Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article

Turk J Math
(2022) 46: 3408 – 3418
© TÜBİTAK
doi:10.55730/1300-0098.3341

Ranks and presentations for order-preserving transformations with one fixed point

Jörg KOPPITZ[1], **Somnuek WORAWISET**[2,*]
[1]Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, Bulgaria
[2]Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen, Thailand

Received: 26.01.2022 • **Accepted/Published Online:** 20.10.2022 • **Final Version:** 09.11.2022

Abstract: In the present paper, we consider the semigroup $O_{n,p}$ of all order-preserving full transformations α on an n-elements chain X_n, where $p \in X_n$ is the only fixed point of α. The nilpotent semigroup $O_{n,p}$ was first studied by Ayik et al. in 2011. Moreover, $O_{n,1}$ is the maximal nilpotent subsemigroup of the Catalan Monoid C_n. Its rank is the difference of the $(n-1)$th and the $(n-2)$th Catalan number. The aim of the present paper is to provide further fundamental information about the nilpotent semigroup $O_{n,p}$. We will calculate the rank of $O_{n,p}$ for $p > 1$ and provide a semigroup presentation for $O_{n,1}$.

Key words: Order-preserving transformations, fixed point transformations, presentations, ranks

1. Introduction

The full transformation semigroup T_n on an n-elements set $X_n = \{1, 2, \ldots, n\}$ is the set of all transformations (i.e. self maps) $X_n \to X_n$ of X_n with composition of transformations as multiplication. It is well known that every finite semigroup is isomorphic to a subsemigroup of a suitable finite transformation semigroup (the analogue of the Cayley's theorem for finite groups). Hence, the transformation semigroups have an important role in semigroup theory, as the symmetric groups in group theory. Various properties of T_n are known and many subsemigroups of T_n are studied. Among the most studied subsemigroups of T_n is the monoid O_n of all order-preserving transformations on the chain $X_n = \{1 < 2 < \cdots < n\}$. A transformation $\alpha \in T_n$ is called *order-preserving* if $x < y$ implies $x\alpha \leq y\alpha$. The monoid O_n has long been considered in the literature. In particular, the rank of monoid O_n is n (see [7]). The rank of a semigroup S is the minimal size of a generating set for S, i.e. $rank(S) = \min\{|G| : \langle G \rangle = S\}$. Aĭzenštat (1962) exhibited presentations for O_n (see [1]). If Y is a set then we denote by Y^+ the free semigroup on Y. If $\mathcal{R} \subseteq Y^+ \times Y^+$, then we denote by $\mathcal{R}^\#$ the congruence on Y^+ generated by $\mathcal{R}$. To say that a semigroup S has semigroup presentation $\langle Y \mid \mathcal{R} \rangle$ is to say that $S \cong Y^+/_{\mathcal{R}^\#}$ or, equivalently, that there is a semigroup epimorphism $\varphi : Y^+ \to S$ with kernel $\mathcal{R}^\#$ (i.e. $ker\varphi = \{(x, y) \in Y^+ \times Y^+ : x\varphi = y\varphi\} = \mathcal{R}^\#$). If φ is such an epimorphism, then we say that S has presentation $\langle Y \mid \mathcal{R} \rangle$ via φ. The elements of Y are called *generators* and the elements of $\mathcal{R}$ are called *relations*. In practice, a relation $(x, y) \in \mathcal{R}$ will be written as $x \approx y$.

*Correspondence: wsomnu@kku.ac.th; koppitz@math.bas.bg
2010 *AMS Mathematics Subject Classification:* 20M20

Besides the already mentioned fundamental paper by Aĭzenštat [1], in 2010, East has given a presentation for the singular part of T_n, i.e. for $T_n \setminus S_n$, where S_n is the symmetric group on X_n [4]. In the literature, one can find presentations for several semigroups of partial transformations (see, e.g., [5, 6]).

A subset $A \subseteq X_n$ is called invariant for a transformation $\alpha \in T_n$ if $A\alpha = \{x\alpha : x \in A\} \subseteq A$. In 1966, Magill Jr. has considered the monoid $T_n(A) = \{\alpha \in T_n : A\alpha \subseteq A\}$ [12]. However, the elements in A do not have to be included in the set $Fix(\alpha) = \{x \in X_n : x\alpha = x\}$ of all fixed points of α. The number $|\{\alpha \in O_n : |Fix(\alpha)| = r\}|$ of all transformations on O_n with r fixed points, for a fixed positive integer r, was given by Higgins in 1993 in [9]. Honyam and Sanwong have studied the semigroup $Fix(X_n, a)$ of all $\alpha \in T_n$ such that $A \subseteq Fix(\alpha)$, for a fixed set $A \subseteq X_n$ [10]. They have shown that $Fix(X_n, a)$ is a regular monoid. Chinram and Yonthanthun have given a necessary and sufficient condition that $Fix(X_n, a)$ is left-regular and right-regular, respectively [3]. If we restrict ourselves to the case that $Fix(\alpha) = A$, we miss that the corresponding semigroup is a monoid, whence $A \neq X_n$. In [2], the cardinality of the semigroup

$$O_{n,A} = \{\alpha \in O_n : Fix(\alpha) = A\}$$

was determined by Ayik et al.. This paper extends the considerations about the cardinalities of semigroups of order-preserving transformations on a finite set in [9]. We will write $O_{n,p}$ instead of $O_{n,A}$, whenever A is the singleton set $\{p\}$ $(A = \{p\})$ for some $p \in X_n$. However, in [9], algebraic properties of $O_{n,A}$ are not studied. The study of the nilpotent semigroup $O_{n,p}$ gives more information about the subsemigroups of the well studied monoid O_n. In the present paper, we will study algebraic properties of $O_{n,p}$. We will calculate the ranks and give a presentation for $O_{n,1}$. The transformations in $O_{n,p}$ can be described as follows:

Proposition 1.1 ([2], Proposition 4) *Let $p \in X_n$ and let $\alpha \in O_{n,p}$.*
i) If $1 \leq x < p$, then $x + 1 \leq x\alpha$.
ii) If $p < x \leq n$, then $x\alpha \leq x - 1$.

