Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 3083 – 3094
© TÜBİTAK
doi:10.55730/1300-0098.3321

k-generalized Pell numbers which are repdigits in base b

Zafer ŞİAR[1,*], **Refik KESKİN**[2]
[1]Department of Mathematics, Faculty of Arts and Science, Bingöl University, Bingöl, Turkey
[2]Department of Mathematics, Faculty of Arts and Science, Sakarya University, Sakarya, Turkey

Received: 08.10.2021 • **Accepted/Published Online:** 03.07.2022 • **Final Version:** 09.11.2022

Abstract: Let $k \geq 2$ be an integer and let $(P_n^{(k)})_{n \geq 2-k}$ be the k-generalized Pell sequence defined by

$$P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + ... + P_{n-k}^{(k)}$$

for $n \geq 2$ with initial conditions

$$P_{-(k-2)}^{(k)} = P_{-(k-3)}^{(k)} = \cdots = P_{-1}^{(k)} = P_0^{(k)} = 0, P_1^{(k)} = 1.$$

In this study, we deal with the Diophantine equation

$$P_n^{(k)} = d\left(\frac{b^m - 1}{b - 1}\right)$$

in positive integers n, m, k, b, d such that $m \geq 2,\ 2 \leq b \leq 9$ and $1 \leq d \leq b - 1$. We show that the repdigits in the base b in the $k-$generalized Pell sequence, which have at least two digits, are the numbers

$$\begin{aligned} P_7^{(4)} &= 228 = (444)_7,\ P_4^{(2)} = 12 = (22)_5,\ P_6^{(2)} = 70 = (77)_9; \\ P_4^{(k)} &= 13 = (111)_3 \end{aligned}$$

for $k \geq 3$ and

$$P_3^{(k)} = 5 = (11)_4$$

for $k \geq 2$.

Key words: Repdigit, Fibonacci and Lucas numbers, Exponential Diophantine equations, linear forms in logarithms; Baker's method

1. Introduction

Let k, r be integers with $k \geq 2$ and $r \neq 0$. Let the linear recurrence sequence $\left(G_n^{(k)}\right)_{n \geq 2-k}$ of order k be defined by

$$G_n^{(k)} = rG_{n-1}^{(k)} + G_{n-2}^{(k)} + \ldots + G_{n-k}^{(k)} \tag{1.1}$$

*Correspondence: zsiar@bingol.edu.tr

2010 *AMS Mathematics Subject Classification:* 11B39, 11D61, 11J86

for $n \geq 2$ with the initial conditions $G_{-(k-2)}^{(k)} = G_{-(k-3)}^{(k)} = \cdots = G_{-1}^{(k)} = 0$, $G_0^{(k)} = a$, and $G_1^{(k)} = b$. For $(a,b,r) = (0,1,1)$ and $(a,b,r) = (2,1,1)$, the sequence $\left(G_n^{(k)}\right)_{n\geq 2-k}$ is called the $k-$generalized Fibonacci sequence $\left(F_n^{(k)}\right)_{n\geq 2-k}$ and the $k-$generalized Lucas sequence $\left(L_n^{(k)}\right)_{n\geq 2-k}$, respectively (see [3, 4]). Also, the sequence $\left(F_n^{(3)}\right)_{n\geq -1}$ is called the Tribonacci sequence. For $(a,b,r) = (0,1,2)$ and $(a,b,r) = (2,2,2)$, the sequence $\left(G_n^{(k)}\right)_{n\geq 2-k}$ is called the $k-$generalized Pell sequence $\left(P_n^{(k)}\right)_{n\geq 2-k}$ and the $k-$generalized Pell-Lucas sequence $\left(Q_n^{(k)}\right)_{n\geq 2-k}$, respectively (see [15]). The terms of these sequences are called $k-$generalized Fibonacci numbers, $k-$generalized Lucas numbers, $k-$generalized Pell numbers and $k-$generalized Pell-Lucas numbers, respectively. When $k = 2$, we have Fibonacci, Lucas, Pell and Pell-Lucas sequences, denoted by

$$(F_n)_{n\geq 0},\ (L_n)_{n\geq 0},(P_n)_{n\geq 0}, \text{ and } (Q_n)_{n\geq 0},$$

respectively.

