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# On a subclass of the analytic and bi-univalent functions satisfying subordinate condition defined by $q$-derivative

**Nizami MUSTAFA, Semra KORKMAZ***
Department of Mathematics, Faculty of Science and Letters, Kafkas University, Kars, Turkey

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**Abstract:** In this paper, we introduce and examine certain subclass $M_{q,\Sigma}(\varphi,\beta)$ of analytic and bi-univalent functions on the open unit disk in the complex plane. Here, we give coefficient bound estimates, upper bound estimate for the second Hankel determinant and Fekete-Szegö inequality for the function belonging to this class. Some interesting special cases of the results obtained here are also discussed.

**Key words:** Subordination, convex function, quasi close-to-$q$-convex function, $q$-derivative

## 1. Introduction and preliminaries

Let $\mathrm{A}$ denote the class of all complex valued functions $f$ given by

$$f(z) = z + a_2 z^2 + a_3 z^3 + ... + a_n z^n + ... = z + \sum_{n=2}^{\infty} a_n z^n, \ a_n \in \mathbb{C}, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane. By $S$ we define the class of all univalent functions in $\mathrm{A}$.

For $\alpha \in [0,1)$, some of the important and well-investigated subclasses of S include the classes $\mathrm{S}^*(\alpha)$ and $\mathrm{C}(\alpha)$, respectively, starlike and convex function classes of order $\alpha$ in $\mathbb{U}$.

By definition, we have

$$\mathrm{S}^*(\alpha) = \left\{ f \in \ \mathrm{S} : \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \ z \in \mathbb{U} \right\}$$

and

$$\mathrm{C}(\alpha) = \left\{ f \in \mathrm{S} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \ z \in \mathbb{U} \right\}.$$

In his fundamental paper [16] for $q \in (0,1)$, Jackson introduce $q-$ derivative operator $D_q$ of an analytic

*Correspondence: korkmazsemra36@outlook.com
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function $f$ as follows:

$$D_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z}, & \text{if} \quad z \neq 0, \\ f^{'}(0), & \text{if} \quad z = 0. \end{cases} \tag{1.2}$$

The formulas for $q-$derivative of a product and a quotient of functions are

$$D_q z^n = [n]_q \, z^{n-1}, \; n \in \mathbb{N},$$

where $[n]_q = \sum\limits_{k=1}^{n} q^{k-1}$ is $q-$ analogue of the natural numbers $n$. It can be easily shown that $[n]_q = \frac{1-q^n}{1-q}$, $[0]_q = 0, \; [1]_q = 1, \; \lim\limits_{q \to 1^-} [n]_q = n.$

It follows from (1.2) that

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q \, a_n z^{n-1}$$

and

$$D_q^2 f(z) = D_q \left( D_q f(z) \right) = \sum_{n=2}^{\infty} [n]_q \, [n-1]_q \, a_n z^{n-2}$$

for the function $f \in A$. Also, it is clear that $\lim\limits_{q \to 1^-} D_q f(z) = f^{'}(z)$ for an analytic function $f$.

Let us define the following subclasses of analytic functions.

**Definition 1.1** *For $q \in (0,1)$ and $\alpha \in [0,1)$, a function $f \in A$ is said to be in $q$-starlike function class $S_q^* (\alpha)$ of order $\alpha$, if satisfied the following condition*

$$\Re \left( \frac{z D_q f\left(z\right)}{f\left(z\right)} \right) > \alpha, \; z \in \mathbb{U} \; .$$

**Definition 1.2** *For $q \in (0,1)$ and $\alpha \in [0,1)$, a function $f \in A$ is said to be in $q$- convex function class $C_q (\alpha)$ of order $\alpha$, if satisfied the following condition*

$$\Re \left( 1 + \frac{z D_q^2 f\left(z\right)}{D_q f\left(z\right)} \right) > \alpha, \; z \in \mathbb{U} \; .$$

Studies on $q-$ derivative were firstly initiated by Jackson [15, 16], Carmichael [9], Mason [18], Adams [1] and Trjizinsky [32]. This topic was forgotten for a long time. Later, some properties related with function theory involving $q-$ theory were introduced by Ismail et al. [14]. Recently, many studies were done on this subject (see [2, 28, 29]). As the study [2] suggests, there is a lot that can be done for this research topic. For example, $q-$ analogy of starlikeness and convexity of analytic functions in the open unit disk and in arbitrary simply connected domains would be interesting for researchers in this field.

