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Subsequence characterization of statistical boundedness

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Abstract: In this paper, we present some relationships between statistical boundedness and statistical monotonicity of a given sequence and its subsequences. The results concerning statistical boundedness and monotonicity presented here are also closely related to earlier results regarding statistical convergence and are dealing with the Lebesgue measure and with the Baire category.

Key words: Statistical convergence, statistical boundedness, statistical monotonicity, subsequences, measure, Baire categor

1. Introduction

The convergence of sequences has undergone numerous generalizations in order to provide deeper insights into summability theory. The convergence of sequences has different generalizations. One of the most important generalizations is statistical convergence. This type of convergence has been introduced by Fast [9] by using asymptotic density and has been studied by many authors in various directions.

Buck [5] has initiated the study of the relationship between the convergence of a given sequence and the summability of its subsequences. Later Agnew [1], Buck [6], Buck and Pollard [7], Miller and Orhan [14], Zeager [16] have studied this relation changing the concept of convergence. Also, Dawson [8] and Fridy [10] have studied analogous results by replacing subsequences with stretching and rearrangements, respectively.

The concept of statistical boundedness was first introduced by Fridy and Orhan [12]. Theorems providing insights into the properties of statistical boundedness and its relation to statistical convergence were proved by Tripathy [15], and Bhardwaj and Gupta [3]. Additionally, in an analogous manner, the idea of statistical monotonicity of sequences was discussed by Aytar and Pehlivan [2] and further connected to statistical boundedness and statistical convergence.

In the present paper, we are concerned with the relationships between the statistical boundedness of a given sequence and its subsequences in the sense of different measures, Lebesgue measure, and Baire category. We also obtain a similar set of results regarding statistical monotonicity and subsequences.

Now let us recall some known notions. Let $A \subseteq \mathbb{N}$. If $n, m \in \mathbb{N}$, by $A(n, m)$ we denote the cardinality of the set of numbers i in A such that $n \leq i \leq m$. The numbers

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$$\underline{d}(A) = \liminf_{n\to\infty} \frac{A(1,n)}{n},\ \bar{d}(A) = \limsup_{n\to\infty} \frac{A(1,n)}{n}$$

are called the lower and the upper asymptotic density of the set A, respectively. If $d(A) = \bar{d}(A)$ then it is said that $d(A) = \underline{d}(A) = \bar{d}(A)$ is the asymptotic density of A.

The concept of statistical convergence has been introduced in [9] as follows: Let $x = (x_n)$ be a sequence of real numbers. The sequence x is said to be statistically convergent to a real number L provided that for every $\varepsilon > 0$ we have $d(A_\varepsilon) = 0$, where $A_\varepsilon = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$. If $x = (x_n)$ converges statistically to L, then we write $st - \lim x = L$.

Miller [13] has shown that $x = \{x_n\}$ converges to L statistically if and only if "almost all" of the subsequences of x converge to L statistically, by establishing a one-to-one correspondence between the interval $(0,1]$ and the collection of all subsequences of the sequence. Namely, if $t \in (0,1]$, then t has a unique binary expansion

$$t = \sum_{n=1}^{\infty} e_n(x) 2^{-n}, e_n(x) \in \{0,1\},$$

with infinitely many ones. For each $t \in (0,1]$, let $x(t)$ denote the subsequence of t obtained by the following rule: x_n is in the subsequence if and only if $e_n(t) = 1$. Clearly the mapping $t \to x(t)$ is a one-to-one onto mapping between $(0,1]$ and the collection of all subsequences of x. In [13] it is shown that $x = \{x_n\}$ converges to L statistically if and only if the set of $t \in (0,1]$, for which $x(t)$ converges to L statistically has Lebesgue measure 1.

Let N denote the set of normal numbers in $(0,1]$, i.e. the set of $t \in (0,1]$, $t = 0.t_1 t_2 ... t_n ...$ (binary representation with infinitely many 1's) for which the asymptotic density of 1's (0's) is exactly $\frac{1}{2}$. It is well known that $m(N) = 1$ (see [4]) .

2. Results on statistically bounded sequences

We say that a sequence of reals $x = \{x_n\}$ is statistically bounded if there exists $L > 0$ such that $d(\{n, |x_n| \geq L\}) = 0$. Tripathy [15] proved that $x = \{x_n\}$ is statistically bounded if and only if there exists a sequence of positive integers $n_1 < n_2 < \ldots < n_k < \ldots$ for which $d(\{n_k : k \in \mathbb{N}\}) = 1$ and $\{x_{n_k}\}$ is bounded. If $d(\{n_k : k \in \mathbb{N}\}) = 1$, we say that $\{x_{n_k}\}$ is statistically dense in $x = \{x_n\}$. Bhardwaj and Gupta [3] proved that $x = \{x_n\}$ is statistically bounded if and only if all of its statistically dense subsequences are statistically bounded.