A base b repdigit is a positive integer N whose digits are all equal when written in base b. Particularly, we say, to simplify notation, for $b = 10$ that N is a repdigit. Recently, some mathematicians have investigated the repdigits or the repdigits in base b in the above sequences for $k = 2$ or general k. In [16], Luca determined that the largest repdigits in the sequences $(F_n)_{n\geq 0}$ and $(L_n)_{n\geq 0}$ are $F_{10} = 55$ and $L_5 = 11$. All base b repdigits in the Fibonacci and Lucas sequences have been investigated by Erduvan et al. in [12, 13] for $2 \leq b \leq 10$. In [14], the authors have found all repdigits in the sequences $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$. Here, they showed that the largest repdigits in these sequences are $P_5 = 5$ and $Q_2 = 6$. In [17], Marques proved that the largest repdigits in the Tribonacci sequence $\left(F_n^{(3)}\right)_{n\geq -1}$ is $F_8^{(3)} = 44$. Furthermore, in [6], Bravo and Luca handled the Diophantine equation

$$F_n^{(k)} = d\left(\frac{10^m - 1}{9}\right) \tag{1.2}$$

and they showed that this equation has only the solutions $(n,k,d,m) = (10,2,5,2)$, $(8,3,4,2)$ in positive integers n,m,k,d with $k \geq 2$, $1 \leq d \leq 9$ and $m \geq 2$. The same authors [4] considered the equation (1.2) for the $k-$generalized Lucas sequence and they have shown that all the solutions of a such equation are given by $(n,k,d,m) = (5,2,1,2),(5,4,2,2)$.

In this paper, motivated by the mentioned above, we will deal with the Diophantine equation

$$P_n^{(k)} = d\left(\frac{b^m - 1}{b-1}\right). \tag{1.3}$$

We think it is difficult to find an upper bound for b in this equation. Although b can be chosen in any interval $[2,c]$, we only consider the case $c = 9$. That is, we will handle the Diophantine equation (1.3) in positive integers n,m,k,b,d with $k,m \geq 2$, $2 \leq b \leq 9$ and $1 \leq d \leq b-1$. We determine all base b repdigits in the $k-$generalized Pell sequence, which have at least two digits. They are the numbers

$$\begin{aligned} P_7^{(4)} &= 228 = (444)_7,\ P_4^{(2)} = 12 = (22)_5,\ P_6^{(2)} = 70 = (77)_9; \\ P_4^{(k)} &= 13 = (111)_3 \end{aligned}$$

for $k \geq 3$ and

$$P_3^{(k)} = 5 = (11)_4$$

for $k \geq 2$. In addition to this, the authors in [7] and (*) have solved the equation (1.3) independently for $b = 10$ and they have shown that $P_5^{(3)} = 33$ and $P_6^{(4)} = 88$ are the only repdigits in the $k-$generalized Pell sequence, which have at least two digits.

2. Preliminaries

It can be seen that the characteristic polynomial of the $k-$generalized Pell sequence is given by

$$\Psi_k(x) = x^k - 2x^{k-1} - \cdots - x - 1. \tag{2.1}$$

We know from Lemma 1 given in [20] that this polynomial has exactly one positive real root located between 2 and 3. We denote the roots of the polynomial in (2.1) by $\alpha_1, \alpha_2, \ldots, \alpha_k$. Particularly, let $\alpha = \alpha_1$ denote the positive real root of $\Psi_k(x)$. The positive real root $\alpha = \alpha(k)$ is called dominant root of $\Psi_k(x)$. The other roots are strictly inside the unit circle. In [5], the Binet- like formula for the $k-$ generalized Pell numbers are given by

$$P_n^{(k)} = \sum_{j=1}^{k} \frac{(\alpha_j - 1)}{\alpha_j^2 - 1 + k(\alpha_j^2 - 3\alpha_j + 1)} \alpha_j^n. \tag{2.2}$$

It has been shown in [5] that the contribution of the roots inside the unit circle to the formula (2.1) is very small, namely that the approximatıon

$$\left| P_n^{(k)} - g_k(\alpha)\alpha^n \right| < \frac{1}{2} \tag{2.3}$$

holds for all $n \geq 2 - k$, where

$$g_k(z) := \frac{z - 1}{(k+1)z^2 - 3kz + k - 1}. \tag{2.4}$$

We will use the inequality

$$\left| \frac{(\alpha_j - 1)}{\alpha_j^2 - 1 + k(\alpha_j^2 - 3\alpha_j + 1)} \right| < 1 \tag{2.5}$$

for $k \geq 2$, where α_j's are the roots of the polynomial in (2.1) for $j = 1, 2, \ldots, k$. The proof of (2.5) can be found in [7] .