In [3], by using applications of $q-$ derivative, it was shown that Szasz Mirakyan operators are convex when convex functions are taken such that their result generalizes well known results for $q = 1$. Also, in [3] the authors showed that $q-$ derivatives of these operators approach $q-$ derivatives of approximated functions.

Very soon, by Uçar [33] and Uçar et al. [34] studied some properties of $q$- close-to-convex functions. By Polatoğlu in [27] was investigated $q-$ starlike functions and gave growth and distortion theorems for this class. Quasi starlike and quasi-convex functions were studied by Altıntaş in [5]. Altıntaş and Mustafa studied quasi $q-$ starlike and quasi $q-$ convex functions (see [6]).

Very recently, in [4] Altıntaş and Aydoğan studied quasi $q-$ convex functions.

Now, let's give some preliminary information and definitions that we use in our study.

It is well-known that every function $f \in \mathrm{S}$ given by (1.1) has an inverse $f^{-1}$ defined as follows

$$\begin{aligned} f^{-1}(f(z)) &= z, z \in \mathbb{U}, f^{-1}(f(w)) = w, w \in \mathbb{U}_{r_0} = \{w \in \mathbb{C} : |w| < r_0(f)\}, \\ r_0(f) &\geq 1/4 \end{aligned}$$

and

$$f^{-1}(w) = w + A_2 w^2 + A_3 w^3 + A_4 w^4 + ..., \ w \in \mathbb{U}_{r_0}, \tag{1.3}$$

where

$$A_2 = -a_2, A_3 = 2a_2^2 - a_3, A_4 = -5a_2^3 + 5a_2 a_3 - a_4.$$

Also, it is well known that a function $f \in \mathrm{A}$ is called bi-univalent function in $\mathbb{U}$, if both $f$ and $f^{-1}$ are univalent in $\mathbb{U}$ and $\mathbb{U}_{r_0}$, respectively. Let $\Sigma$ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). For a short history and examples of functions belonging to the class $\Sigma$ see [30].

For the functions $f$ and $g$ analytic in $\mathbb{U}$, $f$ is said to be subordinate to $g$ and denoted as $f(z) \prec g(z)$, if there exists an analytic function $\omega$ such that

$$\omega(0) = 0, \ \ |\omega(z)| < 1 \text{ and } \ f(z) = g(\omega(z)) \ .$$

In particular, when $g$ is univalent in $U$

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}), \ z \in \mathbb{U}.$$

Firstly, by Lewin in [17] he introduced a subclass of bi-univalent functions and obtained the estimate $|a_2| \leq 1.51$ for the function belonging to this class. Subsequently, Brannan and Clunie in [7] developed the result of Lewin to $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Later, Netanyahu [19] showed that $|a_2| \leq \frac{4}{3}$. Brannan and Taha [8] introduced certain subclasses of bi-univalent function class $\Sigma$, namely bi-starlike function of order $\alpha$ denoted $\mathrm{S}^*_\Sigma(\alpha)$ and bi-convex function of order $\alpha$ denoted $\mathrm{C}_\Sigma(\alpha)$ corresponding to the function classes $\mathrm{S}^*(\alpha)$ and $\mathrm{C}(\alpha)$, respectively. For each of the function classes $\mathrm{S}^*_\Sigma(\alpha)$ and $\mathrm{C}_\Sigma(\alpha)$, nonsharp estimates on the first two Taylor-Maclaurin coefficients were found in [8, 31]. Many researchers have introduced and investigated several interesting subclasses of bi-univalent function class $\Sigma$ and they have found nonsharp estimates on the first two Taylor-Maclaurin coefficients (see [30, 36]).

Among the important tools in the theory of analytic functions is Hankel determinant, which is defined by coefficients of the function $f \in S$. The Hankel determinants $H_m(n), n = 1, 2, 3, ..., m = 1, 2, 3, ...$ of the function $f \in S$ are defined by (see [20]).

$$H_m(n) = \begin{pmatrix} a_n & a_{n+1} & \cdots & a_{n+m-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+m-1} & a_{n+m} & \cdots & a_{n+m-2} \end{pmatrix}, \ a_1 = 1.$$

Generally, these determinants were investigated by researchers with $m = 2$.

