $\{\alpha, \beta\}$-quadratic matrix with respect to an idempotent matrix P if there exist $\alpha, \beta \in \mathbb{C}$ such that

$$(A - \alpha P)(A - \beta P) = \mathbf{0} \quad AP = PA = A[11, 18, 23]. \tag{1.2}$$

In case $P = I$, it is called that the matrix A in (1.2) is an $\{\alpha, \beta\}$-quadratic matrix. From now on, the set of all $\{\alpha, \beta\}$-quadratic matrices with $\alpha \neq \beta$, the set of all generalized $\{\alpha, \beta\}$-quadratic matrices with respect to an idempotent matrix P where $\alpha \neq \beta$, and the set of all $\{\alpha, \beta, \gamma\}$-cubic matrices with $\alpha \neq \beta$, $\beta \neq \gamma$, $\alpha \neq \gamma$ will be denoted by $\Omega(\alpha, \beta)$, $\mathcal{L}(P; \alpha, \beta)$, and $\kappa(\alpha, \beta, \gamma)$, respectively.

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Notice that any generalized $\{\alpha, \beta\}$-quadratic matrix with respect to an idempotent matrix P satisfies the equality $A^3 = (\alpha + \beta)A^2 - \alpha\beta A$. Similarly, any $\{\alpha, \beta, \gamma\}$-cubic matrix A satisfies the equality $A^3 = (\alpha + \beta + \gamma)A^2 - (\alpha\beta + \alpha\gamma + \beta\gamma)A + \alpha\beta\gamma I$. Thus, it is clear that any generalized $\{\alpha, \beta\}$-quadratic matrix is a special cubic matrix with $\gamma = 0$.

On the other hand, we know from [11] that the set of all generalized quadratic matrices covers the sets of all generalized involutive matrices, i.e. $A^2 = P$ and all generalized skew involutive matrices, i.e. $A^2 = -P$, respectively, where $P \neq I$. In addition, as we have explained above, the set of all cubic matrices covers the set of all generalized quadratic matrices, and also, the set of all generalized quadratic matrices covers the set of all quadratic matrices. Moreover, the set of all quadratic matrices contains the set of all idempotent matrices, i.e. $A^2 = A$, all involutive matrices, i.e. $A^2 = I$, and all scalar-potent matrices, i.e. $A^2 = \lambda A$ for some $\lambda \in \mathbb{C}$. Thus, the set of all cubic matrices covers all mentioned above.

In the last years, the problem of characterizing all situations, in which a linear combination of two special types of matrices is again a special type of matrix, are widely considered in the literature, for example, [2–7, 9, 13–15, 18–24]. The main purpose of this work is to characterize all situations in which a linear combination of a quadratic matrix and a cubic matrix that commute is a quadratic matrix. In addition, some special results derived from the main result obtained are given. From these results, it is seen that the main result covers many of the results in the literature related to characterization of linear combinations of special types of matrices.

Now, we want to introduce two additional notations.

Firstly, we know that if A is an $\{\alpha, \beta\}$-quadratic matrix with $\alpha \neq \beta$, then there exists an idempotent matrix Q such that

$$A = (\alpha - \beta)Q + \beta I \tag{1.3}$$

by Theorem 2.1 in [16]. Notice that the matrix $(\alpha - \beta)Q$ in (1.3) is an $(\alpha - \beta)$-scalar-potent matrix. If we denote this matrix by B, then we shall say that the matrix A in (1.3) is an $\{\alpha, \beta\}$-quadratic matrix corresponding to the scalar-potent matrix B. We shall denote the set of all such matrices by $\Omega(\alpha, \beta, B)$.

Secondly, we know that if A is an $\{\alpha, \beta, \gamma\}$-cubic matrix with $\alpha \neq \beta$, $\beta \neq \gamma$, and $\alpha \neq \gamma$, then there exist two disjoint idempotent matrices X and Y such that

$$A = (\alpha - \gamma)X + (\beta - \gamma)Y + \gamma I \tag{1.4}$$

by Lemma 1.1 in [21]. Let us denote the matrix $(\alpha - \gamma)X + (\beta - \gamma)Y$ by C. It is clear that the matrix C is a generalized $\{\alpha - \gamma, \beta - \gamma\}$-quadratic matrix with respect to the idempotent matrix $X + Y := P$ in view of the item (ii) of Theorem 1.1 in [18]. Thus, we shall say that the matrix A in (1.4) is an $\{\alpha, \beta, \gamma\}$-cubic matrix corresponding to the generalized quadratic matrix C. We shall denote the set of all such matrices by $\kappa(\alpha, \beta, \gamma, C, P)$.

2. Results

As pointed out before, this section is designed in two stages. It first presents some auxiliary results which will be used to get the main result. Next, the main result is given within the framework of these results.

Theorem 2.1 *Let* $A_1 \in \kappa(\alpha_1, \beta_1, \gamma_1, B_1, P_1)$, $A_2 \in \Omega(\alpha_2, \beta_2, B_2)$ *with* $A_1, A_2 \in \mathbb{C}_n$ *and* $A_1A_2 = A_2A_1$. *Then there exists a nonsingular matrix* S, *an* $\{\alpha_1 - \gamma_1, \beta_1 - \gamma_1\}$*-quadratic matrix* K, *and* $(\alpha_2 - \beta_2)$*-scalar potent* *matrices X and T such that*

$$B_1 = S\begin{pmatrix} K & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}S^{-1} \quad \textit{and} \quad B_2 = S\begin{pmatrix} X & \mathbf{0} \\ \mathbf{0} & T \end{pmatrix}S^{-1}$$

with $rk(P_1) = r$, $K, X \in \mathbb{C}_r$, and $T \in \mathbb{C}_{n-r}$.

Proof According to the hypotheses, it can be written

$$A_1 = B_1 + \gamma_1 I \quad \text{and} \quad A_2 = B_2 + \beta_2 I. \tag{2.1}$$

The commutativity of the matrices A_1 and A_2 leads to the commutativity of the matrices B_1 and B_2 in view of (2.1). Moreover, since the matrix B_1 is an $\{\alpha_1 - \gamma_1, \beta_1 - \gamma_1\}$-generalized quadratic matrix with respect to the idempotent matrix P_1, we have

$$(B_1 - (\alpha_1 - \gamma_1)P_1)(B_1 - (\beta_1 - \gamma_1)P_1) = \mathbf{0},\ B_1P_1 = P_1B_1 = B_1. \tag{2.2}$$

Similarly, since the matrix B_2 is an $(\alpha - \beta)$-scalar potent matrix, there exists an idempotent matrix W such that

$$B_2 = (\alpha_2 - \beta_2)W. \tag{2.3}$$

Now, because of the idempotency of the matrix P_1, there exists a nonsingular matrix S such that

$$P_1 = S(I_r \oplus \mathbf{0})S^{-1} \tag{2.4}$$

with $r = rk(P_1)$. Let us write the matrix B_1 as

$$B_1 = S\begin{pmatrix} K & L \\ M & N \end{pmatrix}S^{-1},\ K \in \mathbb{C}_r. \tag{2.5}$$

From the second equality of (2.2) and the equalities (2.4) and (2.5), we get $L = \mathbf{0}$, $M = \mathbf{0}$, and $N = \mathbf{0}$. Thus, we can write the matrix B_1 as

$$B_1 = S\begin{pmatrix} K & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}S^{-1}. \tag{2.6}$$

If the equalities (2.4) and (2.6) are substituted into the first equality of (2.2), then the following equality is obtained:

$$(K - (\alpha_1 - \gamma_1)I_r)(K - (\beta_1 - \gamma_1)I_r) = \mathbf{0}. \tag{2.7}$$

It is clearly seen from (2.7) that the matrix K is an $\{\alpha_1 - \gamma_1, \beta_1 - \gamma_1\}$-quadratic matrix. Furthermore, the matrix K is nonsingular because $\alpha_1 \neq \gamma_1$ and $\beta_1 \neq \gamma_1$.