Throughout this paper, α denotes the positive real root of the polynomial given in (2.1). The following relation between α and $P_n^{(k)}$ given in [5] is valid for all $n \geq 1$:

$$\alpha^{n-2} \leq P_n^{(k)} \leq \alpha^{n-1}. \tag{2.6}$$

Furthermore, Kılıç [15] proved that

$$P_n^{(k)} = F_{2n-1} \tag{2.7}$$

for all $1 \leq n \leq k+1$.

*Şiar Z, Keskin R. Repdigits in k-generalized Pell sequences (preprint). arXiv:2009.13387

Lemma 2.1 ([5], Lemma 3.2)*Let $k, l \geq 2$ be integers. Then*

(a) *If $k > l$, then $\alpha(k) > \alpha(l)$, where $\alpha(k)$ and $\alpha(l)$ are the values of α relative to k and l, respectively.*

(b) $\varphi^2(1-\varphi^{-k}) < \alpha < \varphi^2$, *where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden section.*

(c) $g_k(\varphi^2) = \frac{1}{\varphi+2}$.

(d) $0.276 < g_k(\alpha) < 0.5$.

For solving the equation (1.3), we use linear forms in logarithms and Baker's theory. For this, we will give some notions, theorem, and lemmas related to linear forms in logarithms and Baker's theory.

Let η be an algebraic number of degree d with minimal polynomial

$$a_0x^d + a_1x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} \left(x - \eta^{(i)}\right) \in \mathbb{Z}[x],$$

where the a_i's are integers with $\gcd(a_0, \ldots, a_n) = 1$ and $a_0 > 0$ and the $\eta^{(i)}$'s are conjugates of η. Then

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\left\{|\eta^{(i)}|, 1\right\}\right)\right) \tag{2.8}$$

is called the logarithmic height of η. In particular, if $\eta = a/b$ is a rational number with $\gcd(a, b) = 1$ and $b \geq 1$, then $h(\eta) = \log(\max\{|a|, b\})$.

We give some properties of the logarithmic height whose proofs can be found in [10]:

$$h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2, \tag{2.9}$$

$$h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma), \tag{2.10}$$

$$h(\eta^m) = |m|h(\eta). \tag{2.11}$$

From the proof of Lemma 6 given in [8], we can write the inequality

$$h(g_k(\alpha)) < 5 \log k \text{ for } k \geq 2, \tag{2.12}$$

which will be used in the main theorem, where $g_k(\alpha)$ is as defined in (2.4). Now we give a theorem deduced from Corollary 2.3 of Matveev [18] and provides a large upper bound for the subscript n in the equation (1.3) (also see Theorem 9.4 in [9]).

Theorem 2.2 *Assume that $\gamma_1, \gamma_2, \ldots, \gamma_t$ are positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D, $b_1, b_2, \ldots, b_t$ are rational integers, and*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1$$

is not zero. Then

$$|\Lambda| > \exp\left(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2(1 + \log D)(1 + \log B)A_1A_2 \cdots A_t\right),$$

where

$$B \geq \max\{|b_1|, \ldots, |b_t|\},$$

and $A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}$ *for all* $i = 1, \ldots, t$.

Now we give a lemma which was proved in [2]. It is a version of the lemma given by Dujella and Pethő [11]. The lemma given in [11] is a variation of a result of Baker and Davenport [1]. This lemma will be used to reduce the upper bound for the subscript n in the equation (1.3). For any real number x, we let $||x|| = \min\{|x-n| : n \in \mathbb{Z}\}$ be the distance from x to the nearest integer.

Lemma 2.3 *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational number γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\epsilon := ||\mu q|| - M||\gamma q||$. If $\epsilon > 0$, then there exists no solution to the inequality*

$$0 < |u\gamma - v + \mu| < AB^{-w},$$

in positive integers u, v, and w with

$$u \leq M \text{ and } w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

The following lemma can be found in [19].

Lemma 2.4 *Let $a, x \in \mathbb{R}$. If $0 < a < 1$ and $|x| < a$, then*

$$|\log(1+x)| < \frac{-\log(1-a)}{a} \cdot |x|$$

and

$$|x| < \frac{a}{1-e^{-a}} \cdot |e^x - 1|.$$

3. Main theorem

Theorem 3.1 *All solutions (n, m, b, d, k) of Diophantine equation (1.3) in positive integers n, m, k, b, d such that $k, m \geq 2$, $2 \leq b \leq 9$ and $1 \leq d \leq b-1$ are given by $(n, m, b, d, k) = (7, 3, 7, 4, 4), (4, 2, 5, 2, 2), (6, 2, 9, 7, 2)$ and $(n, m, b, d, k) = (4, 3, 3, 1, k)$ for $k \geq 3$ and $(n, m, b, d, k) = (3, 2, 4, 1, k)$ for $k \geq 2$.*

