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On unbounded order continuous operators

Bahri TURAN*, **Birol ALTIN**, **Hüma GÜRKÖK**
Department of Mathematics, Faculty of Science, Gazi University, Ankara, Turkey

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Abstract: Let U and V be two Archimedean Riesz spaces. An operator $S : U \to V$ is said to be unbounded order continuous (uo-continuous), if $r_\alpha \xrightarrow{uo} 0$ in U implies $Sr_\alpha \xrightarrow{uo} 0$ in V. In this paper, we give some properties of the uo-continuous dual $U_{uo}^\sim$ of U. We show that a nonzero linear functional f on U is uo-continuous if and only if f is a linear combination of finitely many order continuous lattice homomorphisms. The result allows us to characterize the uo-continuous dual $U_{uo}^\sim$. In general, by giving an example that the uo-continuous dual $U_{uo}^\sim$ is not a band in $U^\sim$, we obtain the conditions for the uo-continuous dual of a Banach lattice U to be a band in $U^\sim$. Then, we examine the properties of uo-continuous operators. We show that S is an order continuous operator if and only if S is an unbounded order continuous operator when S is a lattice homomorphism between two Riesz spaces U and V. Finally, we proved that if an order bounded operator $S : U \to V$ between Archimedean Riesz space U and atomic Dedekind complete Riesz space V is uo-continuous, then $|S|$ is uo-continuous.

Key words: Riesz space, order convergence, unbounded order convergence, unbounded order continuous operator, unbounded order continuous dual

1. Introduction

In the literature on the Riesz space theory, the order convergence of nets is defined in different ways, and the relations of these different definitions with each other are examined in [2]. We consider the following definition: A net (r_α) in a Riesz space U is said to be order convergent to $r \in U$, if there is a net (q_β) in U with $q_\beta \downarrow 0$ and that for every β, there exists α_0 satisfying $|r_\alpha - r| \leq q_\beta$ for all $\alpha \geq \alpha_0$, and it is denoted by $r_\alpha \xrightarrow{o} r$. If U is a Dedekind complete Riesz space, and $(r_\alpha) \subseteq U$ is order bounded net, then instead of (q_β), we can choose (q_α) which has the same index as (r_α). A net (r_α) in a Riesz space U is unbounded order convergent to $r \in U$ if $|r_\alpha - r| \wedge u \xrightarrow{o} 0$ for all $u \in U_+$. We denote this convergence by $r_\alpha \xrightarrow{uo} r$ and write that (r_α) uo-converge to r. The unbounded order convergence was studied firstly by Nakano in [17], and later the name ”unbounded order convergence” was proposed by De Marr in [7]. It is known that the uo-convergence and coordinate-wise convergence of nets are equivalent in any atomic Riesz space [6, Lemma 3.1], and the uo-convergence of a sequence (r_n) and its a.e. convergence are equivalent in $L_p(\mu)$ $(1 \leq p < \infty)$ space. Since these relations widen the area of applications, the concept of uo-converge has recently been intensively studied in many papers [$4, 6, 8 - 14, 18, 19$].

Let U and V be Riesz spaces. We denote the set of all order bounded (order continuous) operators from U into V by $L_b(U,V)$ $(L_n(U,V))$. Also, if $V = \mathbb{R}$ then $L_b(U,V)$ $(L_n(U,V))$ is denoted by $U^\sim$ $(U_n^\sim)$, and it

*Correspondence: bturan@gazi.edu.tr
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is called the order (order continuous) dual of U. An operator $S : U \to V$ between two Riesz spaces is said to be unbounded order continuous (uo-continuous) if $r_\alpha \xrightarrow{uo} 0$ in U implies $Sr_\alpha \xrightarrow{uo} 0$ in V. The uo-continuous operators were studied by Bahramnezhad and Azar in [5]. The space of all uo-continuous order bounded operators is denoted by $L_{uo}(U,V)$. The space $L_{uo}(U,\mathbb{R})$ is denoted by $U^\sim_{uo}$, and it is called unbounded order continuous dual of U.

In this paper, our central focus is the class of uo-continuous order bounded operators. The paper rests essentially on two parts. The first part is dedicated to the properties of $U^\sim_{uo}$. The second is devoted to the properties of uo-continuous order bounded operators.