Now, let us write the matrix B_2 as

$$B_2 = S\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}S^{-1}, \tag{2.8}$$

where $X \in \mathbb{C}_r$ and $T \in \mathbb{C}_{n-r}$. Since $B_1B_2 = B_2B_1$, from the equalities (2.6) and (2.8), the matrix B_2 turns into

$$B_2 = S\begin{pmatrix} X & \mathbf{0} \\ \mathbf{0} & T \end{pmatrix}S^{-1}. \tag{2.9}$$

Thus, the desired result is obtained. □

Theorem 2.2 *Let X, T, and K be the matrices in Theorem 2.1. Then there exist idempotent matrices M_1, M_2, and Z_1 such that*

$$X = (\alpha_2 - \beta_2)M_1, \quad T = (\alpha_2 - \beta_2)M_2, \quad \textit{and} \quad K = (\alpha_1 - \beta_1)Z_1 + (\beta_1 - \gamma_1)I,$$

under the hypotheses of Theorem 2.1.

Proof From the equalities (2.3) and (2.9), the following can be written:

$$S^{-1}WS = \begin{pmatrix} \frac{1}{\alpha_2-\beta_2}X & \mathbf{0} \\ \mathbf{0} & \frac{1}{\alpha_2-\beta_2}T \end{pmatrix}.$$

If we denote the upper left corner element and lower right corner element of $S^{-1}WS$ by M_1 and M_2, respectively, then we directly see that

$$X = (\alpha_2 - \beta_2)M_1 \text{ and } T = (\alpha_2 - \beta_2)M_2. \tag{2.10}$$

Since the matrices X and T are $(\alpha_2-\beta_2)$-scalar potent, it is clear that the matrices M_1 and M_2 are idempotent. In addition, since the matrix K is an $\{\alpha_1 - \gamma_1, \beta_1 - \gamma_1\}$-quadratic matrix, from the item (iv) of Theorem 2.1 in [16], there exists an idempotent matrix Z_1 such that

$$K = (\alpha_1 - \beta_1)Z_1 + (\beta_1 - \gamma_1)I. \tag{2.11}$$

Thus, the proof is completed. □

Now, consider a linear combination of the form

$$A_3 = a_1A_1 + a_2A_2, \tag{2.12}$$

where $a_1, a_2 \in \mathbb{C}^*$, the matrices A_1 and A_2 are as in Theorem 2.1. We will investigate necessary and sufficient conditions for the $\{\alpha_3, \beta_3\}$-quadraticity of the linear combination matrix A_3 with $\alpha_3, \beta_3 \in \mathbb{C}$. The equality (2.12) is equivalent to

$$A_3 = a_1B_1 + a_2B_2 + (a_1\gamma_1 + a_2\beta_2)I. \tag{2.13}$$

The matrix A_3 of the form (2.13) is an $\{\alpha_3, \beta_3\}$-quadratic matrix if and only if

$$(a_1B_1 + a_2B_2 + (a_3 - \alpha_3)I)\,(a_1B_1 + a_2B_2 + (a_3 - \beta_3)I) = \mathbf{0}, \tag{2.14}$$

where $a_3 = a_1\gamma_1 + a_2\beta_2$. From (2.14) and the commutativity of the matrices B_1 and B_2, we obtain

$$\begin{aligned} &{a_1}^2{B_1}^2 + 2a_1a_2B_1B_2 + {a_2}^2{B_2}^2 + a_1\,(2a_3 - \alpha_3 - \beta_3)\,B_1 \\ &+a_2\,(2a_3 - \alpha_3 - \beta_3)\,B_2 + (a_3 - \alpha_3)\,(a_3 - \beta_3)\,I = \mathbf{0}. \end{aligned} \tag{2.15}$$

Substituting the equalities (2.2) and $B_2^2 = (\alpha_2 - \beta_2)B_2$ into (2.15) leads to

$$c_1B_1 + c_2B_2 + 2a_1a_2B_1B_2 + c_3P_1 + c_4I = \mathbf{0}, \tag{2.16}$$

where $c_1 = a_1^2(\alpha_1+\beta_1-2\gamma_1)+a_1(2a_3-\alpha_3-\beta_3)$, $c_2 = a_2^2(\alpha_2-\beta_2)+a_2(2a_3-\alpha_3-\beta_3)$, $c_3 = -a_1^2(\alpha_1-\gamma_1)(\beta_1-\gamma_1)$, and $c_4 = (a_3 - \alpha_3)(a_3 - \beta_3)$. If we substitute the equalities (2.4), (2.6), and (2.9) into (2.16), then we get the following system.

$$c_1K + c_2X + 2a_1a_2KX + (c_3 + c_4)I = \mathbf{0} \quad \text{and} \quad c_2T + c_4I = \mathbf{0}. \tag{2.17}$$

On the other hand, from (2.6) and (2.9), it is clear that $KX = XK$ due to the fact that $B_1B_2 = B_2B_1$. Moreover, since $\alpha_1 \neq \beta_1$ and $\alpha_2 \neq \beta_2$, from the first equality of (2.10) and the equality (2.11), it is obtained that $Z_1M_1 = M_1Z_1$. If we substitute the matrix X in (2.10) and the matrix K in (2.11) into the first equality of (2.17), then the following equality is obtained:

$$\begin{aligned} &c_1(\alpha_1-\beta_1)Z_1+(c_2(\alpha_2-\beta_2)+2a_1a_2(\beta_1-\gamma_1)(\alpha_2-\beta_2))M_1 \\ &+2a_1a_2(\alpha_1-\beta_1)(\alpha_2-\beta_2)Z_1M_1+(c_1(\beta_1-\gamma_1)+c_3+c_4)I=\mathbf{0}. \end{aligned} \tag{2.18}$$

If we write the matrix T in the second equality of (2.10) into the second equality of (2.17), then we get

$$c_2(\alpha_2-\beta_2)M_2+c_4I=\mathbf{0}. \tag{2.19}$$

Thus, we have the following corollary.

Corollary 2.3 *Let A_1 and A_2 be as in Theorem 2.1. Then the linear combination $A_3 = a_1A_1 + a_2A_2$ with $a_1, a_2 \in \mathbb{C}^*$ is an $\{\alpha_3, \beta_3\}$-quadratic matrix with $\alpha_3, \beta_3 \in \mathbb{C}$ if and only if the equalities (2.18) and (2.19) hold.*

Now, we will investigate the cases in which the equalities (2.18) and (2.19) are satisfied. We first handle the cases related to (2.19).

Theorem 2.4 *Under the conditions of Theorem 2.1, the necessary and sufficient condition to hold the equality (2.19) is that any one of the following sets of additional conditions holds, where M_2 is the matrix in Theorem 2.2.*

(i) $c_4 = 0$ *and* $M_2 = \mathbf{0}$,

(ii) $c_2(\alpha_2-\beta_2)+c_4 = 0$ *and* $M_2 = I$,

(iii) $c_2 = c_4 = 0$ *and* $M_2 \sim I \oplus \mathbf{0}$.

Proof Multiplying the equality (2.19) by M_2 leads to

$$(c_2(\alpha_2-\beta_2)+c_4)\,M_2=\mathbf{0}.$$

Now, we have two possibilities: $M_2 = \mathbf{0}$ or $M_2 \neq \mathbf{0}$.

In the case $M_2 = \mathbf{0}$, from the equality (2.19), we get $c_4 = 0$, which is the item (i).

In the case $M_2 \neq \mathbf{0}$, there are two possibilities for diagonal form of the idempotent matrix M_2: $M_2 \sim I \oplus I$ (that is, $M_2 = I$) or $M_2 \sim I \oplus \mathbf{0}$. Thus, from the equality (2.19), we get $c_2(\alpha_2-\beta_2)+c_4 = 0$ or $c_2 = c_4 = 0$, which are the items (ii) or (iii), respectively. Hence, the desired results are obtained. □

Next, let us handle the cases related to (2.18).