Proof Assume that $P_n^{(k)} = d\left(\frac{b^m-1}{b-1}\right)$ with $n \geq 1$, $m, k \geq 2$, $2 \leq b \leq 9$ and $1 \leq d \leq b-1$. If $1 \leq n \leq k+1$, then we have

$$d\left(\frac{b^m-1}{b-1}\right) = P_n^{(k)} = F_{2n-1}$$

by (2.7). In this case we get $(n, m, b, d) = (3, 2, 4, 1), (4, 3, 3, 1)$ by Corollary 5 given in [12]. Now we can suppose that $n \geq k+2$. Since $k \geq 2$, we have $n \geq 4$. Let α be positive real root of $\Psi_k(x)$ given in (2.1). Then $2 < \alpha < \varphi^2 < 3$ by Lemma 2.1 (b). Besides, it is seen that $2^{m-1} < P_n^{(k)} < 9^m$. Thus, using the inequality (2.6), we get

$$(n-2)\frac{\log 2}{\log 9} < m < (n-1)\frac{\log 3}{\log 2} + 1,$$

which implies that

$$\frac{3n}{20} < m < \frac{9n}{5} \tag{3.1}$$

for $n \geq 4$. Now, rearranging the equation (1.3) as

$$P_n^{(k)} - g_k(\alpha)\alpha^n + \frac{d}{b-1} = d\frac{b^m}{b-1} - g_k(\alpha)\alpha^n$$

and taking the absolute value of both sides, we get

$$\left| d\left(\frac{b^m}{b-1}\right) - g_k(\alpha)\alpha^n \right| = \left| P_n^{(k)} - g_k(\alpha)\alpha^n + \frac{d}{b-1} \right|.$$

Thus, using the inequality (2.3), it is seen that

$$\left| d\left(\frac{b^m}{b-1}\right) - g_k(\alpha)\alpha^n \right| < \frac{3}{2}. \tag{3.2}$$

If we divide both sides of the inequality (3.2) by $g_k(\alpha)\alpha^n$, from Lemma 2.1, we get

$$\left| b^m\alpha^{-n}\frac{d(g_k(\alpha))^{-1}}{b-1} - 1 \right| < \frac{3}{2g_k(\alpha)\alpha^n} < \frac{3}{0.552\cdot\alpha^n} < \frac{5.5}{\alpha^n}. \tag{3.3}$$

In order to use Theorem 2.2, we take

$$(\gamma_1, b_1) := (b, m)\,,\ (\gamma_2, b_2) := (\alpha, -n)\,,\ (\gamma_3, b_3) := \left(\frac{(b-1)\cdot g_k(\alpha)}{d}, -1\right).$$

The number field containing γ_1, γ_2, and γ_3 is $\mathbb{K} = \mathbb{Q}(\alpha)$, which has degree $D = k$. We show that the number

$$\Lambda_1 := b^m\alpha^{-n}\frac{d(g_k(\alpha))^{-1}}{b-1} - 1$$

is nonzero. Contrast to this, assume that $\Lambda_1 = 0$. Then

$$d\frac{b^m}{b-1} = \alpha^n g_k(\alpha) = \frac{\alpha - 1}{(k+1)\alpha^2 - 3k\alpha + k - 1}\alpha^n.$$

Conjugating the above equality by some automorphisim of the Galois group of the splitting field of $\Psi_k(x)$ over $\mathbb{Q}$ and taking absolute values, we get

$$d\frac{b^m}{b-1} = \left| \frac{\alpha_i - 1}{(k+1)\alpha_i^2 - 3k\alpha_i + k - 1}\alpha_i^n \right|$$

for some $i > 1$, where $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_k$ are the roots of $\Psi_k(x)$. Using (2.5) and that $|\alpha_i| < 1$, we obtain from the last equality that

$$d\left(\frac{b^m}{b-1}\right) = \left| \frac{\alpha_i - 1}{(k+1)\alpha_i^2 - 3k\alpha_i + k - 1} \right| |\alpha_i|^n < 1,$$

which is impossible since $m \geq 2$. Therefore, $\Lambda_1 \neq 0$. Moreover, since

$$h(b) = \log b \leq \log 9, h(\gamma_2) = \frac{\log \alpha}{k} < \frac{\log 3}{k}$$