Now we give the following theorem essentially showing that almost all subsequences of a statistically bounded sequence are statistically bounded.

Theorem 2.1 *Suppose $x = \{x_n\}$ is a sequence of reals and let $X := \{t \in (0,1] : x(t)$ is statistically bounded$\}$. Then x is statistically bounded if and only if $m(X) = 1$, where m denotes the Lebesgue measure.*

Proof

First, suppose x is statistically bounded by some positive L. Suppose $t \in (0,1]$ is normal. For n natural number, let n' denote the index for which $x(t)_n$ coincides with $x_{n'}$. Then

$$\lim_{n\to\infty} \frac{|\{i, 1 \le i \le n : |x(t)_i| \ge L\}|}{n} = \lim_{n\to\infty} \frac{|\{i, 1 \le i \le n : |x(t)_i| \ge L\}|}{n'} \cdot \frac{n'}{n} \le$$

$$\lim_{n\to\infty} \frac{|\{i, 1 \le i \le n' : |x_i| \ge L\}|}{n'} \cdot \frac{n'}{n} = 0 \cdot 2 = 0$$

since x is statistically bounded by L and t is normal. Therefore $x(t)$ is statistically bounded (by L), whenever t is normal. Since $m(N) = 1$, the first part of the theorem is proved.

Now suppose that X has measure 1.

Since $m(X) = 1$ implies that $m(1 - X) = 1$ where $1 - X = \{1 - t : t \in (0, 1]\}$, we can fix some $t \in X \bigcap (1 - X) \bigcap N$. We can now fix a positive L so that both $x(t)$ and $x(1 - t)$ are statistically bounded by L.

Now suppose n is arbitrarily fixed. Let n_1 denote the number of 1's among the first n indices of t, and n_2 the number of 0's among the first n indices of t.

Then:

$$\frac{|\{1 \le i \le n, |x_i| \ge L\}|}{n} =$$

$$\frac{|\{1 \le i \le n_1, |(x(t))_i| \ge L\}|}{n} + \frac{|\{1 \le i \le n_2, |(x(1-t))_i| \ge L\}|}{n} =$$

$$\frac{|\{1 \le i \le n_1, |(x(t))_i| \ge L\}|}{n_1} \cdot \frac{n_1}{n} + \frac{|\{1 \le i \le n_2, |(x(1-t))_i| \ge L\}|}{n_2} \cdot \frac{n_2}{n}.$$

Now if we let $n \to \infty$, we have that $n_1 \to \infty$, $n_2 \to \infty$, and that $\frac{n_1}{n} \to \frac{1}{2}$, $\frac{n_2}{n} \to \frac{1}{2}$.

Hence, since $x(t)$ and $x(1 - t)$ are statistically bounded by L, from the above we can conclude that

$$\lim_{n\to\infty} \frac{|\{1 \le i \le n, |x_i| \ge L\}|}{n} = 0$$

and hence x is statistically bounded. The proof is complete. □

The following theorem is a consequence of Theorem 2.1.

Theorem 2.2 *Suppose $x = \{x_n\}$ is a sequence of reals and let $X := \{t \in (0, 1] : x(t)$ is statistically bounded$\}$. Then x is not statistically bounded if and only if $m(X) = 0$.*

Proof

Since X is tail set, i.e. if $t \in X$ and if t' has the same digits as t except for finitely many, then $t' \in X$, X is nonmeasurable or $m(X) = 0$ *or* 1. To show X is measurable:

$$X = \bigcup_{M\in\mathbb{N}} \{t \in (0, 1] : \lim_{n\to\infty} \frac{|\{i : 1 \le i \le n, |x(t)_i| > M\}|}{n} = 0\}$$

$$= \bigcup_{M\in\mathbb{N}} \bigcap_{j\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \bigcap_{n=N}^{\infty} \{t \in (0, 1] : \frac{|\{i : 1 \le i \le n, |x(t)_i| > M\}|}{n} < \frac{1}{j}\}.$$

But for any M, j, n each set

$$\{t \in (0,1] : \frac{|\{i : 1 \leq i \leq n, |x(t)_i| > M\}|}{n} < \frac{1}{j}\}$$

is of the form $G \setminus S$ where G is an open set and S has Lebesgue measure 0. Therefore X is measurable so $m(X) = 0$ *or* 1. Now by applying Theorem 2.1, the conclusion follows.