By Proposition 1.1, it is easy to verify that $(p-1)\alpha = p$ (whenever $p > 1$) and $(p+1)\alpha = p$ (whenever $p < n$).

An element a of a finite semigroup S with 0 is called nilpotent if $a^m = 0$ for some positive integer m. The set of all nilpotent elements of S is denoted by $N(S)$. The semigroup S with 0 is called nilpotent if $S^m = \{0\}$ for some positive integer m. It is a well known fact that S is nilpotent if and only if $N(S) = S$ (see [8]). In the present paper, we deal with nilpotent semigroups. Let c_x be the constant map to x, for $x \in X_n$.

In particular, the semigroup $O_{n,1}$ is already well studied because it is the maximal nilpotent subsemigroup of the Catalan Monoid C_n. A transformation $\alpha \in T_n$ is called order-deceasing if $x\alpha \leq x$ for all $x \in X_n$. The set of all order-decreasing transformations in T_n is a denoted by D_n, which is a subsemigroup of T_n. The subsemigroup $C_n = O_n \cap D_n$ is called Catalan Monoid since its cardinality is the nth Catalan number $\mathcal{C}_n = \frac{1}{n+1}\begin{pmatrix} 2n \\ n \end{pmatrix} = \frac{1}{n}\begin{pmatrix} 2n \\ n-1 \end{pmatrix}$, where $\mathcal{C}_0 = 1$ (see [14]). We observe that c_1 is the zero element in C_n. Moreover, an element $\alpha \in C_n$ is nilpotent if and only if $Fix(\alpha) = \{1\}$ (see [11], Lemma 2.2). From [2], Proposition 4, we have the fact $O_{n,1} = N(C_n)$.

Lemma 1.2 *$O_{n,p}$ is a nilpotent semigroup.*

Proof Since $O_{n,1}$ is equal to the nilpotent semigroup $N(C_n)$, it is enough to consider the case $p \geq 2$. Let $\alpha \in O_{n,p}$. By Proposition 1.1, using $1\alpha \geq 2$ and $(p-1)\alpha = (p+1)\alpha = p$, we can conclude that $x\alpha^{p-1} = p$ for all $x \in \{1,\ldots,p-2\}$ and $x\alpha^{n-p} = p$ for all $x \in \{p+2,\ldots,n\}$. This shows that $x\alpha^m = p$ for $m = \max\{p-1, n-p\}$. Consequently, we have $\alpha^m = c_p$. We obtain $N(O_{n,p}) = O_{n,p}$, i.e. $O_{n,p}$ is a nilpotent semigroup with zero element c_p. □

It is easy to verify that $O_{n,r}$ and $O_{n,n-r+1}$ are isomorphic for $r \in \{1,\ldots,\lfloor \frac{n}{2} \rfloor\}$. In particular, $O_{n,1}$ and $O_{n,n}$ are isomorphic. Thus, we get

Lemma 1.3 ([2]) $|O_{n,1}| = |O_{n,n}| = \mathcal{C}_{n-1}$.

In general, it holds:

Lemma 1.4 ([2], Lemma 6) $|O_{n,p}| = \mathcal{C}_{p-1}\mathcal{C}_{n-p}$.

Using the fact that $O_{n,1} = N(C_n)$, where $rank(N(C_n)) = \mathcal{C}_{n-1} - \mathcal{C}_{n-2}$ ([15], Theorem 2), we obtain

Proposition 1.5 ([15]) $rank(O_{n,1}) = rank(O_{n,n}) = \mathcal{C}_{n-1} - \mathcal{C}_{n-2}$.

As $O_{n,1}$ and $O_{n,n}$, we can show that $O_{n,2}$ and $O_{n-1,1}$ are isomorphic.

Lemma 1.6 *$O_{n,2}$ and $O_{n-1,1}$ are isomorphic.*

Proof Let $\varphi : O_{n,2} \to O_{n-1,1}$ with $x(\alpha\varphi) = (x+1)\alpha - 1$ for all $x \in \{1,\ldots,n-1\} = X_{n-1}$ and $\alpha \in O_{n,2}$. Firstly, we show that φ maps into $O_{n-1,1}$. Let $\alpha \in O_{n,2}$. Then $1(\alpha\varphi) = (1+1)\alpha - 1 = 2 - 1 = 1$. Moreover, for $x < y \in X_{n-1}$, we have $x(\alpha\varphi) = (x+1)\alpha - 1 \leq (y+1)\alpha - 1 = y(\alpha\varphi) \leq n-1$. Further, we have $x(\alpha\varphi) = (x+1)\alpha - 1 < x + 1 - 1 = x$ for all $x \in \{2,3,4,\ldots,n-1\}$. This shows that $\alpha\varphi \in O_{n-1,1}$.

On the other hand, for $\alpha \in O_{n-1,1}$, let $\overline{\alpha} \in T_n$ with $1\overline{\alpha} = 2$ and $x\overline{\alpha} = (x-1)\alpha + 1 (\leq n)$ for all $x \in \{2,\ldots,n\}$. We have $2\overline{\alpha} = (2-1)\alpha + 1 = 1 + 1 = 2$. It is routine to show that $\overline{\alpha}$ is order-preserving and $x\bar{\alpha} < x$ for $x \in \{2,\ldots,n\}$, so $\overline{\alpha} \in O_{n,2}$. Further, we have $x(\overline{\alpha}\varphi) = (x+1)\overline{\alpha} - 1 = (x+1-1)\alpha + 1 - 1 = x\alpha$ for $x \in X_{n-1}$, i.e. $\overline{\alpha}\varphi = \alpha$, which provides that φ is surjective. Next, we verify that φ is injective. Let $\alpha, \beta \in O_{n,2}$ with $\alpha\varphi = \beta\varphi$. For $x \in X_{n-1}$, the assumption $x\alpha\varphi = x\beta\varphi$ implies $(x+1)\alpha = (x+1)\beta$. Moreover, we have $1\alpha = 2 = 1\beta$, which completes the argumentation that $\alpha = \beta$.