We refer to $[1,3,15,16,20]$ for unexplained terminology and facts on Riesz spaces and positive operators which are not explained here. All Riesz spaces in this paper are considered Archimedean.

2. The unbounded order continuous dual

In [12], a linear functional f on U is said to be bounded uo-continuous if $f(r_\alpha) \to 0$ for any norm bounded uo-null net (r_α) in U. They considered the space of boundedly uo-continuous functionals as unbounded order continuous dual of U and they studied its properties. In another study [18], Li et al. considered the same definition. In this study, we only take (order bounded) uo-continuous functionals as unbounded order continuous dual of U.

Remark 2.1 *It is easy to see that every uo-continuous functional is an o-continuous functional, and so $U^\sim_{uo}$ is a subspace of $U^\sim_n$. Since every o-continuous operator is an order bounded operator, every uo-continuous functional is an order bounded functional. Therefore, the order boundedness condition can be removed in the definition of $U^\sim_{uo}$.*

It is clear that $r_\alpha \xrightarrow{o} 0$ in U implies $r_\alpha \xrightarrow{uo} 0$, the converse is not true. Let $U = c_0$, and let (e_n) be the standard basis of U. Then (e_n) uo-converges to 0, but it is not order convergent to zero. Let us give in which case these convergences are equivalent.

Example 2.2 *Order convergence and unbounded order convergence are equivalent in the real number $\mathbb{R}$ and $\mathbb{R}^n$, where $n \geq 1$. By Theorem 26.11 in [15], any Archimedean Riesz space U of finite dimension n, where $n \geq 1$, is Riesz isomorphic to n-dimensional real number space $\mathbb{R}^n$ with the usual coordinate-wise ordering. Also, order convergence and unbounded order convergence are equivalent in Archimedean Riesz space U of finite dimension n.*

On the other hand, considering Theorem 4 in [8], the following proposition is obtained.

Proposition 2.3 *Let U be an Archimedean Riesz space. Order convergence and unbounded order convergence are equivalent in U iff U is a finite dimension space.*

Now we examine the properties of the unbounded order continuous dual of U. Recall that a vector $u > 0$ in a Riesz space is said to be an atom whenever $0 \leq r \leq u$, $0 \leq q \leq u$ and $r \wedge q = 0$ imply that either $r = 0$ or $q = 0$, and a discrete vector whenever for each $0 \leq r \leq u$, there exists some $\lambda \geqslant 0$ such that $r = \lambda u$. In an Archimedean Riesz space, a positive vector is an atom if and only if it is a discrete vector. Moreover, in this case, the vector space $\{\lambda u : \lambda \in \mathbb{R}\}$ generated by an atom u is a projection band [1, Lemma 2.30]. Let u be an atom in a Riesz space U and let $P_u : U \to B_u = \{\lambda u : \lambda \in \mathbb{R}\}$ be the band projection. Then there is a unique positive linear functional f_u on U such that $P_u(r) = f_u(r)u$ for all $r \in U$. The functional f_u is called the coordinate functional of the atom u. Clearly, the coordinate functional f_u is order continuous lattice homomorphism. By Exercise 6 in [1, p.88], $0 < f \in U^\sim$ is an atom if and only if f is a lattice homomorphism. Therefore, if U has an atom then $U_n^\sim$ has an atom. A Riesz space is said to be atomless (or nonatomic) if it does not have any atoms, and it is said to be atomic if it has a complete disjoint system consisting of atoms of U. Let us define the sets $\mathcal{F}$ and $\mathcal{L}$ as follows:

$$\mathcal{F} = \left\{ \sum_{i=1}^{n} \lambda_i f_i : n \in \mathbb{N}, \lambda_i \in \mathbb{R},\ f_i : U \to \mathbb{R} \text{ order continuous lattice homomorphism} \right\}$$

if $U_n^\sim$ has at least one atom, and $\mathcal{F} = \{0\}$ if $U_n^\sim$ does not have any atoms,

$$\mathcal{L} = \left\{ \sum_{i=1}^{n} \lambda_i f_i : n \in \mathbb{N}, \lambda_i \in \mathbb{R},\ f_i : U \to \mathbb{R} \text{ the coordinate functionals on } U \right\}$$

if U has at least one atom, and $\mathcal{L} = \{0\}$ if U does not have any atoms. It is clear that $\mathcal{L} \subseteq \mathcal{F}$.