Recently, the upper bounds of second Hankel determinant $|H_2(2)| = \left|a_2 a_4 - a_3^2\right|$ for the classes $S_\Sigma^*(\alpha)$ and $\mathcal{C}_\Sigma(\alpha)$ were obtained by Deniz et al. [10] . Very soon, Orhan et al. [26] reviewed the study of bound for the second Hankel determinant for the subclass $M_\Sigma^\alpha(\beta)$ of bi-univalent functions. Mustafa et al. [23] improved the results obtained in [10].

One of the important tools in the theory of analytic functions is the functional $H_2(1) = a_3 - a_2^2$, which is known as the Fekete-Szegö functional and one usually considers the further generalized functional $a_3 - \mu a_2^2$, where $\mu$ is a complex or real number (see [12]). Estimating the upper bound of $\left|a_3 - \mu a_2^2\right|$ is known as the Fekete-Szegö problem in the theory of analytic functions. The well-known result due to them states that if $f \in A$, then

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} 3 - 4\mu, & \text{if} \quad \mu \le 0, \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right), & \text{if} \quad 0 \le \mu \le 1, \\ 4\mu - 3, & \text{if} \quad 1 \le \mu. \end{cases}$$

In 1969 Koegh and Merkes ([21]) solved the Fekete- Szegö problem for the classes of starlike and convex functions for some real $\mu$. The Fekete-Szegö problem has been investigated by many mathematicians for several subclasses of analytic functions ([20–22, 25]). Zaprawa (see [35]) has studied on Fekete-Szegö problem for some subclasses of bi-univalent functions. Very soon, Mustafa and Mrugusundaramoorthy [24] solved the Fekete-Szegö problem for the subclass of bi-univalent functions related to shell shaped region.

We define the following subclasses of analytic functions.

**Definition 1.3** *For $q \in (0,1)$, a function $f \in A$ given by (1.1) is said to be in the class $S_q^*(\varphi)$, which we will call $q-$ starlike function class with subordination if the following condition is satisfied*

$$\frac{zD_q f(z)}{f(z)} \prec \varphi(z),$$

*where $\varphi(z) = z + \sqrt{1+z^2}$ and the branch of the square root is chosen to be the principal one, that is $\varphi(0) = 1$.*

**Definition 1.4** *For $q \in (0,1)$, a function $f \in A$ given by (1.1) is said to be in the class $C_q(\varphi)$, which we will call $q-$ convex function class with subordination, if the following condition is satisfied*

$$1 + \frac{zD_q^2 f(z)}{D_q f(z)} \prec \varphi(z),$$

*where $\varphi(z) = z + \sqrt{1+z^2}$ and the branch of the square root is chosen to be the principal one, that is $\varphi(0) = 1$.*

It can be easily seen that the function $\varphi(z) = z + \sqrt{1+z^2}$ maps the unit disc $\mathbb{U}$ onto a shell shaped region on the right half plane and it is analytic and univalent in $\mathbb{U}$. The range $\varphi(\mathbb{U})$ is symmetric respect to real axis and $\varphi$ is a function with positive real part in $\mathbb{U}$, with $\varphi(0) = \varphi'(0) = 1$. Moreover, it is a starlike domain with respect to point $\varphi(0) = 1$.

Now, we define a subclass of analytic and bi-univalent functions as follows.

**Definition 1.5** *For* $q \in (0,1)$ *and* $\beta \geq 0$, *a function* $f \in \Sigma$ *given by (1.1) is said to be in the class* $\mathrm{M}_{q,\Sigma}(\varphi, \beta)$, *if the following conditions are satisfied*

$$(1-\beta)\frac{zD_qf(z)}{f(z)} + \beta\left(1 + \frac{zD_q^2 f(z)}{D_q f(z)}\right) \prec \varphi(z) = z + \sqrt{1+z^2},\ z \in \mathbb{U}$$

*and*

$$(1-\beta)\frac{wD_qg(w)}{g(w)} + \beta\left(1 + \frac{zD_q^2 g(w)}{D_q g(w)}\right) \prec \varphi(w) = w + \sqrt{1+w^2},\ w \in \mathbb{U}_{r_0},$$