Theorem 2.5 *Under the conditions of Theorem 2.1, the necessary and sufficient condition to hold the equality (2.18) is that any one of the following sets of additional conditions holds, where Z_1 and M_1 are the matrices in Theorem 2.2.*

(a) $c_1(\beta_1-\gamma_1)+c_2(\alpha_2-\beta_2)+2a_1a_2(\beta_1-\gamma_1)(\alpha_2-\beta_2)+c_3+c_4=0,$

$Z_1 = \mathbf{0}$, *and* $M_1 = I$,

(b) $c_1(\beta_1 - \gamma_1) = -c_3 - c_4,\ c_2 = -2a_1a_2(\beta_1 - \gamma_1),\ Z_1 = \mathbf{0},$ *and* $M_1 \sim I \oplus \mathbf{0},$

(c) $c_1(\beta_1 - \gamma_1) = -c_3 - c_4,\ Z_1 = \mathbf{0},$ *and* $M_1 = \mathbf{0},$

(d) $c_1(\alpha_1 - \gamma_1) = -c_3 - c_4,\ Z_1 = I,$ *and* $M_1 = \mathbf{0},$

(e) $c_1 = c_3 + c_4 = 0,\ Z_1 \sim I \oplus \mathbf{0},$ *and* $M_1 = \mathbf{0},$

(f) $c_3 + c_4 = -c_1(\alpha_1 - \gamma_1)$

$= -c_2(\alpha_2 - \beta_2) - 2a_1a_2(\beta_1 - \gamma_1)(\alpha_2 - \beta_2) - c_1(\beta_1 - \gamma_1),$

$Z_1 \sim I \oplus \mathbf{0},$ *and* $M_1 \sim \mathbf{0} \oplus I,$

(g) $c_1 = 0,\ c_2 = -2a_1a_2(\beta_1 - \gamma_1),\ c_3 = -c_4,$

$Z_1 \sim I \oplus \mathbf{0} \oplus \mathbf{0},$ *and* $M_1 \sim \mathbf{0} \oplus I \oplus \mathbf{0},$

(h) $c_1(\alpha_1 - \gamma_1) + c_2(\alpha_2 - \beta_2) + 2a_1a_2(\alpha_1 - \gamma_1)(\alpha_2 - \beta_2) + c_3 + c_4 = 0,$

$Z_1 = I,$ *and* $M_1 = I,$

(k) $c_1(\alpha_1 - \gamma_1) = -c_3 - c_4,\ c_2 = -2a_1a_2(\alpha_1 - \gamma_1),$

$Z_1 = I,$ *and* $M_1 \sim I \oplus \mathbf{0},$

(l) $c_1(\beta_1 - \gamma_1) = c_1(\alpha_1 - \gamma_1) + c_2(\alpha_2 - \beta_2) + 2a_1a_2(\alpha_1 - \gamma_1)(\alpha_2 - \beta_2)$

$= -c_3 - c_4,\ Z_1 \sim I \oplus \mathbf{0},$ *and* $M_1 \sim I \oplus \mathbf{0},$

(m) $c_1 = -2a_1a_2(\alpha_2 - \beta_2),\ c_2(\alpha_2 - \beta_2) = -c_3 - c_4,$

$Z_1 \sim I \oplus \mathbf{0},$ *and* $M_1 = I,$

(n) $c_1 = -2a_1a_2(\alpha_2 - \beta_2),\ c_1(\alpha_1 - \gamma_1) = -c_3 - c_4,\ c_2 = -2a_1a_2(\alpha_1 - \gamma_1),$

$Z_1 \sim I \oplus I \oplus \mathbf{0},$ *and* $M_1 \sim I \oplus \mathbf{0} \oplus I,$

(p) $c_1 = 0,\ c_2 = -2a_1a_2(\alpha_1 - \gamma_1),\ c_3 = -c_4,$

$Z_1 \sim I \oplus I \oplus \mathbf{0},$ *and* $M_1 \sim I \oplus \mathbf{0} \oplus \mathbf{0},$

(r) $c_1(\beta_1 - \gamma_1) = -c_3 - c_4,\ c_2 = -2a_1a_2(\beta_1 - \gamma_1),$

$c_1(\alpha_1 - \gamma_1) + c_2(\alpha_2 - \beta_2) + 2a_1a_2(\alpha_1 - \gamma_1)(\alpha_2 - \beta_2) + c_3 + c_4 = 0,$

$Z_1 \sim I \oplus \mathbf{0} \oplus \mathbf{0},$ *and* $M_1 \sim I \oplus I \oplus \mathbf{0}.$

Proof Premultiplying (2.18) by the idempotent matrix Z_1 leads to the equality

$$(c_1(\alpha_1 - \gamma_1) + c_3 + c_4)\, Z_1 + (c_2(\alpha_2 - \beta_2) + 2a_1a_2(\alpha_1 - \gamma_1)(\alpha_2 - \beta_2))\, Z_1M_1 = \mathbf{0}. \tag{2.20}$$

If the equality (2.24) is postmultiplied by the idempotent matrix M_1, then the equality

$$(c_1(\alpha_1 - \gamma_1) + c_2(\alpha_2 - \beta_2) + 2a_1a_2(\alpha_1 - \gamma_1)(\alpha_2 - \beta_2) + c_3 + c_4)\, Z_1M_1 = \mathbf{0}$$

is obtained.

Now, there are two possibilities: $Z_1M_1 = \mathbf{0}$ or $Z_1M_1 \neq \mathbf{0}$.

Firstly, in the case $Z_1M_1 = \mathbf{0}$, from (2.24), we get

$$(c_1(\alpha_1 - \gamma_1) + c_3 + c_4)\, Z_1 = \mathbf{0}. \tag{2.21}$$

There are two possibilities for the matrix Z_1 from the equality (2.25): $Z_1 = \mathbf{0}$ or $Z_1 \neq \mathbf{0}$.

In the case $Z_1 = \mathbf{0}$, from the equality (2.18), the equality

$$(c_2(\alpha_2 - \beta_2) + 2a_1a_2(\beta_1 - \gamma_1)(\alpha_2 - \beta_2))\, M_1 + (c_1(\beta_1 - \gamma_1) + c_3 + c_4)\, I = \mathbf{0} \tag{2.22}$$

is obtained. On the other hand, it is seen that, taking into account the diagonal forms of the idempotent matrix M_1, all possibilities for the matrix M_1 are as in the following:

$$M_1 = I \quad \text{or} \quad M_1 \sim I \oplus \mathbf{0} \quad \text{or} \quad M_1 = \mathbf{0}.$$

Thus, from the equality (2.26), the equalities

$c_1(\beta_1 - \gamma_1) + c_2(\alpha_2 - \beta_2) + 2a_1a_2(\beta_1 - \gamma_1)(\alpha_2 - \beta_2) + c_3 + c_4 = 0$ or

$c_1(\beta_1 - \gamma_1) = -c_3 - c_4,\ c_2 = -2a_1a_2(\beta_1 - \gamma_1)$ or

$c_1(\beta_1 - \gamma_1) = -c_3 - c_4$,

respectively, are obtained. Thus, we have the items (a), (b), and (c), respectively.

In the case $Z_1 \neq \mathbf{0}$, from the equality (2.25), we get

$$c_1(\alpha_1 - \gamma_1) + c_3 + c_4 = 0. \tag{2.23}$$

In addition, since $Z_1M_1 = \mathbf{0}$, all possibilities for the pairs of the idempotent matrices Z_1 and M_1 are as in the following:

$Z_1 = I$ and $M_1 = \mathbf{0}$ or

$Z_1 \sim I \oplus \mathbf{0}$ and $M_1 = \mathbf{0}$ or $M_1 \sim \mathbf{0} \oplus I$ or

$Z_1 \sim I \oplus \mathbf{0} \oplus \mathbf{0}$ and $M_1 \sim \mathbf{0} \oplus I \oplus \mathbf{0}$.

Thus, in view of the equality (2.27), from the equality (2.18), the following equalities are obtained:

$c_1(\alpha_1 - \gamma_1) = -c_3 - c_4$ or

$c_1 = c_3 + c_4 = 0$ or

$c_3 + c_4 = -c_1(\alpha_1 - \gamma_1) = -c_2(\alpha_2 - \beta_2) - 2a_1a_2(\beta_1 - \gamma_1)(\alpha_2 - \beta_2) - c_1(\beta_1 - \gamma_1)$ or

$c_1 = 0, c_2 = -2a_1a_2(\beta_1 - \gamma_1), c_3 = -c_4$.