Proposition 2.4 *Let U be a Riesz space. Then $\mathcal{F}$ is an ideal in $U_n^\sim$ (or $U^\sim$).*

Proof It is easy to see that $\mathcal{F}$ is a vector subspace of $U_n^\sim$. If $\mathcal{F} = \{0\}$, then $\mathcal{F}$ is an ideal. Let $\mathcal{F} \neq \{0\}$. Let $g \in U_n^\sim$, and suppose $|g| \leq |f|$ for some $f \in \mathcal{F}$. By the definition of $\mathcal{F}$, there exist $n \in \mathbb{N}$, $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R}$ and $f_1, f_2, ..., f_n$ lattice homomorphisms in $U_n^\sim$ such that $f = \sum_{i=1}^{n} \lambda_i f_i$. By Theorem 1.13 in [3], there exist $g_1, g_2, ..., g_n, h_1, h_2, ..., h_n \in U_n^\sim$ satisfying $g^+ = g_1 + g_2 + ... + g_n$, $0 \leq g_i \leq |\lambda_i f_i| = |\lambda_i| f_i$ and $g^- = h_1 + h_2 + ... + h_n$, $0 \leq h_i \leq |\lambda_i f_i| = |\lambda_i| f_i$ for each $i = 1, ..., n$, because $0 \leq g^+ \leq |g| \leq \sum_{i=1}^{n} |\lambda_i| f_i$ and $0 \leq g^- \leq |g| \leq \sum_{i=1}^{n} |\lambda_i| f_i$. Since f_i is atom in $U_n^\sim$, there exist $\beta_i, \gamma_i \in \mathbb{R}$ such that $g_i = \beta_i f_i$ and $h_i = \gamma_i f_i$ for all $i = 1, ..., n$. Therefore, we have $g = g^+ - g^- = \sum_{i=1}^{n} (\beta_i - \gamma_i) f_i \in \mathcal{F}$. □

Proposition 2.5 *Let U be a Riesz space. Then $\mathcal{F} = U_{uo}^\sim = \mathcal{L}$.*

Proof It is clear that every order continuous lattice homomorphism from U to $\mathbb{R}$ is *uo*-continuous, then we have $\mathcal{F} \subseteq U_{uo}^\sim$. By Proposition 2.2 in [12], we have $U_{uo}^\sim \subseteq \mathcal{L}$. Thus, $\mathcal{F} \subseteq U_{uo}^\sim \subseteq \mathcal{L} \subseteq \mathcal{F}$ also holds, and the proof is finished. □

By Theorem 1 and Proposition 2 in [5], it is known that $U_{uo}^\sim$ is an ideal of $U^\sim$. Using the above Proposition, we give an alternative proof. In addition, we easily obtain the following results and examples from the same proposition.

Corollary 2.6 *Let U be a Riesz space. Then $U_{uo}^\sim$ is an ideal of $U_n^\sim$ (or $U^\sim$).*

Corollary 2.7 *Let U be a Riesz space. Then the following conditions are equivalent.*
(i) U is atomless.
(ii) $\mathcal{L} = \{0\}$.
(iii) $\mathcal{F} = \{0\}$.
(iv) $U_{uo}^\sim = \{0\}$.

Corollary 2.8 *For a Riesz space U the following statements hold:*
(i) If U is atomless, then its order continuous dual $U_n^\sim$ is likewise an atomless Riesz space.
(ii) If $U^\sim$ is atomless, then U is also atomless.

The above Corollary 2.8 is the general case of Lemma 2.31 given in [1]. Namely, if U is Banach lattice with order continuous norm, then $U' = U^\sim = U_n^\sim$ holds. Considering these equations, and the above Corollary 2.8, we obtain Lemma 2.31 given in [1].

Corollary 2.9 *For a Banach lattice U the following statements hold:*
(1) If U has an order continuous norm and is atomless, then its norm dual U' is likewise an atomless Banach lattice.
(2) If U' is atomless, then U is also atomless.