Finally, let $\alpha, \beta \in O_{n,2}$. Then for $x \in X_{n-1}$, we have $x((\alpha\beta)\varphi) = (x+1)\alpha\beta - 1 = (((x+1)\alpha - 1) + 1)\beta - 1 = ((x+1)\alpha - 1)(\beta\varphi) = (x(\alpha\varphi))(\beta\varphi)$. This shows that $(\alpha\beta)\varphi = (\alpha\varphi)(\beta\varphi)$, which completes the proof. □

2. Rank of $O_{n,p}$

As already mentioned, the rank of $O_{n,1}$ as well as the rank of $O_{n,n}$ is $\mathcal{C}_{n-1} - \mathcal{C}_{n-2}$. Since $O_{n,2}$ is isomorphic to both $O_{n-1,1}$ and $O_{n,n-1}$, we can conclude that $rank(O_{n,2}) = rank(O_{n,n-1}) = \mathcal{C}_{n-2} - \mathcal{C}_{n-3}$. Since $O_{n,p}$ is a nilpotent semigroup, we have $rank(O_{n,p}) = \left|O_{n,p} \setminus O_{n,p}^2\right|$ (see [13], Lemma 2.1). Let

$$G_{n,p} = \{\alpha \in O_{n,p} : |x\alpha - x| = 1 \text{ for at least one } x \in X_n \setminus \{p-1, p+1\}\}$$

and we will show that $O_{n,p} \setminus O_{n,p}^2 = G_{n,p}$. For this, we show first that $O_{n,p}^2 \cap G_{n,p} = \emptyset$.

Lemma 2.1 *If $\alpha, \beta \in O_{n,p}$, then $\alpha\beta \notin G_{n,p}$.*

Proof Assume that there are $\alpha, \beta \in O_{n,p}$ such that $\alpha\beta \in G_{n,p}$. Then there is $x \in X_n \backslash \{p-1, p, p+1\}$ such that $|x\alpha\beta - x| = 1$.

If $x < p-1$, then $x\alpha \geq x+1$ by Proposition 1.1, where $x+1 < p$. This implies $x\alpha\beta \geq (x+1)\beta \geq x+2$. Thus, $x\alpha\beta - x \geq x+2-x = 2$, a contradiction.

If $x > p+1$, then $x\alpha \leq x-1$ by Proposition 1.1, where $x-1 > p$. Thus, $x\alpha\beta \leq (x-1)\beta \leq x-2$, i.e. $x\alpha\beta - x \leq x-2-x = -2$, a contradiction. □

Proposition 2.2 $O_{n,p} \setminus O^2_{n,p} = G_{n,p}$.

Proof By Lemma 2.1, we can conclude that $O_{n,p} \setminus O^2_{n,p} \supseteq G_{n,p}$. In order to show the converse inclusion, we verify that $\alpha \in O^2_{n,p}$ for all $\alpha \in O_{n,p} \setminus G_{n,p}$. Let $\alpha \in O_{n,p} \backslash G_{n,p}$. We define $\alpha^* : X_n \to X_n$ by

$$x\alpha^* = \begin{cases} x\alpha - 1 & \text{for } x \leq p-2; \\ x\alpha + 1 & \text{for } x \geq p+2; \\ p & \text{otherwise}. \end{cases}$$

Since $\alpha \notin G_{n,p}$ implies $|x - x\alpha| \geq 2$ for all $x \in X_n \backslash \{p-1, p, p+1\}$, it is easy to see that $\alpha^* \in O_{n,p}$. We show that $\alpha = \alpha^*\gamma$, where

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & \cdots & p-2 & p-1 & p & p+1 & p+2 & \cdots & n \\ 2 & 3 & 4 & \cdots & p-1 & p & p & p & p+1 & \cdots & n-1 \end{pmatrix}.$$

It is clear that $\gamma \in O_{n,p}$. Let $x \in X_n$. Suppose that $x \leq p-2$. We have $x\alpha^*\gamma = (x\alpha - 1)\gamma$, where $x\alpha \leq p$ since α is order-preserving. If $x\alpha = p$, then $(x\alpha - 1)\gamma = (p-1)\gamma = p = x\alpha$. If $x\alpha < p$, then $(x\alpha - 1)\gamma = (x\alpha - 1) + 1 = x\alpha$. Thus, $x\alpha^*\gamma = x\alpha$. If $x \in \{p-1, p, p+1\}$, then $x\alpha = p = p\gamma = x\alpha^*\gamma$. Suppose that $x \geq p+2$. Then $x\alpha^*\gamma = (x\alpha + 1)\gamma$. Since α is order-preserving, we have $x\alpha \geq p$. If $x\alpha = p$, then $x\alpha^*\gamma = (x\alpha+1)\gamma = (p+1)\gamma = p = x\alpha$. If $x\alpha > p$, then $x\alpha+1 > p+1$ and $x\alpha^*\gamma = (x\alpha+1)\gamma = (x\alpha+1-1) = x\alpha$. □

Proposition 2.2 shows that $\alpha \in O^2_{n,p}$ if and only if $|x\alpha - x| \geq 2$ for all $x \in X_n \setminus \{p-1, p, p+1\}$. We will use this fact for the calculation of the size of $O^2_{n,p}$.

Lemma 2.3 $\left|O^2_{n,p}\right| = |O_{n-2,p-1}|$.

Proof Let $\alpha \in O^2_{n,p}$. We define $\alpha^* \in T_{n-2}$ by

$$x\alpha^* = \begin{cases} x\alpha - 1 & \text{for} \quad x \leq p-1; \\ (x+2)\alpha - 1 & \text{for} \quad p \leq x \leq n-2. \end{cases}$$

Clearly, $x\alpha^* \leq n-1$ for all $x \in X_{n-2}$. Since $|x\alpha - x| \geq 2$ for all $x \in X_n \backslash \{p-1, p, p+1\}$, we have $|x\alpha^* - x| \geq 1$ for all $x \in \{1, \ldots, p-2, p, \ldots, n-2\}$. Moreover, $(p-1)\alpha = p$ implies $(p-1)\alpha^* = (p-1)\alpha - 1 = p-1$. Obviously, α^* is order-preserving. Hence, we can conclude that $\alpha^* \in O_{n-2,p-1}$. We define a mapping $\varphi : O^2_{n,p} \to O_{n-2,p-1}$ by $\alpha\varphi = \alpha^*$. It is easy to verify that φ is a bijection, i.e. $\left|O^2_{n,p}\right| = |O_{n-2,p-1}|$. □

Now we are able to calculate the rank of the nilpotent semigroup $O_{n,p}$.