by (2.8) and

$$\begin{aligned} h(\gamma_3) &= h\left(\frac{(b-1)\cdot g_k(\alpha)}{d}\right) \leq h((b-1)/d) + h(g_k(\alpha)) \\ &< \log 9 + 5\log k < 10\log k \end{aligned}$$

by (2.12), we can take $A_1 := k\log 9$, $A_2 := \log 3$, and $A_3 := 10k\log k$. Also, since $m < 9n/5$ by (3.1), it follows that $B := 9n/5$. Thus, taking into account the inequality (3.3) and using Theorem 2.2, we obtain

$$\frac{5.5}{\alpha^n} > |\Lambda_1| > \exp\left(-1.4\cdot 30^6 \cdot 3^{4.5}\cdot k^2(1+\log k)(1+\log(9n/5))\,(k\log 9)\,(\log 3)\,(10k\log k)\right)$$

and so

$$n\log\alpha - \log(5.5) < 1.4\cdot 30^6\cdot 3^{4.5}\cdot k^2(3\log k)(3\log n)\,(k\log 9)\,(\log 3)\,(10k\log k)\,,$$

where we have used the fact that $1+\log k < 3\log k$ for all $k \geq 2$ and $1+\log(9n/5) < 3\log n$ for $n \geq 3$. From the last inequality, a quick computation with Mathematica yields

$$n\log\alpha < 3.12\cdot 10^{13}\cdot k^4\cdot(\log k)^2\cdot\log n$$

or

$$n < 4.51\cdot 10^{13}\cdot k^4\cdot(\log k)^2\cdot\log n. \tag{3.4}$$

The inequality (3.4) can be rearranged as

$$\frac{n}{\log n} < 4.51\cdot 10^{13}\cdot k^4\cdot(\log k)^2.$$

Using the fact that

$$\text{if } A \geq 3 \text{ and } \frac{n}{\log n} < A, \text{ then } n < 2A\log A,$$

we obtain

$$\begin{aligned} n &< 2\cdot 4.51\cdot 10^{13}\cdot k^4\cdot(\log k)^2\log\left(4.51\cdot 10^{13}\cdot k^4\cdot(\log k)^2\right) \qquad (3.5) \\ &< 9.02\cdot 10^{13}\cdot k^4\cdot(\log k)^2(31.5+4\log k+2\log(\log k)) \\ &< 9.02\cdot 10^{13}\cdot k^4\cdot(\log k)^2(52\log k) \\ &< 4.7\cdot 10^{15}\cdot k^4\cdot(\log k)^3, \end{aligned}$$

where we have used the fact that $31.5+4\log k+2\log(\log k) < 52\log k$ for all $k \geq 2$.

Let $k \in [2, 450]$. Now, let us reduce the upper bound on n applying Lemma 2.3. Let

$$z_1 := m\log b - n\log\alpha + \log\left[\frac{d}{b-1}\left(g_k(\alpha)\right)^{-1}\right].$$

and $x := e^{z_1} - 1$. Then, from (3.3), we get

$$|x| = |e^{z_1} - 1| < \frac{5.5}{\alpha^n} < 0.35$$

for $n \geq 4$. Choosing $a :=\ 0.35$, we obtain the inequality

$$|z_1| = |\log(x+1)| < \frac{\log(100/65)}{0.35} \cdot \frac{5.5}{\alpha^n} < \frac{6.77}{\alpha^n}$$

by Lemma 2.4. Thus, it follows that

$$0 < \left| m \log b - n \log \alpha + \log \left[\left(\frac{d}{b-1} \right) (g_k(\alpha))^{-1} \right] \right| < \frac{6.77}{\alpha^n}.$$

Dividing this inequality by $\log \alpha$, we get

$$0 < |m\gamma - n + \mu| < A \cdot B^{-w}, \tag{3.6}$$

where

$$\gamma := \frac{\log b}{\log \alpha} \notin \mathbb{Q},\ \ \mu := \frac{\log \left(\frac{d}{b-1} (g_k(\alpha))^{-1} \right)}{\log \alpha},\ \ A := 9.77,\ \ B := \alpha, \text{ and } w := n.$$