Thus, we get the items (d), (e), (f), and (g), respectively.

Secondly, in the case $Z_1M_1 \neq \mathbf{0}$, the matrices Z_1 and M_1 are both nonzero. Thus, it is seen that all possibilities for the pairs of the idempotent matrices Z_1 and M_1 are as in the following:

$Z_1 = I$ and $M_1 = I$ or

$Z_1 = I$ and $M_1 \sim I \oplus \mathbf{0}$ or

$Z_1 \sim I \oplus \mathbf{0}$ and $M_1 \sim I \oplus \mathbf{0}$ or $M_1 = I$ or

$Z_1 \sim I \oplus I \oplus \mathbf{0}$ and $M_1 \sim I \oplus \mathbf{0} \oplus I$ or

$Z_1 \sim I \oplus I \oplus \mathbf{0}$ and $M_1 \sim I \oplus \mathbf{0} \oplus \mathbf{0}$ or

$Z_1 \sim I \oplus \mathbf{0} \oplus \mathbf{0}$ and $M_1 \sim I \oplus I \oplus \mathbf{0}$.

If we write the pairs of matrices above in (2.18) in view of $\alpha_1 \neq \beta_1$, $\alpha_2 \neq \beta_2$, and $c_3 \neq 0$, then we get the following equalities, respectively:

$c_1(\alpha_1-\gamma_1)+c_2(\alpha_2-\beta_2)+2a_1a_2(\alpha_1-\gamma_1)(\alpha_2-\beta_2)+c_3+c_4=0$ or
$c_1(\alpha_1-\gamma_1)=-c_3-c_4, c_2=-2a_1a_2(\alpha_1-\gamma_1)$ or
$c_1(\beta_1-\gamma_1)=c_1(\alpha_1-\gamma_1)+c_2(\alpha_2-\beta_2)+2a_1a_2(\alpha_1-\gamma_1)(\alpha_2-\beta_2)=-c_3-c_4$ or
$c_1=-2a_1a_2(\alpha_2-\beta_2), c_2(\alpha_2-\beta_2)=-c_3-c_4$ or
$c_1=-2a_1a_2(\alpha_2-\beta_2), c_1(\alpha_1-\gamma_1)=-c_3-c_4, c_2=-2a_1a_2(\alpha_1-\gamma_1)$ or
$c_1=0, c_2=-2a_1a_2(\alpha_1-\gamma_1), c_3=-c_4$ or
$c_1(\beta_1-\gamma_1)=-c_3-c_4, c_2=-2a_1a_2(\beta_1-\gamma_1),\ c_1(\alpha_1-\gamma_1)+c_2(\alpha_2-\beta_2)+2a_1a_2(\alpha_1-\gamma_1)(\alpha_2-\beta_2)+c_3+c_4=0.$

Hence, the items (h), (k), (l), (m), (n), (p), and (r), respectively, are obtained. Thus, the proof is completed. □

Considering Corollary 2.3, Theorem 2.4, and Theorem 2.6, now we can give the following theorem which is the main result of the work. Next, let us handle the cases related to (2.18).

Theorem 2.6 *Under the conditions of Theorem 2.1, the necessary and sufficient condition to hold the equality (2.18) is that any one of the following sets of additional conditions holds, where Z_1 and M_1 are the matrices in Theorem 2.2.*

(a) $c_1(\beta_1-\gamma_1)+c_2(\alpha_2-\beta_2)+2a_1a_2(\beta_1-\gamma_1)(\alpha_2-\beta_2)+c_3+c_4=0,$
$Z_1=\mathbf{0},$ *and* $M_1=I,$

(b) $c_1(\beta_1-\gamma_1)=-c_3-c_4,\ c_2=-2a_1a_2(\beta_1-\gamma_1),\ Z_1=\mathbf{0},$ *and* $M_1\sim I\oplus\mathbf{0},$

(c) $c_1(\beta_1-\gamma_1)=-c_3-c_4,\ Z_1=\mathbf{0},$ *and* $M_1=\mathbf{0},$

(d) $c_1(\alpha_1-\gamma_1)=-c_3-c_4,\ Z_1=I,$ *and* $M_1=\mathbf{0},$

(e) $c_1=c_3+c_4=0,\ Z_1\sim I\oplus\mathbf{0},$ *and* $M_1=\mathbf{0},$

(f) $c_3+c_4=-c_1(\alpha_1-\gamma_1)$
$=-c_2(\alpha_2-\beta_2)-2a_1a_2(\beta_1-\gamma_1)(\alpha_2-\beta_2)-c_1(\beta_1-\gamma_1),$
$Z_1\sim I\oplus\mathbf{0},$ *and* $M_1\sim\mathbf{0}\oplus I,$

(g) $c_1=0,\ c_2=-2a_1a_2(\beta_1-\gamma_1),\ c_3=-c_4,$
$Z_1\sim I\oplus\mathbf{0}\oplus\mathbf{0},$ *and* $M_1\sim\mathbf{0}\oplus I\oplus\mathbf{0},$

(h) $c_1(\alpha_1-\gamma_1)+c_2(\alpha_2-\beta_2)+2a_1a_2(\alpha_1-\gamma_1)(\alpha_2-\beta_2)+c_3+c_4=0,$
$Z_1=I,$ *and* $M_1=I,$

(k) $c_1(\alpha_1-\gamma_1)=-c_3-c_4,\ c_2=-2a_1a_2(\alpha_1-\gamma_1),$
$Z_1=I,$ *and* $M_1\sim I\oplus\mathbf{0},$

(l) $c_1(\beta_1-\gamma_1)=c_1(\alpha_1-\gamma_1)+c_2(\alpha_2-\beta_2)+2a_1a_2(\alpha_1-\gamma_1)(\alpha_2-\beta_2)$
$=-c_3-c_4,\ Z_1\sim I\oplus\mathbf{0},$ *and* $M_1\sim I\oplus\mathbf{0},$

(m) $c_1=-2a_1a_2(\alpha_2-\beta_2),\ c_2(\alpha_2-\beta_2)=-c_3-c_4,$
$Z_1\sim I\oplus\mathbf{0},$ *and* $M_1=I,$

(n) $c_1 = -2a_1a_2(\alpha_2 - \beta_2),\ c_1(\alpha_1 - \gamma_1) = -c_3 - c_4,\ c_2 = -2a_1a_2(\alpha_1 - \gamma_1),$
$Z_1 \sim I \oplus I \oplus \mathbf{0},\ \textit{and}\ M_1 \sim I \oplus \mathbf{0} \oplus I,$

(p) $c_1 = 0,\ c_2 = -2a_1a_2(\alpha_1 - \gamma_1),\ c_3 = -c_4,$
$Z_1 \sim I \oplus I \oplus \mathbf{0},\ \textit{and}\ M_1 \sim I \oplus \mathbf{0} \oplus \mathbf{0},$

(r) $c_1(\beta_1 - \gamma_1) = -c_3 - c_4,\ c_2 = -2a_1a_2(\beta_1 - \gamma_1),$
$c_1(\alpha_1 - \gamma_1) + c_2(\alpha_2 - \beta_2) + 2a_1a_2(\alpha_1 - \gamma_1)(\alpha_2 - \beta_2) + c_3 + c_4 = 0,$
$Z_1 \sim I \oplus \mathbf{0} \oplus \mathbf{0},\ \textit{and}\ M_1 \sim I \oplus I \oplus \mathbf{0}.$

Proof Premultiplying (2.18) by the idempotent matrix Z_1 leads to the equality

$$(c_1(\alpha_1 - \gamma_1) + c_3 + c_4)\, Z_1 + (c_2(\alpha_2 - \beta_2) + 2a_1a_2(\alpha_1 - \gamma_1)(\alpha_2 - \beta_2))\, Z_1M_1 = \mathbf{0}. \tag{2.24}$$

If the equality (2.24) is postmultiplied by the idempotent matrix M_1, then the equality

$$(c_1(\alpha_1 - \gamma_1) + c_2(\alpha_2 - \beta_2) + 2a_1a_2(\alpha_1 - \gamma_1)(\alpha_2 - \beta_2) + c_3 + c_4)\, Z_1M_1 = \mathbf{0}$$

is obtained.