Theorem 2.4 $rank(O_{n,p}) = \mathcal{C}_{p-1}\mathcal{C}_{n-p} - \mathcal{C}_{p-2}\mathcal{C}_{n-p-1}$.

Proof Note that $rank(O_{n,p}) = |O_{n,p} \setminus O_{n,p}^2| = |O_{n,p}| - |O_{n,p}^2| = |O_{n,p}| - |O_{n-2,p-1}|$ by Lemma 2.3. Because of $|O_{n,p}| = \mathcal{C}_{p-1}\mathcal{C}_{n-p}$ and $|O_{n-2,p-1}| = \mathcal{C}_{p-2}\mathcal{C}_{n-p-1}$ by Lemma 1.4, we get $rank(O_{n,p}) = |O_{n,p} \setminus O_{n,p}^2| = \mathcal{C}_{p-1}\mathcal{C}_{n-p} - \mathcal{C}_{p-2}\mathcal{C}_{n-p-1}$. □

3. Presentations for $O_{n,1}$

The goal of this section is to give a presentation for $O_{n,1}$. Let A_n be set of all mappings $g \in \{0, 1, \dots, n-1\}^{X_n}$ with $1g = 0$, $xg \geq 1$ for $x \in \{2, \dots, n\}$, where $lg = 1$ for some $l \in \{3, \dots, n\}$, and either $(x+1)g \leq xg$ or $(x+1)g = xg + 1$ for all $x \in \{1, \dots, n-1\}$.

Lemma 3.1 *Let $g \in A_n$. Then $x - xg \geq 1$ for all $x \in X_n$.*

Proof We have $1 - 1g = 1 - 0 = 1$. Suppose that $x - xg \geq 1$ for some $x \in X_n$ and we will show that $(x+1) - (x+1)g \geq 1$. We have $(x+1)g \leq xg$ or $(x+1)g = xg + 1$. If $(x+1)g \leq xg$, then $1 \leq x - xg < x + 1 - xg \leq (x+1) - (x+1)g$. If $(x+1)g = xg + 1$, then $(x+1) - xg = (x+1) - (x+1)g + 1$; thus, $1 \leq x - xg = (x+1) - (x+1)g$. □

For each $g \in A_n$, let $\alpha_g \in T_n$ with

$$x\alpha_g = x - xg \text{ for all } x \in X_n.$$

Lemma 3.2 $G_{n,1} = \{\alpha_g : g \in A_n\}$.

Proof Let $g \in A_n$ and we will show that $\alpha_g \in G_{n,1}$. Clearly, $1\alpha_g = 1 - 1g = 1 - 0 = 1$ and $|x - x\alpha_g| = xg \geq 1$ for all $x \in \{2, 3, \dots, n\}$. Note that there is $l \in \{3, \dots, n\}$ such that $lg = 1$. This provides $l\alpha_g = l - lg = l - 1$; thus, $|l - l\alpha_g| = 1$.

Next, we show that $\alpha_g \in O_{n,1}$, i.e. α_g is order-preserving. Let $x < y \in X_n$. Then $y = x + k$ for some $k \in X_n$. Firstly, we verify that $x\alpha_g \leq (x+1)\alpha_g$. We have $xg \geq (x+1)g$ or $(x+1)g = xg + 1$. If $xg \geq (x+1)g$, then $x\alpha_g = x - xg \leq x - (x+1)g < (x+1) - (x+1)g = (x+1)\alpha_g$. If $(x+1)g = xg + 1$, then $x\alpha_g = x - xg = x - (x+1)g + 1 = (x+1) - (x+1)g = (x+1)\alpha_g$. By the same arguments, we obtain $x\alpha_g \leq (x+1)\alpha_g \leq (x+2)\alpha_g \leq \cdots \leq (x+k)\alpha_g$, i.e. $x\alpha_g \leq y\alpha_g$.

Therefore, $\alpha_g \in G_{n,1}$. Altogether, we have shown that $\{\alpha_g : g \in A_n\} \subseteq G_{n,1}$.

For the converse inclusion, let $\beta \in G_{n,1}$ and we define $g_\beta \in \{0, 1, \dots, n-1\}^{X_n}$ by

$$xg_\beta = x - x\beta \text{ for all } x \in X_n.$$

We will show that $g_\beta \in A_n$. Clearly, $1g_\beta = 1 - 1\beta = 1 - 1 = 0$ and $xg_\beta = x - x\beta \geq 1$ for all $x \in \{2, \dots, n\}$ by Proposition 1.1. Further, there is $p \in \{3, \dots, n\}$ such that $1 = p - p\beta = pg_\beta$. Let $x \in \{1, \dots, n-1\}$ with $(x+1)g_\beta > xg_\beta$. This implies $(x+1) - (x+1)\beta > x - x\beta$ and so $1 - (x+1)\beta > -x\beta$, i.e. $(x+1)\beta - 1 < x\beta$. Since β is order-preserving, $(x+1)\beta - 1 < x\beta$ is only possible if $(x+1)\beta = x\beta$, i.e. $(x+1) - (x+1)\beta = x - x\beta + 1$. This shows $(x+1)g_\beta = xg_\beta + 1$. Consequently, $g_\beta \in A_n$.

Finally, since $x\alpha_{g_\beta} = x - xg_\beta = x - (x - x\beta) = x\beta$, for all $x \in X_n$, we can conclude that $\beta = \alpha_{g_\beta} \in \{\alpha_g : g \in A_n\}$. Consequently, we have shown the converse inclusion $G_{n,1} \subseteq \{\alpha_g : g \in A_n\}$, which completes the proof. □

Together with Proposition 2.2 we obtain:

Corollary 3.3 *$\{\alpha_g : g \in A_n\}$ is a generating set of $O_{n,1}$.*

For $g, h \in A_n$, let $g \oplus h \in \{0, 1, 2, \ldots, n-1\}^{X_n}$ with $1(g \oplus h) = 0, 2(g \oplus h) = 1$ and $x(g \oplus h) = (x - xg)h + xg - 1$ for $x \in \{3, 4, \ldots, n\}$. Note that $2g = 1$ for all $g \in A_n$.