If we take

$$M := \left\lfloor 4.7 \cdot 10^{15} \cdot k^4 \cdot (\log k)^3 \right\rfloor,$$

which is an upper bound on m since $m < n < 4.7 \cdot 10^{15} \cdot k^4 \cdot (\log k)^3$ by (3.5), we found that q_{221}, the denominator of the 221 st convergent of γ exceeds $6M$. Furthermore, a quick computation with Mathematica gives us that the value

$$\frac{\log \left(A q_{221}/\epsilon \right)}{\log B}$$

is less than 442.9. So, if the inequality (3.6) has a solution, then

$$n < \frac{\log \left(A q_{221}/\epsilon \right)}{\log B} < 442.9,$$

that is, $n \leq 442$. In this case, $m < 796$ by (3.1). A quick computation with Mathematica gives us that the equation

$$P_n^{(k)} = d \left(\frac{b^m - 1}{b - 1} \right)$$

has only the solutions

$$(n, m, b, d, k) = (7, 3, 7, 4, 4), (4, 2, 5, 2, 2), (6, 2, 9, 7, 2)$$

in the intervals $n \in [4, 442]$, $m \in [2, 796)$, and $k \in [2, 450]$. Thus, this completes the analysis in the case $k \in [2, 450]$.

From now on, we can assume that $k > 450$. Then we can see from (3.5) that the inequality

$$n < 4.7 \cdot 10^{15} \cdot k^4 \cdot (\log k)^3 < \varphi^{k/2} \tag{3.7}$$

holds for $k > 450$.

By Lemma 2 given in [6], we have

$$g_k(\alpha)\alpha^n = \frac{\varphi^{2n}}{\varphi+2}(1+\eta), \tag{3.8}$$

where

$$|\eta| < \frac{4}{\varphi^{k/2}}. \tag{3.9}$$

So, using (3.2), (3.8), and (3.9), we obtain

$$\begin{aligned}\left|\frac{d \cdot b^m}{b-1} - \frac{\varphi^{2n}}{\varphi+2}\right| &= \left|\frac{d \cdot b^m}{b-1} - g_k(\alpha)\alpha^n + \frac{\varphi^{2n}}{\varphi+2}\eta\right| \\ &\leq \left|\frac{d \cdot b^m}{b-1} - g_k(\alpha)\alpha^n\right| + \frac{\varphi^{2n}}{\varphi+2}|\eta| \\ &< \frac{3}{2} + \frac{4\varphi^{2n}}{\varphi^{k/2}(\varphi+2)}.\end{aligned} \tag{3.10}$$

Dividing both sides of the above inequality by $\dfrac{\varphi^{2n}}{\varphi+2}$, we get

$$\begin{aligned}\left|\frac{b^m\varphi^{-2n} \cdot d}{b-1}(\varphi+2) - 1\right| &< \frac{3(\varphi+2)}{2\varphi^{2n}} + \frac{4}{\varphi^{k/2}} \\ &< \frac{0.01}{\varphi^{k/2}} + \frac{4}{\varphi^{k/2}} = \frac{4.01}{\varphi^{k/2}},\end{aligned} \tag{3.11}$$

where we have used the fact that

$$\frac{3(\varphi+2)}{2\varphi^{2n}} < \frac{0.01}{\varphi^{k/2}}$$

for $k > 450$ and $n \geq k+2$. In order to use the result of Theorem 2.2, we take

$$(\gamma_1, b_1) := (b, m), (\gamma_2, b_2) := (\varphi, -2n), (\gamma_3, b_3) := \left(\frac{d(\varphi+2)}{b-1}, 1\right).$$

The number field containing γ_1, γ_2, and γ_3 is $\mathbb{K} = \mathbb{Q}(\sqrt{5})$, which has degree $D = 2$. We show that the number

$$\Lambda_2 := b^m\varphi^{-2n}\frac{d}{b-1}(\varphi+2) - 1$$

is nonzero. Contrast to this, assume that $\Lambda_2 = 0$. Then

$$b^m\frac{d}{b-1}(\varphi+2) = \varphi^{2n}$$

and conjugating this relation in $\mathbb{Q}(\sqrt{5})$, we get $b^m\left(\frac{d}{b-1}\right)(\beta+2) = \beta^{2n}$, where $\beta = \frac{1-\sqrt{5}}{2} = \frac{-1}{\varphi}$. So, we have

$$\frac{\varphi^{2n}}{\varphi+2} = \frac{\beta^{2n}}{\beta+2},$$

which implies that

$$\frac{\varphi^{4n}}{\varphi+2}=\frac{1}{\beta+2}<1.$$

The last inequality is impossible for $n\geq 4$. Therefore, $\Lambda_2\neq 0$. Moreover, since

$$h(\gamma_1)=h(b)\leq \log 9, h(\gamma_2)=h(\varphi)\leq \frac{\log\varphi}{2}$$