□

Now, given a sequence of reals $x = \{x_n\}$, let us observe the size of the earlier defined X in terms of the Baire category. The conclusion here is very different. If x is simply bounded, then clearly all of its subsequences are bounded and therefore also statistically bounded. In every other case, we have the following theorem.

Theorem 2.3 *Suppose $x = \{x_n\}$ is an unbounded sequence of reals and let* $X := \{t \in (0,1] : x(t)$ *is statistically bounded*$\}$. *Then X is meager i.e. of the first Baire category.*

Proof

Since x is unbounded, we can fix a sequence $n_1 < n_2 < ... < n_i < ...$ of integers in $\mathbb{N}$ such that $x_{n_i} \to \infty$ or $-\infty$. Without loss of generality, we can assume that the limit is ∞.

For $m, k \in \mathbb{N}$ define

$$A_{m,k} = \{t \in (0,1] : \exists n \geq m, \frac{|\{i : 1 \leq i \leq n : |x(t)_i| > k\}|}{n} > \frac{1}{2}\}.$$

We will show that $A_{m,k}$ is comeager for $m, k \in \mathbb{N}$.

Let m and k be fixed. We will show that $A_{m,k}$ is comeager. Let $\bar{t} = (t_1, t_2, ..., t_d)$ be an arbitrary fixed finite sequence of $0's$ and $1's$. It is sufficient to show that there exists a finite extension t^* of $\bar{t}$ such that every $t \in [0,1)$ starting with t^* is in $A_{m,k}$. Now fix i so that $n_i > d$ and $x_{n_j} > k$ for $j \geq i$. Fix s such that $s > n_i$ and $s > m$.

Consider the following extension of $\bar{t}$

$$t^* = (t_1, t_2, ..., t_d, ..., t_{n_i}, ..., t_{n_{i+s}})$$

where for $j > d$: $t_j = 1$ for $j \in \{n_i, n_{i+1}, n_{i+2} ..., n_{i+s}\}$ and $t_j = 0$, $otherwise$. Then it is easy to see that every $t \in [0,1)$ that extends t^* is in $A_{m,k}$. Hence $A_{m,k}$ is comeager.

Now we can conclude that $\bigcap_k \bigcap_m A_{m,k}$ is comeager. Also if $t \in \bigcap_k \bigcap_m A_{m,k}$, clearly $x(t)$ is not statistically bounded. Therefore X and $\bigcap_k \bigcap_m A_{m,k}$ are disjoint and consequently X is meager.

□

3. Results on statistically monotone sequences

A sequence $x = \{x_n\}$ will be called increasing (decreasing) if $x_n \leq x_{n+1}$ (for $x_n \geq x_{n+1}$) for $n \in \mathbb{N}$ respectively. The notion of statistical monotonicity of a sequence was discussed by Aytar and Pehlivan [2] as follows. We will say that $x = \{x_n\}$ is statistically increasing (decreasing) if there exists a subsequence of x, $x(t)$, where $t = 0.t_1t_2 \ldots t_i \ldots \in (0,1]$ is such that $x(t)$ is increasing (decreasing) and $d(\{i : t_i = 1\}) = 1$. In this case, we say that x is statistically monotone and that $x(t)$ is statistically dense in x (as was mentioned earlier). Now we give the following theorem essentially showing that almost all subsequences of a statistically increasing (decreasing) sequence are statistically increasing (decreasing).

Theorem 3.1 *Suppose $x = \{x_n\}$ is a sequence of reals and let $X := \{t \in (0,1] : x(t)$ is statistically increasing$\}$. Then x is statistically increasing if and only if $m(X) = 1$, where m denotes the Lebesgue measure. The same statement is true if we replace statistically increasing with statistically decreasing.*

Proof

Suppose x is statistically increasing. Then we can fix $n_1 < n_2 < \ldots n_k < n_{k+1} \ldots$ so that x_{n_k} is increasing and $d(\{n_k : k \in \mathbb{N}\}) = 1$. Let $N_1 = \{n_k : k \in \mathbb{N}\}$. Let $N_2 = \mathbb{N} \setminus N_1$. Clearly $d(N_2) = 0$.