Lemma 3.4 *Let $g, h \in A_n$. Then $g \oplus h \in A_n$.*

Proof For $x \in \{3, 4, \ldots, n\}$, we have $xg \geq 1$. If $xg = 1$, then $x - xg \geq 2$; thus, $(x - xg)h \geq 1$. This provides $(x - xg)h + xg - 1 \geq 1$. If $xg > 1$, then $xg - 1 \geq 1$; thus $(x - xg)h + xg - 1 \geq 1$. We take $x = 3$. Then $3g \in \{1, 2\}$ by Lemma 3.1. If $3g = 1$, then $3(g \oplus h) = (3 - 1)h + 1 - 1 = 2h = 1$. If $3g = 2$, then $3(g \oplus h) = (3 - 2)h + 2 - 1 = 1h + 2 - 1 = 1$. Let $x \in \{2, 3, 4, \ldots, n-1\}$ such that $(x+1)(g \oplus h) > x(g \oplus h)$. We show that $(x+1)(g \oplus h) = x(g \oplus h) + 1$.

If $xg > (x+1)g$, then $xg = r + (x+1)g$ for some $r \in \{1, 2, \ldots, n\}$. Then $(x+1)(g \oplus h) = (x+1-(x+1)g)h + (x+1)g - 1$ and $x(g \oplus h) = (x - xg)h + xg - 1 = (x - (r + (x+1)g))h + r + (x+1)g - 1$, that is $(x+1-(x+1)g)h > (x-(r+(x+1)g))h + r$. Thus, $(x+1-(x+1)g)h \geq (x-(r+(x+1)g))h + r + 1$. On the other hand, we can calculate that $(x+1-(x+1)g)h \leq (x+1-(x+1)g-(r+1))h + (r+1) = (x-(x+1)g-r)h + (r+1)$. This implies $(x+1-(x+1)g)h = (x - r - (x+1)g)h + r + 1$. We get $(x+1-(x+1)g)h + (x+1)g - 1 = (x - r - (x+1)g)h + r + (x+1)g = (x - r - (xg - r))h + r + xg - r = (x - xg)h + xg - 1 + 1 = x(g \oplus h) + 1$. This implies $(x+1)(g \oplus h) = x(g \oplus h) + 1$.

If $xg = (x+1)g$, then $(x+1)(g \oplus h) = (x + 1 - xg)h + xg - 1 > (x - xg)h + xg - 1 = x(g \oplus h)$. This gives $(x+1-xg)h > (x - xg)h$, i.e. $(x+1-xg)h = (x-xg)h + 1$. This provides $(x+1)(g \oplus h) = x(g \oplus h) + 1$.

If $xg < (x+1)g$, i.e. $(x+1)g = xg + 1$, then we can calculate $(x+1)(g \oplus h) = (x + 1 - (x+1)g)h + (x+1)g - 1 = (x + 1 - xg - 1)h + xg + 1 - 1 = ((x - xg)h + xg - 1) + 1 = x(g \oplus h) + 1$. □

Let us put $\omega \in \{0, 1, 2, \ldots, n-1\}^{X_n}$ with $1\omega = 0$ and $x\omega = 1$ for $x \in \{2, 3, 4, \ldots, n\}$. Clearly, $\omega \in A_n$. It is useful to determine the product of any transformation in $G_{n,1}$ with $\alpha_\omega = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 1 & 2 & \cdots & n-1 \end{pmatrix}$.

Let us define a mapping $f : X_n \cup \{0\} \to X_n$ by $f(0) = 1$ and $f(x) = x$ for $x \in X_n$. In this case, we write the argument right of the mapping for convenient.

Lemma 3.5 *Let $g \in A_n$. Then*

$$\alpha_g \alpha_\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 1 & f(2 - 3g) & f(3 - 4g) & \cdots & f((n-1) - ng) \end{pmatrix}.$$

Proof We have $1\alpha_g\alpha_\omega = 1\alpha_\omega = 1$, $2\alpha_g\alpha_\omega = 1\alpha_\omega = 1$, and

$$x\alpha_g\alpha_\omega = \begin{cases} x - xg - 1 & \text{if } x - xg > 1; \\ 1 & \text{if } x - xg = 1, \end{cases}$$

i.e. $x\alpha_g\alpha_\omega = f(x - xg - 1)$, for all $x \in \{3, 4, 5, \ldots, n\}$. □

For $g \in A_n$ let $\hat{g} \in \{0, 1, 2, \ldots, n-1\}^{X_n}$ with $1\hat{g} = 0$, $2\hat{g} = 1$, and $x\hat{g} = x - 1 - f(f(x - 1 - xg) - 1)$ for $x \in \{3, 4, \ldots, n\}$.

Lemma 3.6 *$\hat{g} \in A_n$ for all $g \in A_n$.*

Proof Let $g \in A_n$. We have $1\hat{g} = 0$ and $2\hat{g} = 1$. If $x \geq 3$, then we obtain $x\hat{g} = x - 1 - f(f(x-1-xg)-1) = x - 1 - x + 1 + xg + 1 = xg + 1 \geq 1$, whenever $f(x - 1 - xg) \geq 2$ and $x\hat{g} = x - 1 - 1 \geq 3 - 2 = 1$, whenever $f(x - 1 - xg) = 1$. Further, $3g \in \{1, 2\}$ implies $f(2 - 3g) = 1$ and $3\hat{g} = 2 - f(f(2 - 3g) - 1) = 2 - f(1 - 1) = 2 - 1 = 1$, i.e. $2\hat{g} = 3\hat{g}$. Let $x \in \{3, \ldots, n-1\}$ with $(x+1)\hat{g} > x\hat{g}$. Since $\alpha_{\hat{g}}\alpha_\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 1 & f(f(2-3g)-1) & f(f(3-4g)-1) & \cdots & f(f((n-1)-ng)-1) \end{pmatrix} \in O_{n,1}$, we obtain $f(f(x - 1 - xg) - 1) \leq f(f(x - (x+1)g) - 1)$, $-f(f(x - (x+1)g) - 1) \leq -f(f(x - 1 - xg) - 1)$; thus, $(x+1)\hat{g} = x - f(f(x-(x+1)g)-1) \leq x - f(f(x-1-xg)-1) = x - 1 - f(f(x-1-xg)-1) + 1 = x\hat{g} + 1$. On the other hand, $x\hat{g} < (x+1)\hat{g}$ implies $x\hat{g} + 1 \leq (x+1)\hat{g}$. Therefore, we have $x\hat{g} + 1 = (x+1)\hat{g}$. □