and

$$h(\ \gamma_3)\leq h(d/(b-1))+h(\varphi)+h(2)+\log 2\leq \log 36+\frac{\log\varphi}{2},$$

by (2.10), we can take $A_1:=2\log 9$, $A_2:=\log\varphi$, and $A_3:=\log(1296\varphi)$. Also, since $m<\frac{9n}{5}$, we can take $B:=2n$. Thus, taking into account the inequality (3.11) and using Theorem 2.2, we obtain

$$(4.01)\cdot\varphi^{-k/2}>|\Lambda_2|>\exp\left(C\cdot(1+\log 2n)\left(2\log 9\right)(\log\varphi)\log\left(1296\varphi\right)\right),$$

where $C=-1.4\cdot 30^6\cdot 3^{4.5}\cdot 2^2\cdot(1+\log 2)$ and we have used the fact that $(1+\log 2n)<2\log 2n$ for $n\geq 2$. This implies that

$$\frac{k}{2}\log\varphi-\log(4.01)<3.14\cdot 10^{13}\cdot\log 2n$$

or

$$k<1.31\cdot 10^{14}\cdot\log 2n. \tag{3.12}$$

On the other hand, from (3.7), we get

$$\begin{aligned}\log 2n &< \log\left(2\cdot 4.7\cdot 10^{15}\cdot k^4\cdot(\log k)^3\right)\\ &< 36.8+4\log k+3\log(\log k)\\ &< 57\log k\end{aligned}$$

for $k\geq 2$. So, from (3.12), we obtain

$$k<1.31\cdot 10^{14}\cdot 57\log k,$$

which implies that

$$k<3.01\cdot 10^{17}. \tag{3.13}$$

To reduce this bound on k, we use Lemma 2.3. Substituting this bound of k into (3.7), we get $n<2.52\cdot 10^{90}$, which implies that $m<4.54\cdot 10^{90}$ by (3.1).

Now, let

$$z_2:=m\log b-2n\log\varphi+\log\left(\frac{d}{b-1}(\varphi+2)\right).$$

and $x:=1-e^{z_2}$. Then, by (3.11), we have

$$|x|=|1-e^{z_2}|<\frac{4.01}{\varphi^{k/2}}<0.1$$

for $k>450$. Choosing $a:=0.1$, we obtain the inequality

$$|z_2| = |\log(x+1)| < \frac{\log(10/9)}{0.1} \cdot \frac{4.01}{\varphi^{k/2}} < \frac{4.23}{\varphi^{k/2}}$$

by Lemma 2.4. That is,

$$0 < \left| m\log b - 2n\log\varphi + \log\left(\frac{d}{9}(\varphi+2)\right)\right| < \frac{4.23}{\varphi^{k/2}}.$$

Dividing both sides of the above inequality by $\log\varphi$, we get

$$0 < |m\gamma - 2n + \mu| < A \cdot B^{-w}, \tag{3.14}$$

where

$$\gamma := \frac{\log b}{\log\varphi} \notin \mathbb{Q},\ \ \mu := \frac{\log\left(\frac{d\cdot(\varphi+2)}{b-1}\right)}{\log\varphi},\ \ A := 8.8,\ \ B := \varphi, \text{ and } w := k/2.$$

If we take $M := 4.54 \cdot 10^{90}$, which is an upper bound on m, we found that q_{190}, the denominator of the 190 th convergent of γ exceeds $6M$. Furthermore, a quick computation with Mathematica gives us that the value

$$\frac{\log\left(Aq_{190}/\epsilon\right)}{\log B}$$

is less than 514.47. So, if the inequality (3.14) has a solution, then

$$\frac{k}{2} < \frac{\log\left(Aq_{190}/\epsilon\right)}{\log B} < 514.47,$$

which implies that $k \leq 1028$. Hence, from (3.7), we get $n < 1.76 \cdot 10^{30}$, which implies that $m < 3.168 \cdot 10^{30}$ since $m < 9n/5$ by (3.1). If we again apply Lemma 2.3 to (3.14) with $M := 3.168 \cdot 10^{30}$, we found that q_{69}, the denominator of the 69 th convergent of γ exceeds $6M$. After doing this, then a quick computation with Mathematica shows that the inequality (3.14) has a solution only for $k \leq 375$. This contradicts the fact that $k > 450$. This completes the proof. □

Thus, we can give the following results easily.