Now, there are two possibilities: $Z_1M_1 = \mathbf{0}$ or $Z_1M_1 \neq \mathbf{0}$.

Firstly, in the case $Z_1M_1 = \mathbf{0}$, from (2.24), we get

$$(c_1(\alpha_1 - \gamma_1) + c_3 + c_4)\, Z_1 = \mathbf{0}. \tag{2.25}$$

There are two possibilities for the matrix Z_1 from the equality (2.25): $Z_1 = \mathbf{0}$ or $Z_1 \neq \mathbf{0}$.

In the case $Z_1 = \mathbf{0}$, from the equality (2.18), the equality

$$(c_2(\alpha_2 - \beta_2) + 2a_1a_2(\beta_1 - \gamma_1)(\alpha_2 - \beta_2))\, M_1 + (c_1(\beta_1 - \gamma_1) + c_3 + c_4)\, I = \mathbf{0} \tag{2.26}$$

is obtained. On the other hand, it is seen that, taking into account the diagonal forms of the idempotent matrix M_1, all possibilities for the matrix M_1 are as in the following:

$$M_1 = I \quad \text{or} \quad M_1 \sim I \oplus \mathbf{0} \quad \text{or} \quad M_1 = \mathbf{0}.$$

Thus, from the equality (2.26), the equalities

$c_1(\beta_1 - \gamma_1) + c_2(\alpha_2 - \beta_2) + 2a_1a_2(\beta_1 - \gamma_1)(\alpha_2 - \beta_2) + c_3 + c_4 = 0$ or
$c_1(\beta_1 - \gamma_1) = -c_3 - c_4,\ c_2 = -2a_1a_2(\beta_1 - \gamma_1)$ or
$c_1(\beta_1 - \gamma_1) = -c_3 - c_4$,

respectively, are obtained. Thus, we have the items (a), (b), and (c), respectively.

In the case $Z_1 \neq \mathbf{0}$, from the equality (2.25), we get

$$c_1(\alpha_1 - \gamma_1) + c_3 + c_4 = 0. \tag{2.27}$$

In addition, since $Z_1M_1 = \mathbf{0}$, all possibilities for the pairs of the idempotent matrices Z_1 and M_1 are as in the following:

$Z_1 = I$ and $M_1 = \mathbf{0}$ or

$Z_1 \sim I \oplus \mathbf{0}$ and $M_1 = \mathbf{0}$ or $M_1 \sim \mathbf{0} \oplus I$ or

$Z_1 \sim I \oplus \mathbf{0} \oplus \mathbf{0}$ and $M_1 \sim \mathbf{0} \oplus I \oplus \mathbf{0}$.

Thus, in view of the equality (2.27), from the equality (2.18), the following equalities are obtained:

$c_1(\alpha_1 - \gamma_1) = -c_3 - c_4$ or

$c_1 = c_3 + c_4 = 0$ or

$c_3 + c_4 = -c_1(\alpha_1 - \gamma_1) = -c_2(\alpha_2 - \beta_2) - 2a_1a_2(\beta_1 - \gamma_1)(\alpha_2 - \beta_2) - c_1(\beta_1 - \gamma_1)$ or

$c_1 = 0, c_2 = -2a_1a_2(\beta_1 - \gamma_1), c_3 = -c_4$.

Thus, we get the items (d), (e), (f), and (g), respectively.

Secondly, in the case $Z_1M_1 \neq \mathbf{0}$, the matrices Z_1 and M_1 are both nonzero. Thus, it is seen that all possibilities for the pairs of the idempotent matrices Z_1 and M_1 are as in the following:

$Z_1 = I$ and $M_1 = I$ or

$Z_1 = I$ and $M_1 \sim I \oplus \mathbf{0}$ or

$Z_1 \sim I \oplus \mathbf{0}$ and $M_1 \sim I \oplus \mathbf{0}$ or $M_1 = I$ or

$Z_1 \sim I \oplus I \oplus \mathbf{0}$ and $M_1 \sim I \oplus \mathbf{0} \oplus I$ or

$Z_1 \sim I \oplus I \oplus \mathbf{0}$ and $M_1 \sim I \oplus \mathbf{0} \oplus \mathbf{0}$ or

$Z_1 \sim I \oplus \mathbf{0} \oplus \mathbf{0}$ and $M_1 \sim I \oplus I \oplus \mathbf{0}$.

If we write the pairs of matrices above in (2.18) in view of $\alpha_1 \neq \beta_1$, $\alpha_2 \neq \beta_2$, and $c_3 \neq 0$, then we get the following equalities, respectively:

$c_1(\alpha_1 - \gamma_1) + c_2(\alpha_2 - \beta_2) + 2a_1a_2(\alpha_1 - \gamma_1)(\alpha_2 - \beta_2) + c_3 + c_4 = 0$ or

$c_1(\alpha_1 - \gamma_1) = -c_3 - c_4, c_2 = -2a_1a_2(\alpha_1 - \gamma_1)$ or

$c_1(\beta_1 - \gamma_1) = c_1(\alpha_1 - \gamma_1) + c_2(\alpha_2 - \beta_2) + 2a_1a_2(\alpha_1 - \gamma_1)(\alpha_2 - \beta_2) = -c_3 - c_4$ or

$c_1 = -2a_1a_2(\alpha_2 - \beta_2), c_2(\alpha_2 - \beta_2) = -c_3 - c_4$ or

$c_1 = -2a_1a_2(\alpha_2 - \beta_2), c_1(\alpha_1 - \gamma_1) = -c_3 - c_4, c_2 = -2a_1a_2(\alpha_1 - \gamma_1)$ or

$c_1 = 0, c_2 = -2a_1a_2(\alpha_1 - \gamma_1), c_3 = -c_4$ or

$c_1(\beta_1-\gamma_1) = -c_3-c_4, c_2 = -2a_1a_2(\beta_1-\gamma_1),\ c_1(\alpha_1-\gamma_1)+c_2(\alpha_2-\beta_2)+2a_1a_2(\alpha_1-\gamma_1)(\alpha_2-\beta_2)+c_3+c_4 = 0.$

Hence, the items (h), (k), (l), (m), (n), (p), and (r), respectively, are obtained. Thus, the proof is completed. □

Considering Corollary 2.3, Theorem 2.4, and Theorem 2.6, now we can give the following theorem which is the main result of the work.

Theorem 2.7 *Let $A_1 \in \kappa(\alpha_1, \beta_1, \gamma_1, B_1, P_1)$, $A_2 \in \Omega(\alpha_2, \beta_2, B_2)$, $A_1, A_2 \in \mathbb{C}_n$, $a_1, a_2 \in \mathbb{C}^*$, and $A_1A_2 = A_2A_1$. Then $A_3 = a_1A_1 + a_2A_2$ is an $\{\alpha_3, \beta_3\}$-quadratic matrix with $\alpha_3, \beta_3 \in \mathbb{C}$ if and only if any one of the following cases holds:*

(a_1) $c_1(\omega_i - \gamma_1) + c_2(\alpha_2 - \beta_2) + c_3 + 2a_1a_2(\omega_i - \gamma_1)(\alpha_2 - \beta_2) = 0,\ c_4 = 0,$

$B_1 = (\omega_i - \gamma_1)P_1,\ B_2 = (\alpha_2 - \beta_2)P_1,\ i = 1 \textit{ or } 2,\ (\omega_1, \omega_2) = (\alpha_1, \beta_1),$

(a_2) $c_2 = -2a_1a_2(\omega_i - \gamma_1),\ c_3 = -c_1(\omega_i - \gamma_1),\ c_4 = 0,$

$B_1 = (\omega_i - \gamma_1)P_1,\ B_1B_2 = (\omega_i - \gamma_1)B_2,\ i = 1 \textit{ or } 2,\ (\omega_1, \omega_2) = (\alpha_1, \beta_1),$

(a_3) $c_3 = -c_1(\omega_i - \gamma_1),\ c_4 = 0,$

$B_1 = (\omega_i - \gamma_1)P_1,\ B_2 = \mathbf{0},\ i = 1 \textit{ or } 2,\ (\omega_1, \omega_2) = (\alpha_1, \beta_1),$