Now we define an alphabet set $Y_n = \{x_g : g \in A_n\}$ and define two sets of relations on Y_n^+. Let

$$\begin{aligned} R_1 &= \{x_g x_\omega^2 \approx x_{\hat{g}} x_\omega : g \in A_n\} \text{ and} \\ R_2 &= \{x_g x_h \approx x_{g \oplus h} x_\omega : g, h \in A_n\}. \end{aligned}$$

It is easy to see that $|R_1| = |A_n|$ and $|R_2| = |A_n|^2$. Since $R_1 \cap R_2 = \emptyset$, we have

$$|R_1 \cup R_2| = |A_n| + |A_n|^2 = (1 + |A_n|) \cdot |A_n| = (1 + |G_{n,1}|)|G_{n,1}| = (1 + \mathcal{C}_{n-1} - \mathcal{C}_{n-2})(\mathcal{C}_{n-1} - \mathcal{C}_{n-2})$$

using Lemma 3.2 and Proposition 2.2.

We write $\sim$ for the congruence on Y_n^+ generated by $R_1 \cup R_2$. Note that $\{\alpha_g : g \in A_n\}$ is a generating set of $O_{n,1}$ (see Corollary 3.3). We define an epimorphism $\varphi : Y_n \to O_{n,1}$ by $x_g\varphi = \alpha_g$ for $g \in A_n$. We aim to show that $ker\varphi = \sim$, so that $O_{n,1}$ has presentation $\langle Y_n | R_1 \cup R_2 \rangle$ via φ.

Lemma 3.7 *We have the inclusion $\sim \subseteq ker\varphi$.*

Proof It is sufficient to show that the relations in $R_1 \cup R_2$ hold as equations in $O_{n,1}$ when the variables are replaced by their images under φ. Note that $x_\omega\varphi = \alpha_\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 1 & 2 & 3 & \cdots & n-1 \end{pmatrix}$.

Let $g \in A_n$, where $\hat{g} \in A_n$ by Lemma 3.6. Then $\alpha_g\alpha_\omega = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 1 & f(2-3g) & \cdots & f((n-1)-ng) \end{pmatrix}$ by Lemma 3.5 and it is easy to verify that $\alpha_g\alpha_\omega = \alpha_h$ with $1h = 0$, $2h = 1$ and $xh = x - f(x - 1 - xg)$ for $x \in \{3, 4, \ldots, n\}$. Now we have

$$(\alpha_g\alpha_\omega)\alpha_\omega = \alpha_h\alpha_\omega = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 1 & f(2-3+f(2-3g)) & \cdots & f(n-1-n+f((n-1)-ng)) \end{pmatrix}$$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 1 & f(f(2-3g)-1) & f(f(3-4g)-1) & \cdots & f(f((n-1)-ng)-1) \end{pmatrix}$. On the other hand, Lemma

3.5 gives

$$\alpha_{\hat{g}}\alpha_\omega = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 1 & f(2-2+f(f(2-3g)-1)) & \cdots & f((n-1)-(n-1)+f(f((n-1)-ng)-1)) \end{pmatrix}$$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 1 & f(f(2-3g)-1) & f(f(3-4g)-1) & \cdots & f(f((n-1)-ng)-1) \end{pmatrix}$ using $f(f(x)) = f(x)$ for $x \in X_n$. Hence, $\alpha_g\alpha_\omega\alpha_\omega = \alpha_{\hat{g}}\alpha_\omega$ and we obtain $x_g x_\omega^2\varphi = x_g\varphi x_\omega\varphi x_\omega\varphi = \alpha_g\alpha_\omega\alpha_\omega = \alpha_{\hat{g}}\alpha_\omega = x_{\hat{g}}\varphi x_\omega\varphi = x_{\hat{g}}x_\omega\varphi$.

Let $g, h \in A_n$, where $g \oplus h \in A_n$ by Lemma 3.4. Then we can easily calculate

$$\alpha_g\alpha_h = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 1 & 3-3g-(3-3g)h & \cdots & n-ng-(n-ng)h \end{pmatrix}.$$

For $x \in \{3,4,5,\ldots,n\}$, we have $x - xg - (x-xg)h \geq 1$ since $x - xg - (x-xg)h$ is in the image of $\alpha_g\alpha_h \in O_{n,1}$; thus, $f(x-xg-(x-xg)h) = x - xg - (x-xg)h$. Hence, $x_{g\oplus h}x_\omega\varphi = x_{g\oplus h}\varphi x_\omega\varphi = \alpha_{g\oplus h}\alpha_\omega$

$$= \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 1 & f(3-3g-(3-3g)h+1-1) & \cdots & f(n-ng-((n-ng)h)+1-1) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 1 & 3-3g-(3-3g)h & 4-4g-(4-4g)h & \cdots & n-ng-(n-ng)h \end{pmatrix}$$ □

$= \alpha_g\alpha_h = x_g\varphi x_h\varphi = (x_g x_h)\varphi$.

Now, we want to show the converse inclusion $ker\varphi \subseteq \sim$.