Corollary 3.2 *Let $2 \leq b \leq 9$. If P_n is a base b repdigit and has at least two digits, then $(n,b) = (3,4), (4,5)$, and $(6,9)$. Namely, $P_3 = 5 = (11)_4$, $P_4^{(2)} = 12 = (22)_5$, $P_6^{(2)} = 70 = (77)_9$.*

Corollary 3.3 *If $m > 1$, then the equation $P_n^{(k)} = b^m - 1$ has no solution for $2 \leq b \leq 9$.*

Corollary 3.4 *The $k-$generalized Pell sequence does not contain any Mersenne number greater than 1.*

References

[1] Baker A, Davenport H. The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$. The Quarterly Journal of Mathematics 1969; 20 (1): 129-137. doi: 10.1093/qmath/20.1.129

[2] Bravo JJ, Gomez CA, Luca F. Powers of two as sums of two k-Fibonacci numbers. Miskolc Mathematical Notes 2016; 17 (1): 85-100. doi: 10.18514/MMN.2016.1505

[3] Bravo JJ, Luca F. Powers of two in generalized Fibonacci sequences. Revista Colombiana de Matemáticas 2012; 46 (1): 67-79.

[4] Bravo JJ, Luca F. Repdigits in k-Lucas sequences. Proceedings Mathematical Sciences 2014; 124 (2): 141-154. doi: 10.1007/s12044-014-0174-7

[5] Bravo JJ, Luca F. On a generalization of the Pell sequence. Mathematica Bohemica 2021; 146 (2): 199-213. doi: 10.21136/MB.2020.0098-19

[6] Bravo JJ, Luca F. On a conjecture about repdigits in k-generalized Fibonacci sequences. Publicationes Mathematicae Debrecen 2013; 82 (3-4): 623-639. doi: 10.5486/PMD.2013.5390

[7] Bravo JJ, Herrera JL. Repdigits in k-generalized Pell sequences. Archivum mathematicum 2020; 56 (4): 249-262. doi: 10.5817/AM2020-4-249

[8] Bravo JJ,, Herrera JL, Luca F. Common values of generalized Fibonacci and Pell sequences. Journal of Number Theory 2021; 226: 51-71. doi: 10.1016/j.jnt.2021.03.001

[9] Bugeaud Y, Mignotte M, Siksek S. Classical and modular approaches to exponential Diophantine equations I. Fibonacci and Lucas perfect powers. Annals of Mathematics 2006; 163 (3): 969-1018. doi: 10.4007/annals.2006.163.969

[10] Bugeaud Y. Linear Forms in Logarithms and Applications. IRMA Lectures in Mathematics and Theoretical Physics. 28, Zurich: European Mathematical Society, 2018.

[11] Dujella A, Pethò A. A generalization of a theorem of Baker and Davenport. The Quarterly Journal of Mathematics 1998; 49 (3): 291-306. doi: 10.1093/qmathj/49.3.291

[12] Erduvan F, Keskin R, Şiar Z. Repdigits base b as products of two Fibonacci numbers. Indian Journal of Pure and Applied Mathematics 2021; 52 (3): 861-868. doi: 10.1007/s13226-021-00041-8

[13] Erduvan F, Keskin R, Şiar Z. Repdigits base b as products of two Lucas numbers. Quaestiones Mathematicae 2021; 44 (10): 1283-1293. doi: 10.2989/16073606.2020.1787539

[14] Faye B, Luca F. Pell and Pell-Lucas numbers with only one distinct digit. Annales Mathematicae et Informaticae 2015; 45: 55-60.

[15] Kılıç E, Taşcı D. The generalized binet formula, representation and sums of the generalized order-k Pell numbers. Taiwanese Journal of Mathematics 2006; 10 (6): 1661-1670. doi: 10.11650/twjm/1500404581

[16] Luca F. Fibonacci and Lucas numbers with only one distinct digit. Portugaliae Mathematica 2000; 57 (2): 243-254.

[17] Marques D. On k-generalized Fibonacci numbers with only one distinct digit, Utilitas Mathematica 2015; 98: 23-31.

[18] Matveev EM. An Explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers II. Izvestiya Rossijskoj Akademii Nauk Seriya Matematicheskaya 2000; 64 (6): 125-180 (Russian). Translation in Izvestiya Mathematics 2000; 64 (6): 1217-1269. doi: 10.1070/IM2000v064n06ABEH000314

[19] Weger de BMM. Algorithms for Diophantine Equations. CWI Tracts 65. Amsterdam: Centrum voor Wiskunde en Informatica, 1989.

[20] Wu Z, Zhang H. On the reciprocal sums of higher-order sequences. Advances in Difference Equations 2013; 2013, Article No: 189. doi: 10.1186/1687-1847-2013-189