*where* $g = f^{-1}$ *as given by (1.3).*

**Remark 1.6** *Taking* $\beta = 0$ *in the Definition 1.5, we have bi-q−starlike function class with subordination* $S^*_{q.\Sigma}(\varphi)$, *which satisfied the following conditions*

$$\frac{zD_qf(z)}{f(z)} \prec \varphi(z) = z + \sqrt{1+z^2},\ z \in \mathbb{U}$$

*and*

$$\frac{wD_qg(w)}{g(w)} \prec \varphi(w) = w + \sqrt{1+w^2},\ w \in \mathbb{U}_{r_0}$$

*where* $g = f^{-1}$ *as given by (1.3).*

**Remark 1.7** *Taking* $\beta = 1$ *in the Definition 1.5, we have bi-q−convex function class with subordination* $C_{q,\Sigma}(\varphi)$, *which satisfied the following conditions*

$$1 + \frac{zD_q^2 f(z)}{D_q f(z)} \prec \varphi(z) = z + \sqrt{1+z^2},\ z \in \mathbb{U}$$

*and*

$$1 + \frac{wD_q^2 g(w)}{D_q g(w)} \prec \varphi(w) = w + \sqrt{1+w^2},\ w \in \mathbb{U}_{r_0},$$

*where* $g = f^{-1}$ *as given by (1.3).*

**Notation 1.8** *It is clear that*

$$\lim_{q\to 1^-} \mathrm{M}_{q,\Sigma}(\varphi,\beta) = \mathrm{M}_{\Sigma}(\varphi,\beta).$$

*The class* $\mathrm{M}_{\Sigma}(\varphi,\beta)$ *was recently investigated by Mustafa and Mrugusundaramoorthy in [24] .*

In this paper, our aim is to give coefficient bound estimates, determine the upper bound estimate for the second Hankel determinant and solve the Fekete-Szegö problem for the function belonging to the class $\mathrm{M}_{\Sigma}(\varphi,\beta)$.

In order to prove our main results, we shall need the following lemmas.

**Lemma 1.9** *([11, 13]) Let* $\mathrm{P}$ *be the class of all analytic functions* $p$ *of the form*

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ... = 1 + \sum_{n=1}^{\infty} p_n z^n \tag{1.4}$$

*satisfying* $\Re(p(z)) > 0, z \in \mathbb{U}$ *and* $p(0) = 1$. *Then,*

$$|p_n| \leq 2, \ \ n = 1,2,3,....$$

*This inequality is sharp for each* $n = 1,2,3,...$ *. In particular, equality holds for the function*

$$p(z) = \frac{1+z}{1-z}$$

*for all* $n = 1,2,3,...$ *.*

**Lemma 1.10** *([11, 13]) Let* $\mathrm{P}$ *be the class of all analytic functions* $p$ *of the form*

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ... = 1 + \sum_{n=1}^{\infty} p_n z^n$$

*satisfying* $\Re(p(z)) > 0, z \in \mathbb{U}$ *and* $p(0) = 1$. *Then,*

$$\begin{aligned} 2p_2 &= p_1^2 + \left(4 - p_1^2\right) x, \\ 4p_3 &= p_1^3 + 2\left(4 - p_1^2\right) p_1 x - \left(4 - p_1^2\right) p_1 x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right) z \end{aligned}$$

*for some* $x, z$ *with* $|x| \leq 1, \ |z| \leq 1$.

**Notation 1.11** *As can be seen from serial expansion of the function* $\varphi$ *given in Definition 1.3, 1.4 and 1.5, this function belongs to the class* $P$.

**Lemma 1.12** *( [13]) The power series given by (1.4) converge in* $U$ *to the function* $p$ *in* $P$ *if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & p_1 & p_2 & \cdots & p_n \\ p_{-1} & 2 & p_1 & \cdots & p_{n-1} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ p_{-n} & p_{-n+1} & p_{-n+2} & \cdots & 2 \end{vmatrix}, \ \ n = 1,2,3,...$$

*and* $p_{-n} = \bar{p_n}$, *are all nonnegative. They are strictly positive except for*

$$p(z) = \sum_{n=1}^{n} \rho_n p_0 \left(e^{it_n z}\right), \ \rho_n > 0, \ t_n$$

*real and* $t_n \neq t_k$ *for* $n \neq k$ *in this case* $D_n > 0$ *for* $n < m-1$ *and* $D_n = 0$ *for* $n \geq m$.