Lemma 3.8 *Let $v_1, v_2, \ldots, v_r \in \{x_g : g \in A_n\}$, $r \geq 2$. Then there is $h \in A_n$ such that $v_1v_2\cdots v_r \sim x_h x_\omega$.*

Proof We prove by induction on r. If $r = 2$, then the statement is satisfied by R_2. Suppose that for $v_1, v_2, \ldots, v_r \in \{x_g : g \in A_n\}$, for some integer $r \geq 2$, there is $h \in A_n$ such that $v_1v_2\cdots v_r \sim x_h x_\omega$. Let $v_1, v_2, \ldots, v_{r+1} \in \{x_g : g \in A_n\}$. Then $v_1(v_2v_2\cdots v_r v_{r+1}) \sim v_1(x_h x_\omega)$ for some $h \in A_n$. Further, $(v_1 x_h)x_\omega \sim (x_{\widehat{h}}x_\omega)x_\omega$ by R_2, for some $\widehat{h} \in A_n$. By R_1, there is $\widetilde{h} \in A_n$ such that $x_{\widehat{h}}x_\omega x_\omega \sim x_{\widetilde{h}}x_\omega$, i.e. $v_1v_2\cdots v_r v_{r+1} \sim x_{\widetilde{h}}x_\omega$. □

We put now $\widehat{A}_n = \{g \in A_n : |x - xg| > 1 \text{ for all } x \in \{3,4,5,\ldots,n\}\}$.

Lemma 3.9 *For all $h \in A_n$, there is $g \in \widehat{A}_n$ such that $x_g x_\omega \sim x_h x_\omega$.*

Proof Let $h \in A_n$. Suppose that $i - ih = 1$ for some $i \in \{3,4,5,\ldots,n\}$. Let $Q = \{x \in \{3,4,5,\ldots,n\} : x - xh = 1\} \neq \emptyset$. Then $xh \geq 2$ for all $x \in Q$. We put

$$xg = \begin{cases} xh & \text{if } x \notin Q; \\ xh - 1 & \text{if } x \in Q. \end{cases}$$

We have to show that $g \in A_n$.
We have $1g = 1h = 0$ and for $x \in \{2,\ldots,n\}$,

$$xg = \begin{cases} xh \neq 0 & \text{if } x \notin Q; \\ xh - 1 \geq 2 - 1 = 1 \neq 0 & \text{if } x \in Q. \end{cases}$$

There is $l \in \{3,4,5,\dots,n\}$ such that $lh = 1$, i.e. $|l - lh| \neq 1$ and $l \notin Q$. Thus, $lg = lh = 1$. Let $x \in \{2,3,4,\dots,n-1\}$ such that $(x+1)g > xg$. Then we have the following cases:

Case a: $xg = xh$. Assume that $(x+1)g = (x+1)h - 1$. Then $(x+1)h - 1 = (x+1)g > xg = xh$. This gives $(x+1)h > (x+1)h - 1 > xh$, i.e. $(x+1)h = xh + 1$. However, $(x+1)h = (x+1)h - 1 + 1 > xh + 1 = (x+1)h$. This is a contradiction. Hence, $(x+1)g = (x+1)h$. Thus, $(x+1)h > xh$, i.e. $(x+1)h = xh + 1$; therefore, $(x+1)g = (x+1)h = xh + 1 = xg + 1$.

Case b: $(x+1)g = (x+1)h$ and $xg = xh - 1$, i.e. $(x+1)h \geq xh$. This implies that $(x+1)h = xh$ or $(x+1)h = xh + 1$. If $(x+1)h = xh + 1$, then $1 = x + 1 - (x+1)h = x + 1 - xh - 1 = x - xh \geq 2$, a contradiction. Thus, we have $(x+1)h = xh$. This means that $(x+1)g = (x+1)h = xh$ and then $(x+1)g = xg + 1$.

Case c: $xg = xh - 1$ and $(x+1)g = (x+1)h - 1$. Then $(x+1)g > xg$ implies $(x+1)h > xh$, i.e. $(x+1)h = xh + 1$, $(x+1)h - 1 = xh - 1 + 1$; thus, $(x+1)g = xg + 1$.

Altogether, this shows that $g \in A_n$.

Let $x \in \{3,4,5,\dots,n\}$. If $x \notin Q$, then $x - xg = x - xh > 1$. If $x \in Q$, then $x - xg = x - xh + 1 = 2 > 1$. This means that $x - xg > 1$ for all $x \in \{3,4,5,\dots,n\}$. Therefore, $g \in \widehat{A}_n$.

Next, we show that $x_g x_\omega \sim x_h x_\omega$. We have $x_h x_\omega \approx x_{h \oplus \omega} x_\omega \in R_2$ and $x_g x_\omega \approx x_{g \oplus \omega} x_\omega \in R_2$. If $x \in Q$, then $x - xh = 1$ and $x - xg = x - xh + 1 = 2$. This gives $(x - xh)\omega + xh - 1 = 0 + xh - 1 = xh - 1 = 1 + xh - 1 - 1 = 1 + xg - 1 = (x - xg)\omega + xg - 1$. If $x \notin Q$, then $xh = xg$ and $(x - xh)\omega + xh - 1 = (x - xg)\omega + xg - 1$. This provides $x_{h \oplus \omega} x_\omega = x_{g \oplus \omega} x_\omega$. Then we obtain $x_h x_\omega \sim x_g x_\omega$ by transitivity. □

We define a function $\mu : \widehat{A}_n \to O_{n,1}$ by $g \mapsto \alpha_g \alpha_\omega$ for all $g \in \widehat{A}_n$.

Lemma 3.10 *μ is an injection from $\widehat{A}_n$ into $O_{n,1} \setminus G_{n,1}$.*

Proof We show that μ is a function from $\widehat{A}_n$ to $O_{n,1} \setminus G_{n,1}$. For this, we take $g \in \widehat{A}_n$ such that $g\mu = \alpha_g \alpha_\omega \in O_{n,1}$. This shows that $\alpha_g \alpha_\omega \notin G_{n,1}$ by Lemma 2.1.