Example 2.10 *Since $L_1[0,1]$ and $C[0,1]$ are atomless, then $L_1[0,1]_{uo}^\sim = \{0\}$ and $C[0,1]_{uo}^\sim = \{0\}$.*

Example 2.11 *Let $U = c_0$, then its uo-dual $U_{uo}^\sim = c_{00}$ consists of all sequences which have only finitely many nonzero elements.*

By Example 2.11, $U_{uo}^\sim$ is different from $U_n^\sim$ since $(c_0)_n^\sim = c_0$ holds. We obtained above that $U_{uo}^\sim$ is an ideal in $U^\sim$. If $U = C[0,1]$, then $U_{uo}^\sim = \{0\}$. In these obvious cases, $U_{uo}^\sim$ is a band in $U^\sim$. In general, $U_{uo}^\sim$ is not a band in $U^\sim$.

Example 2.12 *Let $U = l_1$. Let us define*

$$f_k : U \to \mathbb{R},\ (\alpha_i) \to f_k(\alpha_i) = \sum_{i=1}^{k} \alpha_i.$$

f_k is uo-continuous for each k. If we take $f(\alpha_i) = \sum_{i=1}^{\infty} \alpha_i$, then $f \in U^\sim$ and $f_k \uparrow f$, but f is not uo-continuous. Let (e_n) be the standard unit sequence in U. Then the sequence (e_n) is uo-null since every disjoint sequence is uo-null. On the other hand, $f(e_n)$ is not uo-null because $f(e_n) = 1$ for each n. Thus, we obtain $f \notin U_{uo}^\sim$. It follows $U_{uo}^\sim$ is not a band in $U^\sim$.

Alpay et al. defined the *bbuo*-convergence of a net and the *bbuo*-dual ($U_{bbuo}^\sim$ for short) of Riesz space U. They obtained the order continuous dual relation with bbuo-dual of U. It was proved that if $U_n^\sim$ separates points of a Riesz space U then $U_{bbuo}^\sim = U_n^\sim$ [4]. The situation is quite different for $U_{uo}^\sim$ whenever U is a Banach lattice, as the following proposition shows.

Proposition 2.13 *Let U be a Banach lattice and such that $U_{uo}^\sim$ separates the points of U. Then the following conditions are equivalent.*
(i) U is a finite dimension space.
(ii) $U_{uo}^\sim = U_n^\sim$.
(iii) $U_{uo}^\sim$ is a band in $U^\sim$.

Proof (i)$\Rightarrow$(ii) It is easily obtained from Example 2.2.
(ii)$\Rightarrow$(iii) Since $U_n^\sim$ is a band in $U^\sim$, $U_{uo}^\sim$ *is a band in* $U^\sim$.

(iii)$\Rightarrow$(i) We assume that $U_{uo}^{\sim}$ is a band, and U is an infinite dimension space. Then there exists sequence (r_n) in U such that (r_n) is disjoint and $r_n \neq 0$ for each n. Since $U_{uo}^{\sim}$ separates the points of U, there exists sequence (f_n) in $(U_{uo}^{\sim})_+$ such that $f_n(r_n) = 1$ for each n. Let us define

$$g_n : U \to \mathbb{R},\ g_n(r) = \sum_{i=1}^{n} \frac{f_i(r)}{i^2 \|f_i\|}.$$

It is clear that g_n is uo-continuous for each n. If we take $g(r) = \sum_{i=1}^{\infty} \frac{f_i(r)}{i^2\|f_i\|}$ then $g \in U^{\sim}$ and $g_n \uparrow g$, but g is not uo-continuous. Let $(q_n) = n^2 \|f_n\| r_n$. (q_n) is uo-null since every disjoint sequence is uo-null. On the other hand, $g(q_n)$ is not uo-null because $g(q_n) \geq 1$ for each n. Thus, we obtain $g \notin U_{uo}^{\sim}$. This is a contradiction. □

3. The uo-continuous order bounded operators

It is known that for an order bounded net (r_α) in a Dedekind complete vector lattice,

$$r_\alpha \xrightarrow{o} 0 \text{ iff } \inf_\alpha \sup_{\beta\geq\alpha} |r_\beta| = 0 \text{ iff } 0 = \inf_\alpha \sup_{\beta\geq\alpha} |r_\beta| = \sup_\alpha \inf_{\beta\geq\alpha} |r_\beta|.$$