Suppose $t \in (0,1]$ is normal. For n natural number, let n' denote the index for which $x(t)_n$ coincides with $x_{n'}$. Then

$$\lim_{n\to\infty} \frac{|\{i, 1 \le i \le n : i' \in N_2\}|}{n} = \lim_{n\to\infty} \frac{|\{i, 1 \le i \le n : i' \in N_2\}|}{n'} \cdot \frac{n'}{n} \le$$

$$\lim_{n\to\infty} \frac{|\{i, 1 \le i \le n' : i \in N_2\}|}{n'} \cdot \frac{n'}{n} = 0 \cdot 2 = 0$$

since $d(N_2) = 0$ and t is normal. Hence $d(\{n : n' \in N_2\}) = 0$, and consequently $d(\{n : n' \in N_1\}) = 1$. Since $x(t)_n : n' \in N_1$ is increasing, it follows that $x(t)$ is statistically increasing. Now since the set N of all normal t has measure 1, it follows that $m(X) = 1$.

Now suppose that X has Lebesgue measure 1. Suppose, contrary to what we want to show that x is not statistically increasing.

Similar to an earlier argument, since X has measure 1, we can fix $t \in (0,1]$, t normal such that $t, 1-t \in X$. Let $y = y_m$ denote the statistically dense subsequence of $x(t)$ that is increasing and $z = z_n$ denote the statistically dense subsequence of $x(1-t)$ that is increasing. Easily y and z are also normal as subsequences of x i.e. they correspond to normal numbers in $(0,1]$, and they are mutually disjoint. Without loss of generality we can assume that y and z are complements of each other in x (i.e. $y = x(t'), z = x(1-t')$ for some normal $t' \in (0,1]$), since we can replace x with a statistically dense subsequence and this will not affect the course of the argument.

Now introduce the following notation. For m, let m' be the index for which y_m coincides with $x_{m'}$ and similarly for n, let n'' be the index for which z_n coincides with x''_n.

We say that y_m and z_n are out of order in the first sense if $m' < n''$ and $y_m > z_n$ and we say that y_m and z_n are out of order in the second sense if $m' > n''$ and $y_m < z_n$.

Now let m_1 be the smallest index such that there exists n, for which y_{m_1} and z_n are out of order in the first sense. Let n_1 be the smallest index n for which $m'_1 < n''$. Since z is increasing, automatically $y_{m_1} > z_{n_1}$. Next, m_2 be the smallest index such that there exists $n > n_1$, for which y_{m_2} and z_n are out of order in the first sense. Let n_2 be the smallest index $n > n_1$ for which $m'_2 < n''$. Again automatically $y_{m_2} > z_{n_2}$. We continue constructing $m_1 < m_2 < \ < m_k \ldots$, $n_1 < n_2 < \ < n_k \ldots$ so that m_k be the smallest index such that there exists $n > n_{k-1}$, for which y_{m_k} and z_n are out of order in the first sense, and n_k is the smallest $n > n_{k-1}$ for which $m'_k < n''$, and automatically $y_{m_k} > z_{n_k}$.

Likewise we can construct $p_1 < p_2 < \ < p_k \ldots$, $q_1 < q_2 < \ < q_k \ldots$ so that p_k be the smallest index such that there exists $q > q_{k-1}$, for which y_q and z_{p_k} are out of order in the second sense, and q_k is the smallest such index.

Easily, since y and z are increasing the sets $\bigcup_k \{y_{m_k}, z_{n_k}\}$ and $\bigcup_k \{z_{p_k}, y_{q_k}\}$ are disjoint. If we remove the terms from $\bigcup_k \{y_{m_k}, z_{n_k}\}$ and $\bigcup_k \{z_{p_k}, y_{q_k}\}$ from x it is easy to see that the remaining subsequence of x

is increasing. If both sets $\bigcup_k \{m'_k, n''_k\}$, $\bigcup_k \{p'_k, q''_k\}$ have asymptotic density 0 then the remaining subsequence of x would be statistically dense in x and be increasing so x would be statistically increasing, a contradiction. Therefore, at least one of $\bigcup_k \{m'_k, n''_k\}$, $\bigcup_k \{p'_k, q''_k\}$ has positive upper asymptotic density. Without loss of generality assume that

$$\bar{d}(\bigcup_k \{m'_k, n''_k\}) > 0.$$

From the previous construction and the fact that $y = x(t'), z = x(1-t')$ for some normal $t' \in (0,1]$, we can easily conclude that there exist $\delta > 0$ and $N_1 < N_2 < \; < N_j \ldots$ such that

$$\lim_{j\to\infty} \frac{|\{k : m'_k < n''_k < N_j\}|}{N_j} = \delta.$$

By the law of large numbers we can then conclude that for almost all $t \in (0,1]$, $x(t)$ satisfies

$$\limsup_{i\to\infty} \frac{|\{k : y_{m_k}, z_{n_k} \textit{ are contained among } (x(t))_1, (x(t))_2, \ldots (x(t))_i\}|}{i} \geq \frac{\delta}{4}.$$

Let X' denote the set of such t, $m(X') = 1$.