We show μ is injective. For this, let $g_1, g_2 \in \widehat{A}_n$ such that $g_1\mu = g_2\mu$. By Lemma 3.5, we have $g_1\mu = \alpha_{g_1}\alpha_\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 1 & f(3 - 3g_1 - 1) & f(4 - 4g_1 - 1) & \cdots & f(n - ng_1 - 1) \end{pmatrix}$ and $g_2\mu = \alpha_{g_2}\alpha_\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 1 & f(3 - 3g_2 - 1) & f(4 - 4g_2 - 1) & \cdots & f(n - ng_2 - 1) \end{pmatrix}$. For $x \in \{3,4,5,\dots,n\}$, we have $x - xg_1 > 1$ and $x - xg_2 > 1$, this means $x - xg_1 - 1 = f(x - xg_1 - 1) = f(x - xg_2 - 1) = x - xg_2 - 1$. Hence $xg_1 = xg_2$. Altogether, we have $xg_1 = xg_2$ for all $x \in X_n$, that means $g_1 = g_2$. □

Theorem 3.11 *The semigroup $O_{n,1}$ has presentation $\langle Y_n | R_1 \cup R_2 \rangle$ via φ.*

Proof By Lemma 3.7, we have $\sim \subseteq ker\varphi$. It remains to show that $ker\varphi \subseteq \sim$. For this, let $(w_1, w_2) \in ker\varphi$. By Lemmas 3.8 and 3.9, there are $g_1, g_2 \in \widehat{A}_n$ such that $w_1 \sim x_{g_1} x_\omega$ and $w_2 \sim x_{g_2} x_\omega$. Since $\sim \subseteq ker\varphi$, we can calculate that $x_{g_1} x_\omega \varphi = w_1\varphi = w_2\varphi = x_{g_2} x_\omega \varphi$. In particular, $g_1\mu = \alpha_{g_1}\alpha_\omega = x_{g_1} x_\omega \varphi$ and

$g_2\mu = \alpha_{g_2}\alpha_\omega = x_{g_2}x_\omega\varphi$. This provides $g_1\mu = g_2\mu$. Since μ is injective by Lemma 3.10, we obtain $g_1 = g_2$. This gives $w_1 \sim x_{g_1}x_\omega = x_{g_2}x_\omega \sim w_2$, i.e. $w_1 \sim w_2$. □

We want to illustrate Theorem 3.11 for $n = 4$. It is easy to verify that $O_{4,1}$ consists of the five transformations

$$\begin{aligned} a &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix}, b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 \end{pmatrix}, c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix}, \\ d &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 \end{pmatrix}, \text{ and } e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \end{aligned}$$

Let $g_1, g_2 \in \{0,1,2,3\}^{X_4}$ with $1g_1 = 1g_2 = 0$, $2g_1 = 2g_2 = 3g_2 = 4g_1 = 1$, and $3g_1 = 4g_2 = 2$. It is easy to see that $A_4 = \{g_1, g_2, w\}$, i.e. $Y_4 = \{x_{g_1}, x_{g_2}, w\}$. For convenience, we use a, b, and c for x_{g_1}, x_{g_2}, and ω, respectively. The relations in R_1 provide

$$ac^2 \approx bc,\ bc^2 \approx bc, \text{ and } cc^2 \approx bc.$$

The relations in R_2 provide

$$\begin{array}{lll} aa \approx bc, & ba \approx bc, & ca \approx bc, \\ ab \approx ac, & bb \approx bc, & cb \approx ac, \\ ac \approx ac, & bc \approx bc, & cc \approx ac. \end{array}$$

Then we obtain the following presentation for $O_{4,1}$:
$\langle \{a,b,c\} \mid \{ac^2 \approx bc,\ bc^2 \approx bc,\ cc^2 \approx bc,\ aa \approx bc,\ ab \approx ac, ba \approx bc,\ bb \approx bc,\ ca \approx bc,\ cb \approx ac,\ cc \approx ac\} \rangle$.

Acknowledgments

We thank the reviewers for the very important suggestions. In particular, the results in Section 2 are based on the suggestions of the reviewers. This research is supported by Department of Mathematics, Faculty of Science, Khon Kaen University, Fiscal Year 2022, Thailand.

References

[1] Aĭzenštat AJa. Opredeljajuŝie sootnošenija polugruppy èndomorfizmov konečnogo linejno uporjadočennogo množestva. Sibirskii Matematicheskii Zhurnal 1962; 3: 161-169 (in Russian).

[2] Ayık G, Ayık H, Koç M. Combinatorial results for order-preserving and order-decreasing transformations. Turkish Journal of Mathematics 2011; 35 (6): 617-625. doi: 10.3906/mat-1010-432

[3] Chinram R, Yonthanthun W. Regularity on the semigroups of transformations with fixed point set. Thai Journal of Mathematics 2020; 18 (3): 1261-1268.

[4] East J. A presentation for the singular part of the full transformation semigroup. Semigroup Forum 2010; 81: 357-379.

[5] East J. Presentations for singular subsemigroups of the partial transformation semigroup. International Journal of Algebra and Computation 2010; 20 (1): 1-25.

[6] Fernandes VH, Gomes GMS, Jesus MM. Presentations for some monoids of partial transformations on a finite chain. Communications in Algebra 2005; 33 (2): 587-604.

[7] Gomes GMS, Howie JM. On the ranks of certain semigroups of order-preserving transformations. Semigroup Forum 1992; 45: 272-282.

[8] Ganyushkin O, Mazorchuk V. Classical Finite Transformation Semigroups. London, UK: Springer-Verlag, 2009.

[9] Higgins PM. Combinatorial results for semigroups of order-preserving mappings. Mathematical Proceedings of the Cambridge Philosophical Society 1993; 113: 281-296.

[10] Honyam P, Sanwong J. Semigroups of transformations with fixed sets. Quaestiones Mathematicae 2013; 36 (1): 79-92.

[11] Laradji A, Umar A. Combinatorial results for semigroups of order-preserving full transformations. Semigroup Forum 2006; 72: 51-62.

[12] Magill Jr. KD. Subsemigroups of $S(X)$. Mathematica Japonica 1966; 11: 109-115.

[13] McAlister D. Ash's Theorem, Finite Nilpotent Semigroups, and One-Dimensional Tiling Semigroups. Manuscript of talk given at CAUL, October 2001 and Northern Illinois University, July 2002.

[14] Stanley RP. Enumerative Combinatorics. Volume 1: Cambridge University Press, 1997.

[15] Yağci M, Korkmaz E. On nilpotent subsemigroups of the order-preserving and decreasing transformation semigroups. Semigroup Forum 2020; 101: 486-496.