Let U be a Riesz space, and let U^δ be Dedekind completion of U. Then $r_\alpha \xrightarrow{o} 0$ in U iff $r_\alpha \xrightarrow{o} 0$ in U^δ [11, Corollary 2.9]. Thus, we have

$$\begin{aligned}\inf_\alpha \sup_{\beta\geq\alpha} |r_\beta \wedge q| &= 0 \text{ iff } 0 = \inf_\alpha \sup_{\beta\geq\alpha} |r_\beta \wedge q| = \sup_\alpha \inf_{\beta\geq\alpha} |r_\beta \wedge q| \text{ for all } q \in U_+ \\ &\text{iff } r_\alpha \xrightarrow{uo} 0 \text{ in } U^\delta \text{ iff } r_\alpha \xrightarrow{uo} 0 \text{ in } U\end{aligned}$$

for all (r_α) in U. Therefore, without loss of generality, we may assume that U is Dedekind complete in some uo-convergence studies.

Lemma 3.1 *Let (r_α) be a net in a Riesz space U. If (r_α) is $uo-$null in U, then* $\inf_\alpha |r_\alpha| = 0$.

Proof Let $r_\alpha \xrightarrow{uo} 0$ and $0 \leq t \leq |r_\alpha|$ for each α in U. We have to show that $t = 0$. Since $r_\alpha \xrightarrow{uo} 0$, we have $|r_\alpha| \wedge u \xrightarrow{o} 0$ for each $u \in U_+$. Take $u = t$, we see that $0 \leq t \leq |r_\alpha| \wedge t \xrightarrow{o} 0$, the inequality yields $t = 0$, as desired. □

Proposition 3.2 *Let $S : U \to V$ be an order bounded operator between two Riesz spaces with V Dedekind complete. If S is an uo-continuous operator, then S^+, S^-, S, and $|S|$ are o-continuous operators.*

Proof Let us first show that S^+ is order continuous. It is enough to show that $S^+(r_\alpha) \downarrow 0$ when $r_\alpha \downarrow 0$. Let $r_\alpha \downarrow 0$ in U, and let $0 \leq t \leq S^+(r_\alpha) \downarrow$. Fix some index α_0 and for each $0 \leq q \leq r_{\alpha_0}$ we have

$$0 \leq q - q \wedge r_\alpha \leq q \wedge r_{\alpha_0} - q \wedge r_\alpha \leq r_{\alpha_0} - r_\alpha$$

for each $\alpha \geq \alpha_0$. Thus, we obtain

$$\begin{aligned} S(q) - S(q \wedge r_\alpha) &= S(q - q \wedge r_\alpha) \\ &\leq S^+(q - q \wedge r_\alpha) \\ &\leq S^+(r_{\alpha_0} - r_\alpha) \\ &\leq S^+(r_{\alpha_0}) - S^+(r_\alpha) \end{aligned}$$

from which it follows that

$$0 \leq t \leq S^+(r_\alpha) \leq S^+(r_{\alpha_0}) - S(q) + S(q \wedge r_\alpha) \leq S^+(r_{\alpha_0}) - S(q) + |S(q \wedge r_\alpha)|$$

holds for all $\alpha \geq \alpha_0$ and all $0 \leq q \leq r_{\alpha_0}$. From $r_\alpha \downarrow 0$, we see that $q \wedge r_\alpha \downarrow_{\alpha \geq \alpha_0} 0$, and hence $q \wedge r_\alpha \xrightarrow{o} 0$. This implies $q \wedge r_\alpha \xrightarrow{uo} 0$, it then follows from our hypothesis that $S(q \wedge r_\alpha) \xrightarrow{uo} 0$, and hence from Lemma 3.1 $\inf_{\alpha \geq \alpha_0} |S(q \wedge r_\alpha)| = 0$. Using this, and in view $\sup\limits_{0 \leq q \leq r_{\alpha_0}} S(q) = S^+(r_{\alpha_0})$, we get

$$\begin{aligned} 0 \leq t \leq S^+(r_{\alpha_0}) - S(q) &\Rightarrow 0 \leq t + S(q) \leq S^+(r_{\alpha_0}) \\ &\Rightarrow 0 \leq t + \sup_{0 \leq q \leq r_{\alpha_0}} S(q) \leq S^+(r_{\alpha_0}) \\ &\Rightarrow 0 \leq t + S^+(r_{\alpha_0}) \leq S^+(r_{\alpha_0}) \\ &\Rightarrow t = 0 \end{aligned}$$

and the proof is finished. From the identity $S^- = (-S)^+$, $S = S^+ - S^-$, and $|S| = S^+ + S^-$, we obtain S^-, S, and $|S|$ are order continuous operators. □

In the following Theorem, we give the condition that an o-continuous operator and an uo-continuous operator are the same.