Suppose $t \in X'$ is fixed. Then there exist $M_1 < M_2 < \ldots < M_j \ldots$ such that

$$\lim_{j\to\infty} \frac{|\{k, y_{m_k}, z_{n_k} \textit{ are contained among } (x(t))_1, (x(t))_2, \ldots (x(t))_{M_j}\}|}{M_j} = \epsilon \geq \frac{\delta}{4}.$$

Suppose $x(t)_{i_l}$ is an increasing subsequence of $x(t)$. From the above,

$$\liminf_{j\to\infty} \frac{\text{The number of } x(t)_{i_l} \text{ contained in} (x(t))_1, (x(t))_2, \ldots (x(t))_{M_j}}{M_j} \leq 1 - \epsilon.$$

Hence $d(\{i_1, i_2, \ldots, i_l, \ldots\}) \leq 1 - \epsilon < 1$ and we conclude that any increasing subsequence of $x(t)$ is not statistically dense in $x(t)$. Hence $x(t)$ is not statistically increasing for $t \in X'$. But $m(X') = 1$ a contradiction since $m(X) = 1$. Therefore assuming that x is not statistically increasing leads to a contradiction in completing the proof of the second part.

Since the argument for the statistically decreasing case is analogous, the proof is complete.

□

Now, given a sequence of reals $x = \{x_n\}$, let us observe the size of the set X defined in Theorem 3.1 in terms of Baire category. The conclusion here is similar to the one in Theorem 2.3. If x is simply increasing (or decreasing) from some index on, then clearly all of its subsequences are increasing (or decreasing) from some index on and therefore also statistically increasing (decreasing). In every other case, we have the following theorem.

Theorem 3.2 *Suppose $x = \{x_n\}$ is a sequence of reals that is not monotone from some index on and let $X := \{t \in (0,1] : x(t)$ is statistically monotone$\}$. Then X is meager i.e. of the first Baire category.*

Proof

Since x is not monotone from some index on and therefore not increasing from some point on, we can fix

$$n_1 < n_2 < \ldots < n_{2k-1} < n_{2k} \ldots$$

, $k \in \mathbb{N}$, such that $x_{n_{2k-1}} > x_{n_{2k}}$.

For $m \in \mathbb{N}$ define A_m as the set of $t \in (0,1]$ for which:
There exists $k, n_{2k-1} > m$, such that for j, $n_{2k-1} \leq j \leq n_{2(k+n_{2k-1})}$:
$t_j = 1\ for\ j = n_{2i-1}, j = n_{2i}, k \leq i \leq k + n_{2k-1}$ and;
$t_j = 0\ otherwise$.

We will show that A_m is comeager for $m \in \mathbb{N}$.

Let m be fixed. Let $\bar{t} = (t_1, t_2, ..., t_d)$ be an arbitrary fixed finite sequence of $0's$ and $1's$. It is sufficient to show that there exists a finite extension t^* of $\bar{t}$ such that every $t \in [0,1)$ starting with t^* is in A_m. Now fix k so that $n_{2k-1} > m, d$.

Consider the following extension of $\bar{t}$

$$t^* = (t_1, t_2, ..., t_d, ..., t_{n_{2k-1}}, ..., t_{n_{2k}}, ..., t_{n_{2(k+n_{2k-1})-1}}, ..., t_{n_{2(k+n_{2k-1})}})$$

where for $d < j \leq n_{2(k+n_{2k-1})}$:
$t_j = 1\ for\ j = n_{2i-1}, j = n_{2i}, k \leq i \leq k + n_{2k-1}$ and;
$t_j = 0\ otherwise$.

Then it is easy to see that every $t \in [0,1)$ that extends t^* is in A_m. Hence A_m is comeager. Now $\bigcap_m A_m$ is comeager. It is easy to see that if $t \in \bigcap_m A_m$, $x(t)$ cannot contain a statistically dense increasing subsequence and hence is not statistically increasing. Therefore the set of $t \in (0,1]$, $x(t)$ is not statistically increasing is comeager. In a similar manner, we can show that the set of $t \in (0,1]$, $x(t)$ is not statistically decreasing is comeager. Consequently, the set of $t \in (0,1]$, $x(t)$ is not statistically monotone is comeager (as an intersection of two comeager sets), so the theorem is proved.

□

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