(a_4) $c_1 = -2a_1a_2(\alpha_2 - \beta_2),\ c_2 = -2a_1a_2(\omega_i - \gamma_1),$
$c_3 = 2a_1a_2(\omega_i - \gamma_1)(\alpha_2 - \beta_2),\ c_4 = 0,$
$B_1B_2 + (\omega_i - \gamma_1)\left((\alpha_2 - \beta_2)P_1 - B_2\right) = (\alpha_2 - \beta_2)B_1,$
$i = 1\ or\ 2, (\omega_1,\ \omega_2) = (\alpha_1, \beta_1),$

(a_5) $c_1(\omega_i - \gamma_1) + c_3 = 0,\ c_2(\alpha_2 - \beta_2) + c_1(\omega_j - \gamma_1)$
$+ 2a_1a_2(\omega_j - \gamma_1)(\alpha_2 - \beta_2) + c_3 = 0,\ c_4 = 0,$
$(\alpha_2 - \beta_2)B_1 + (\omega_i - \omega_j)B_2 = (\omega_i - \gamma_1)(\alpha_2 - \beta_2)P_1,$
$B_1B_2 = (\omega_j - \gamma_1)B_2,\ (i, j) = (1, 2)\ or\ (2, 1),\ (\omega_1, \omega_2) = (\alpha_1, \beta_1),$

(a_6) $c_1 = -2a_1a_2(\alpha_2 - \beta_2),\ c_3 = -c_2(\alpha_2 - \beta_2),\ c_4 = 0,\ B_2 = (\alpha_2 - \beta_2)P_1,$

(a_7) $c_1(\omega_i - \gamma_1) + c_3 + 2a_1a_2(\omega_i - \gamma_1)(\alpha_2 - \beta_2) = 0,\ c_2(\alpha_2 - \beta_2) + c_4 = 0,$
$B_1 = (\omega_i - \gamma_1)P_1,\ B_2 = (\alpha_2 - \beta_2)I,\ i = 1\ or\ 2,\ (\omega_1, \omega_2) = (\alpha_1, \beta_1),$

(a_8) $c_2 = -2a_1a_2(\omega_i - \gamma_1),\ c_3 = -(\omega_i - \gamma_1)\left(c_1 + 2a_1a_2(\alpha_2 - \beta_2)\right),$
$c_4 = 2a_1a_2(\omega_i - \gamma_1)(\alpha_2 - \beta_2),$
$B_1 = (\omega_i - \gamma_1)P_1, B_1B_2 + (\omega_i - \gamma_1)\left((\alpha_2 - \beta_2)I - B_2 - (\alpha_2 - \beta_2)P_1\right) = \mathbf{0},$
$i = 1\ or\ 2,\ (\omega_1, \omega_2) = (\alpha_1, \beta_1),$

(a_9) $c_1(\omega_i - \gamma_1) + c_3 = c_2(\alpha_2 - \beta_2) = -c_4,$
$B_1 = (\omega_i - \gamma_1)P_1,\ B_2 + (\alpha_2 - \beta_2)(P_1 - I) = \mathbf{0},\ i = 1\ or\ 2,\ (\omega_1, \omega_2) = (\alpha_1, \beta_1),$

(a_{10}) $c_1 = 0,\ c_2 = -2a_1a_2(\omega_i - \gamma_1),\ c_3 = -2a_1a_2(\omega_i - \gamma_1)(\alpha_2 - \beta_2) = -c_4,$
$B_1B_2 + (\omega_i - \gamma_1)\left((\alpha_2 - \beta_2)I - B_2 - (\alpha_2 - \beta_2)P_1\right) = \mathbf{0},$
$i = 1\ or\ 2,\ (\omega_1, \omega_2) = (\alpha_1, \beta_1),$

(a_{11}) $c_1(\omega_i - \gamma_1) + c_3 + c_4 = 0, c_2(\alpha_2 - \beta_2) + c_4 = 0,$
$c_1(\omega_j - \gamma_1) + 2a_1a_2(\omega_j - \gamma_1)(\alpha_2 - \beta_2) + c_3 = 0,$
$(\omega_i - \omega_j)\left(B_2 - (\alpha_2 - \beta_2)I\right) + (\alpha_2 - \beta_2)\left(B_1 - (\omega_j - \gamma_1)P_1\right) = \mathbf{0},$
$(\omega_i - \omega_j)B_1B_2 + (\omega_j - \gamma_1)(\alpha_2 - \beta_2)\left(B_1 - (\omega_i - \gamma_1)P_1\right) = \mathbf{0},$
$(i, j) = (1, 2)\ or\ (2, 1),\ (\omega_1, \omega_2) = (\alpha_1, \beta_1),$

(a_{12}) $c_1 = 0,\ c_2(\alpha_2 - \beta_2) = c_3 = -c_4,\ B_2 + (\alpha_2 - \beta_2)(P_1 - I) = \mathbf{0},$

(a_{13}) $c_1(\omega_i - \gamma_1) + c_3 + 2a_1a_2(\omega_i - \gamma_1)(\alpha_2 - \beta_2) = 0,\ c_2 = 0,\ c_4 = 0,$
$B_1 = (\omega_i - \gamma_1)P_1,\ B_1B_2 = (\omega_i - \gamma_1)(\alpha_2 - \beta_2)P_1,$
$i = 1\ or\ 2,\ (\omega_1, \omega_2) = (\alpha_1, \beta_1),$

(a_{14}) $c_1(\omega_i - \gamma_1) + c_3 = 0,\ c_2 = c_4 = 0,\ B_1 = (\omega_i - \gamma_1)P_1,\ B_1B_2 = \mathbf{0},$

$i = 1 \ or\ 2,\ (\omega_1, \omega_2) = (\alpha_1, \beta_1),$

(a_{15}) $c_1(\omega_j - \gamma_1) + 2a_1a_2(\omega_j - \gamma_1)(\alpha_2 - \beta_2) + c_3 = 0,$

$c_2 = c_4 = 0,\ c_3 + c_1(\omega_i - \gamma_1) = 0,$

$(\omega_i - \omega_j)B_1B_2 + (\omega_j - \gamma_1)(\alpha_2 - \beta_2)\left(B_1 - (\omega_i - \gamma_1)P_1\right) = \mathbf{0},$

$(i, j) = (1, 2)\ or\ (2, 1),\ (\omega_1, \omega_2) = (\alpha_1, \beta_1),$

where $c_1 = a_1^2(\alpha_1+\beta_1-2\gamma_1)+a_1(2a_3-\alpha_3-\beta_3)$, $c_2 = a_2^2(\alpha_2-\beta_2)+a_2(2a_3-\alpha_3-\beta_3)$, $c_3 = -a_1^2(\alpha_1-\gamma_1)(\beta_1-\gamma_1)$, $c_4 = (a_3 - \alpha_3)(a_3 - \beta_3)$, *and* $a_3 = a_1\gamma_1 + a_2\beta_2$.

Proof Let K represent the item (i) or (ii) or (iii) in Theorem 2.4, and L represent the item (a) or (b) or (c) or …(r) in Theorem 2.6. Mutual intersections of the items K and L easily lead to the coefficients included in the items of the theorem. To obtain the matrix equalities included in the items of the theorem, it is enough to put these equalities of coefficients into (2.16) taking into account the diagonal forms of the matrices B_1 and B_2, and also, considering that $c_3 \neq 0$, $\alpha_1 \neq \beta_1$, $\beta_1 \neq \gamma_1$, $\alpha_1 \neq \gamma_1$, and $\alpha_2 \neq \beta_2$. Which intersection corresponds to which item of the theorem is given in Table. □

Note that some intersections of the items K and L are naturally not included in the table because they contradict the corresponding hypotheses of the theorem.

Table . Summary of the intersections of the items K and L, and the corresponding items of the Theorem 2.7.