Theorem 3.3 *Let $S : U \to V$ be a lattice homomorphism between two Riesz spaces. Then S is o-continuous if and only if S is uo-continuous.*

Proof Let S be an order continuous operator and let $0 \leq r_\alpha \xrightarrow{uo} 0$ in U. We shall show that $Sr_\alpha \wedge w \xrightarrow{o} 0$ for each $w \in V_+$. Considering the explanation above, it is equivalent for S to be *uo*-continuous from U to V and for S to be *uo*-continuous from U to Dedekind completion of V. Thus, we may assume that V is Dedekind complete. Step 1: Let $0 \leq w$ be an element of $S(U)$. There exists $u \in U_+$ such that $Su = w$ since S is a lattice homomorphism. Thus by the assumption, we have

$$\begin{aligned} r_\alpha \xrightarrow{uo} 0 &\Rightarrow r_\alpha \wedge u \xrightarrow{o} 0 \\ &\Rightarrow S(r_\alpha \wedge u) \xrightarrow{o} 0 \\ &\Rightarrow S(r_\alpha) \wedge Su \xrightarrow{o} 0 \\ &\Rightarrow S(r_\alpha) \wedge w \xrightarrow{o} 0. \end{aligned}$$

Step 2: Let w be an element of the ideal generated by $S(U)$ in V. There exist $n \in \mathbb{N}$, $u_1, u_2, ..., u_n \in U_+$ and

$\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R}_+$ such that $w \leq \sum_{i=1}^{n} \lambda_i S u_i$. Thus, we have

$$0 \leq S(r_\alpha) \wedge w \leq S(r_\alpha) \wedge \sum_{i=1}^{n} \lambda_i S u_i \leq \sum_{i=1}^{n} S(r_\alpha) \wedge \lambda_i S u_i.$$

By step 1, it is easy to see that $S(r_\alpha) \wedge w \xrightarrow{o} 0$. Step 3: Let w be an element of the band generated by $S(U)$ in V. There exists (w_β) in the ideal generated by $S(U)$ such that $0 \leq w_\beta \uparrow w$. By step 2, we have $S(r_\alpha) \wedge w_\beta \xrightarrow{o} 0$ for every β. Therefore,

$$\begin{aligned}
\inf_\alpha \sup_{\gamma \geqslant \alpha} (S(r_\gamma) \wedge w_\beta) = 0 \quad &\Rightarrow \quad \inf_\alpha \sup_{\gamma \geqslant \alpha} (S(r_\gamma) \wedge w_\beta) \wedge w = 0 \\
&\Rightarrow \quad \inf_\alpha \sup_{\gamma \geqslant \alpha} (S(r_\gamma) \wedge w) \wedge w_\beta = 0 \\
&\Rightarrow \quad \sup_\beta \left[\inf_\alpha \sup_{\gamma \geqslant \alpha} (S(r_\gamma) \wedge w) \wedge w_\beta \right] = 0 \\
&\Rightarrow \quad \inf_\alpha \sup_{\gamma \geqslant \alpha} (S(r_\gamma) \wedge w) \wedge w = 0 \\
&\Rightarrow \quad \inf_\alpha \sup_{\gamma \geqslant \alpha} (S(r_\gamma) \wedge w) = 0 \\
&\Rightarrow \quad S(r_\alpha) \wedge w \xrightarrow{o} 0.
\end{aligned}$$

Finally, let $w \in V_+$ and let B be the band generated by $S(U)$ in V. Since $V = B \oplus B^d$, there exist $0 \leq w_1 \in B$ and $0 \leq w_2 \in B^d$ such that $w = w_1 + w_2$. As $S(r_\alpha) \wedge w_2 = 0$ for each α, we get

$$0 \leq S(r_\alpha) \wedge w = S(r_\alpha) \wedge (w_1 + w_2) \leq (S(r_\alpha) \wedge w_1) + (S(r_\alpha) \wedge w_2) = S(r_\alpha) \wedge w_1.$$

By step 3, we have $S(r_\alpha) \wedge w \xrightarrow{o} 0$. Conversely, it is seen from Proposition 4 in [5]. □

As a consequence of the preceding theorem, we obtain the following result.