The items of Theorem 2.7	**Intersecting situations of K and L**
a_1 when $i = 2$ (when $i = 1$)	(i) and (a) ((i) and (h))
a_2 when $i = 2$ (when $i = 1$)	(i) and (b) ((i) and (k))
a_3 when $i = 2$ (when $i = 1$)	(i) and (c) ((i) and (d))
a_4 when $i = 2$ (when $i = 1$)	(i) and (r) ((i) and (n))
a_5 when $(i, j) = (1, 2)$ (when $(i, j) = (2, 1)$)	(i) and (f) ((i) and (l))
a_6	(i) and (m)
a_7 when $i = 2$ (when $i = 1$)	(ii) and (a) ((ii) and (h))
a_8 when $i = 2$ (when $i = 1$)	(ii) and (b)((ii) and (k))
a_9 when $i = 2$ (when $i = 1$)	(ii) and (c) ((ii) and (d))
a_{10} when $i = 2$ (when $i = 1$)	(ii) and (g) ((ii) and (p))
a_{11} when $(i, j) = (1, 2)$ (when $(i, j) = (2, 1)$)	(ii) and (f)((ii) and (l))
a_{12}	(ii) and (e)
a_{13} when $i = 2$ (when $i = 1$)	(iii) and (a) ((iii) and (h))
a_{14} when $i = 2$ (when $i = 1$)	(iii) and (c) ((iii) and (d))
a_{15} when $(i, j) = (1, 2)$ (when $(i, j) = (2, 1)$)	(iii) and (f) ((iii) and (l))

Observe that if $(\alpha_1, \beta_1, \gamma_1) \in \{(1, -1, 0), (-1, 1, 0), (1, 0, -1), (-1, 0, 1), (0, 1, -1), (0, -1, 1)\}$ and $(\alpha_2, \beta_2) \in \{(1, 0), (0, 1)\}$, and $(\alpha_3, \beta_3) \in \{(1, 0), (0, 1)\}$, we get the following result which gives a detailed analysis of Theorem 2 in [24] in case where the matrices involved in linear combination are commutative.

Corollary 2.8 (Theorem 2.4, [19]) *Let $A_1, A_2 \in \mathbb{C}_n \backslash \{\mathbf{0}\}$ be a tripotent and an idempotent matrix, respectively, with the assumption $A_1A_2 = A_2A_1$, and let $A = a_1A_1 + a_2A_2$ where $a_1, a_2 \in \mathbb{C}^*$. Then the matrix A is idempotent if and only if any of the following sets of conditions holds:*

(a) $(a_1, a_2) = (1, 1)$ *and one of the following matrix equalities:*

(a1) ${A_1}^2 = A_1, A_1 + A_2 = I$,

(a2) ${A_1}^2 = A_1, A_1A_2 = \mathbf{0}$,

(a3) ${A_1}^2 = I, A_2 = \frac{1}{2}(I - A_1)$,

(a4) $A_2 = \frac{1}{2}({A_1}^2 - A_1)$,

(a5) ${A_1}^2 = -A_1 = -A_1A_2$,

(a6) $-A_1A_2 = \frac{1}{2}({A_1}^2 - A_1)$,

(b) $(a_1, a_2) = (-1, 1)$ *and one of the following matrix equalities:*

(b1) ${A_1}^2 = -A_1, -A_1 + A_2 = I$,

(b2) ${A_1}^2 = -A_1, A_1A_2 = \mathbf{0}$,

(b3) ${A_1}^2 = I, A_2 = \frac{1}{2}(I + A_1)$,

(b4) $A_2 = \frac{1}{2}({A_1}^2 + A_1)$,

(b5) ${A_1}^2 = A_1 = A_1A_2$,

(b6) $A_1A_2 = \frac{1}{2}({A_1}^2 + A_1)$,

(c) $(a_1, a_2) = (-1, -1)$ *and one of the following matrix equalities:*

(c1) $A_1 = -I$,

(c2) ${A_1}^2 = -A_1, A_1A_2 = -A_2$,

(d) $(a_1, a_2) = (1, -1)$ *and one of the following matrix equalities:*

(d1) $A_1 = I$,

(d2) ${A_1}^2 = A_1, A_1A_2 = A_2$,

(e) $(a_1, a_2) = (-1, 2)$ *and one of the following matrix equalities:*

(e1) ${A_1}^2 = I, A_2 = \frac{1}{2}(I + A_1)$,

(e2) $A_2 = \frac{1}{2}({A_1}^2 + A_1)$,

(f) $(a_1, a_2) = (1, 2)$ *and one of the following matrix equalities:*

(f1) ${A_1}^2 = I, A_2{=}\frac{1}{2}(I - A_1)$,

(f2) $A_2{=}\frac{1}{2}({A_1}^2 - A_1)$,

(g) $(a_1, a_2) \in \{(\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, \frac{1}{2})\}$ *and* $A_1^2 = A_2$,

(h) $a_1, a_2 \in \mathbb{C}^*$ *with* $a_1 + a_2 = 0$ *or* $a_1 + a_2 = 1$; $A_1 = A_2$,

(i) $a_1, a_2 \in \mathbb{C}^*$ *with* $a_1 - a_2 = 0$ *or* $a_1 - a_2 = -1$; $A_1 = -A_2$.

If we consider the matrix identities that satisfy the condition $A_1^2 \neq \pm A_1$ in Corollary 2.8, then we immediately get the following result.

Corollary 2.9 (Corollary 2.5, [19], The item (a) of Theorem 1, [3]) *Let* A_1 *be an essentially tripotent matrix and let* A_2 *be a nonzero idempotent matrix such that* $A_1A_2 = A_2A_1$. *The linear combination of the form* $a_1A_1 + a_2A_2$ *is an idempotent matrix if and only if any of the following sets of conditions holds:*

(i) $(a_1, a_2) = (1, 1)$ *and* $A_1A_2 = \frac{1}{2}(A_1 - A_1^2)$,

(ii) $(a_1, a_2) = (1, 2)$ *and* $A_2 = \frac{1}{2}(A_1^2 - A_1)$,

(iii) $(a_1, a_2) = (-1, 1)$ *and* $A_1A_2 = \frac{1}{2}(A_1 + A_1^2)$,

(iv) $(a_1, a_2) = (-1, 2)$ *and* $A_2 = \frac{1}{2}(A_1^2 + A_1)$,

(v) $(a_1, a_2) \in \{(\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, \frac{1}{2})\}$ *and* $A_2 = A_1^2$.

If we consider the matrix identities that satisfy the conditions $A_1^2 = A_1$ and $A_1 \neq A_2$ in Corollary 2.8, then we simply obtain the following result.

Corollary 2.10 (Theorem (i), [2]) *Let* A_1 *and* A_2 *be two different nonzero idempotent matrices that commute. Let* A_3 *be their linear combination of the form* $A_3 = a_1A_1 + a_2A_2$ *with* $a_1, a_2 \in \mathbb{C}^*$. *Then there are exactly three situations, where* A_3 *is an idempotent matrix:*

(i) $(a_1, a_2) = (1, 1)$, $A_1A_2 = \mathbf{0}$,

(ii) $(a_1, a_2) = (1, -1)$, $A_1A_2 = A_2$,

(iii) $(a_1, a_2) = (-1, 1)$, $A_1A_2 = A_1$.

If $(\alpha_1, \beta_1, \gamma_1) \in \{(1, -1, 0), (-1, 1, 0), (1, 0, -1), (-1, 0, 1), (0, 1, -1), (0, -1, 1)\}$, $(\alpha_2, \beta_2) \in \{(1, -1), (-1, 1)\}$, and $(\alpha_3, \beta_3) \in \{(1, 0), (0, 1)\}$ in Theorem 2.7, then we easily get the following result.

Corollary 2.11 *Let* $A_1, A_2 \in \mathbb{C}_n$ *be a tripotent and an involutive matrix, respectively, such that* $A_1A_2 = A_2A_1$, *and let* $A = a_1A_1 + a_2A_2$ *where* $a_1, a_2 \in \mathbb{C}^*$. *Then* A *is idempotent if and only if any of the following sets of conditions holds:*

(a) $(a_1, a_2) = (1, 1)$ *and one of the following matrix equalities:*

(a1) $A_1^2 = A_1 = \frac{1}{2}(I - A_2)$,

(a2) $A_1^2 = -A_1$, $A_2 = I$,

(a3) $A_1 + A_2 = I + A_1A_2$, $A_1A_2 = -A_1^2$,

(a4) $A_1^2 + A_1 = I - A_2$,

(a5) $A_1^2 = A_1$, $2A_1 + A_2 = I$,

(a6) $A_1 + A_2 = I + A_1A_2$,

(b) $(a_1, a_2) = (-1, 1)$ *and one of the following matrix equalities:*

(b1) $A_1^2 = -A_1 = \frac{1}{2}(I - A_2)$,

(b2) $A_1^2 = A_1$, $A_2 = I$,

(b3) $-A_1 + A_2 = I - A_1A_2$, $A_1A_2 = A_1^2$,

(b4) $A_1^2 - A_1 = I - A_2$,

(b5) $A_1^2 = -A_1$, $2A_1 - A_2 = -I$,

(b6) $-A_1 + A_2 = I - A_1A_2$,

(c) $(a_1, a_2) = (1, -1)$ *and one of the following matrix equalities:*

(c1) $A_1^2 = A_1 = \frac{1}{2}(I + A_2)$,

(c2) $A_1^2 = -A_1$, $A_2 = -I$,

(c3) $A_1 - A_2 = I - A_1A_2$, $A_1A_2 = A_1^2$,

(c4) $A_1^2 + A_1 = I + A_2$,

(c5) $A_1^2 = A_1$, $2A_1 - A_2 = I$,

(c6) $A_1 - A_2 = I - A_1A_2$,

(d) $(a_1, a_2) = (-1, -1)$ *and one of the following matrix equalities:*

(d1) $A_1^2 = -A_1 = \frac{1}{2}(I + A_2)$,

(d2) $A_1^2 = A_1$, $A_2 = -I$,

(d3) $-A_1 - A_2 = I + A_1A_2$, $A_1A_2 = -A_1^2$,

(d4) $A_1^2 - A_1 = I + A_2$,

(d5) $A_1^2 = -A_1\,,\ 2A_1 + A_2 = -I\,,$

(d6) $-A_1 - A_2 = I + A_1 A_2\,,$

(e) $(a_1, a_2) = (2, 1)$ *and one of the following matrix equalities:*

(e1) $A_1^2 = A_1 = \frac{1}{2}(I - A_2)\,,$

(e2) $A_1 A_2 = -A_1^2\,,$

(e3) $A_1^2 = A_1\,,\ 2A_1 + A_2 = I\,,$

(f) $(a_1, a_2) = (-2, 1)$ *and one of the following matrix equalities:*

(f1) $A_1^2 = -A_1 = \frac{1}{2}(I - A_2)\,,$

(f2) $A_1 A_2 = A_1^2\,,$

(f3) $A_1^2 = -A_1\,,\ 2A_1 - A_2 = -I\,,$

(g) $(a_1, a_2) = (2, -1)$ *and one of the following matrix equalities:*

(g1) $A_1^2 = A_1 = \frac{1}{2}(I + A_2)\,,$

(g2) $A_1 A_2 = A_1^2\,,$

(g3) $A_1^2 = A_1\,,\ 2A_1 - A_2 = I\,,$

(h) $(a_1, a_2) = (-2, -1)$ *and one of the following matrix equalities:*

(h1) $A_1^2 = -A_1 = \frac{1}{2}(I + A_2)\,,$

(h2) $A_1 A_2 = -A_1^2\,,$

(h3) $A_1^2 = -A_1\,,\ 2A_1 + A_2 = -I\,,$

(i) $(a_1, a_2) = (\frac{1}{2}, \frac{1}{2})$ *and one of the following matrix equalities:*

(i1) $A_1^2 = I\,,\ I + A_1 A_2 = A_1 + A_2\,,$

(i2) $A_1^2 = A_2 = I\,,$

(j) $(a_1, a_2) = (-\frac{1}{2}, \frac{1}{2})$ *and one of the following matrix equalities:*

(j1) $A_1^2 = I\,,\ I - A_1 A_2 = -A_1 + A_2\,,$

(j2) $A_1^2 = A_2 = I\,,$

(k) $(a_1, a_2) = (\frac{1}{2}, -\frac{1}{2})$ *and one of the following matrix equalities:*

(k1) $A_1^2 = I\,,\ I - A_1 A_2 = A_1 - A_2\,,$

(k2) $A_1^2 = -A_2 = I$,

(l) $(a_1, a_2) = (-\frac{1}{2}, -\frac{1}{2})$ *and one of the following matrix equalities:*

(l1) $A_1^2 = I$, $I + A_1A_2 = -A_1 - A_2$,

(l2) $A_1^2 = -A_2 = I$,

(m) $a_1, a_2 \in \mathbb{C}^*$ *with* $a_1 + a_2 = 0$; $A_1 = A_2$,

(n) $a_1, a_2 \in \mathbb{C}^*$ *with* $a_1 - a_2 = 0$; $A_1 = -A_2$.

If we consider the matrix identities that satisfy the conditions $A_1^2 = I$ and $A_1 \neq \pm A_2$ in Corollary 2.11, then we get the following result.

Corollary 2.12 (Theorem 2.2 (i), [20]) *Let* A_1 *and* A_2 *be two involutive matrices with* $A_1 \neq \pm A_2$ *and* $A_1A_2 = A_2A_1$. *Consider linear combination* $A_3 = a_1A_1 + a_2A_2$ *with* $a_1, a_2 \in \mathbb{C}^*$. *The matrix* A_3 *is an idempotent matrix if and only if any of the following sets of additional conditions holds:*

(i) $(a_1, a_2) = (-\frac{1}{2}, -\frac{1}{2})$ *and* $-A_1 - A_2 = I + A_1A_2$,

(ii) $(a_1, a_2) = (\frac{1}{2}, \frac{1}{2})$ *and* $A_1 + A_2 = I + A_1A_2$,

(iii) $(a_1, a_2) = (-\frac{1}{2}, \frac{1}{2})$ *and* $-A_1 + A_2 = I - A_1A_2$,

(iv) $(a_1, a_2) = (\frac{1}{2}, -\frac{1}{2})$ *and* $A_1 - A_2 = I - A_1A_2$.

If $(\alpha_1, \beta_1, \gamma_1) \in \{(1,-1,0), (-1,1,0), (1,0,-1), (-1,0,1), (0,1,-1), (0,-1,1)\}$, $(\alpha_2, \beta_2) \in \{(1,0),(0,1)\}$, and $(\alpha_3, \beta_3) \in \{(1,-1),(-1,1)\}$ in Theorem 2.7, and we consider the matrix identities that satisfy $A_1^2 = A_1$, then we get Corollary 2.5 (i) in [18].

If $(\alpha_1, \beta_1, \gamma_1) \in \{(1,-1,0), (-1,1,0), (1,0,-1), (-1,0,1), (0,1,-1), (0,-1,1)\}$, $(\alpha_2, \beta_2) \in \{(1,-1),(-1,1)\}$, and $(\alpha_3, \beta_3) \in \{(1,-1),(-1,1)\}$ in Theorem 2.7, then we get Corollary 2 in [9].

If $(\alpha_1, \beta_1, \gamma_1) \in \{(1,-1,0), (-1,1,0), (1,0,-1), (-1,0,1), (0,1,-1), (0,-1,1)\}$, $(\alpha_2, \beta_2) \in \{(1,0),(0,1)\}$, and $(\alpha_3, \beta_3) \in \{(1,-1),(-1,1)\}$ in Theorem 2.7, and we consider the matrix identities that satisfy $A_1^2 = -A_1$ and $A_1 \neq \pm A_2$, then we get Corollary 3 in [9].

If $(\alpha_1, \beta_1, \gamma_1) \in \{(1,-1,0), (-1,1,0), (1,0,-1), (-1,0,1), (0,1,-1), (0,-1,1)\}$, $(\alpha_2, \beta_2) \in \{(1,0),(0,1)\}$, and $(\alpha_3, \beta_3) \in \{(1,-1),(-1,1)\}$ in Theorem 2.7, and we consider the matrix identities that satisfy $A_1^2 = I$ and $A_1 \neq \pm A_2$, then we get Corollary 4 in [9].

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