Corollary 3.4 *Let U and V be Riesz spaces. For an order bounded disjoint preserving operator S from U to V is o-continuous if and only if it is uo-continuous.*

Proof If $S : U \to V$ is an order bounded disjoint preserving operator, then its modulus exists and $|S| : U \to V$ is a lattice homomorphism [3, Theorem 2.40]. Using Theorem 3.3, the proof is easily obtained. □

Corollary 3.5 *Let U be a Riesz space. For any orthomorphism π is uo-continuous. Therefore, any order projection and central operator are uo-continuous since any order projection and central operator are orthomorphism.*

If $S : U \to V$ is an order bounded operator between two Riesz spaces, then its (order) adjoint $S^\sim : V^\sim \to U^\sim$ is order bounded and o-continuous [3, Theorem 1.73]. In general, $S^\sim$ is not uo-continuous.

Example 3.6 *Let $U = l_1$ and let $V = s$ the Riesz space of all real-valued sequences with pointwise order. Consider the operator $S : U \to V$ defined by*

$$S(r_n) = (r_1, r_1 + r_2, ..., \sum_{i=1}^{n} r_i, ...).$$

Let us define the $f_i : F \to \mathbb{R}$ *functionals as follows:*

$$f_i(r) = f_i(r_n) = r_i$$

for all $i \in \mathbb{N}$ *and* $r = (r_n) \in V$. *It is clear that the sequence* (f_n) *is in* $V^{\sim}$ *and* (f_n) *is disjoint sequence, and hence* $f_n \xrightarrow{uo} 0$. *Since* $S^{\sim}(f_n) \uparrow (1, 1, ...1, ...)$ *in* $l_\infty \cong l_1^{\sim}$, $S^{\sim}(f_n)$ *is not uo-null.*

Corollary 3.7 *If* $S : U \to V$ *is an interval preserving operator between two Riesz spaces, then* $S^{\sim}$ *is uo-continuous.*

Proof Let $S : U \to V$ be an interval preserving operator, then its order adjoint $S^{\sim} : V^{\sim} \to U^{\sim}$ is an order continuous lattice homomorphism [3, Theorem 2.19]. By Theorem 3.3, $S^{\sim}$ is *uo*-continuous. □

The following proposition gives us a partial solution to the open problem given in [5, Problem 1].

Proposition 3.8 *Let* U *be a Riesz space and* V *be an atomic Dedekind complete Riesz space. If an operator* $S : U \to V$ *is order bounded and uo-continuous, then* $|S|$ *is uo-continuous.*

Proof Let $0 \leq r_\alpha \xrightarrow{uo} 0$ in U. We shall show that $|S| r_\alpha \wedge w \xrightarrow{o} 0$ for each $w \in V_+$. For this, it is enough to show that $t = 0$ when $t = \inf_\alpha \sup_{\beta \geqslant \alpha} (|S| (r_\beta) \wedge w)$. Fix a maximal disjoint collection $(q_\gamma)_{\gamma \in \Gamma}$ of atoms in V, and let P_γ be band projections on B_γ generated by q_γ. By Corollary 3.5, P_γ is *uo*-continuous and it is easy to see that $P_\gamma \circ |S| = |P_\gamma \circ S|$. Furthermore, since S is *uo*-continuous and B_γ is one-dimension $|P_\gamma \circ S|$ is *uo*-continuous for each γ. Then, we have

$$\begin{aligned} P_\gamma(t) &= P_\gamma(\inf_\alpha \sup_{\beta \geqslant \alpha} (|S| (r_\beta) \wedge w)) \\ &= \inf_\alpha \sup_{\beta \geqslant \alpha} P_\gamma[(|S| (r_\beta) \wedge w)] \\ &= \inf_\alpha \sup_{\beta \geqslant \alpha} [P_\gamma(|S| (r_\beta)) \wedge P_\gamma(w)] \\ &= \inf_\alpha \sup_{\beta \geqslant \alpha} [(|P_\gamma \circ S| (r_\beta)) \wedge P_\gamma(w)] \\ &= 0 \end{aligned}$$

for each γ. Thus, we obtain $t \in B_\gamma^d$ for all γ, and hence $t \perp q_\gamma$ for all γ this yields $t = 0$. □

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