**Notation 1.13** *According to Lemma 1.12* $p_n \geq 0$ *for each* $n = 1, 2, 3, \ldots$ , *if* $p \in P$. *On the other hand, according to Lemma 1.9* $|p_n| \leq 2$ *for each* $n = 1, 2, 3, \ldots$, *if* $p \in P$. *For these reasons, we will assume that* $\left|4 - p_1^2\right| = \left|4 - |p_1|^2\right| = 4 - |p_1|^2$ *for* $p_1$, *which is the first coefficient in (1.4), throughout our study.*

## 2. Coefficients bound estimates

In this section, we prove the following theorem on upper bound estimates for the first three coefficients of the functions belonging to the class $\mathrm{M}_{q,\Sigma}(\varphi, \beta)$.

**Theorem 2.1** *Let the function* $f$ *given by (1.1) be in the class* $\mathrm{M}_{q,\Sigma}(\varphi, \beta)$. *Then,*

$$|a_2| \leq \frac{1}{[2]_q - 1 + \beta},$$

$$|a_3| \leq \max\left\{a_3^{(1)}(q,\beta), a_3^{(2)}(q,\beta)\right\},$$

*where*

$$a_3^{(1)}(q,\beta) = \frac{1}{\left([2]_q - 1 + \beta\right)^2}$$

*and*

$$a_3^{(2)}(q,\beta) = \frac{1}{[3]_q - 1 + \left(\left([2]_q 1\right)[3]_q + 1\right)\beta},$$

$$|a_4| \leq \max\left\{a_4^{(1)}(q,\beta), a_4^{(2)}(q,\beta)\right\},$$

*where*

$$a_4^{(1)}(q,\beta) = \frac{1}{[4]_q - 1 + \left(\left([3]_q 1\right)[4]_q + 1\right)\beta}$$

*and*

$$a_4^{(2)}(q,\beta) = \frac{[3]_q - 1 + \left\{\left([2]_q - 1\right)[3]_q + \left([3]_q - [2]_q\right)[2]_q^2 + 1\right\}\beta}{\left[[4]_q - 1 + \left(\left([3]_q 1\right)[4]_q + 1\right)\beta\right]\left([2]_q - 1 + \beta\right)^3}$$

*The results obtained here are sharp for* $\beta \in [0,1]$.

**Proof** Let $f \in \mathrm{M}_{q,\Sigma}(\varphi,\beta), \beta \geq 0$ and $g = f^{-1}$. Then, there are analytic functions $\omega : \mathbb{U} \to \mathbb{U}, \varpi : \mathbb{U}_{r_0} \to \mathbb{U}_{r_0}$ with $\omega(0) = 0 = \varpi(0), |\omega(z)| < 1, |\varpi(w)| < 1$ satisfying the following conditions

$$(1-\beta)\frac{zD_q f(z)}{f(z)} + \beta\left(1 + \frac{zD_q^2 f(z)}{D_q f(z)}\right) = \varphi(\omega(z)) = \omega(z) + \sqrt{1+\omega^2(z)}\ ,\ z \in \mathbb{U} \tag{2.1}$$

and

$$(1-\beta)\frac{wD_q g(w)}{g(w)} + \beta\left(1 + \frac{wD_q^2 g(w)}{D_q g(w)}\right) = \varphi(\varpi(w)) = \varpi(w) + \sqrt{1+\varpi^2(w)}\ ,\ w \in \mathbb{U}_{r_0}\ . \tag{2.2}$$

We define the functions $p, q \in \mathrm{P}$ as follows:

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ... = 1 + \sum_{n=1}^{\infty} p_n z^n,\ z \in \mathbb{U}$$

and

$$q(w) = \frac{1+\varpi(w)}{1-\varpi(w)} = 1 + q_1 w + q_2 w^2 + q_3 w^3 + ... = 1 + \sum_{n=1}^{\infty} q_n w^n,\ w \in \mathbb{U}_{r_0}.$$

It follows that

$$\omega(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right)z^2 + \left(p_3 - p_1 p_2 + \frac{p_1^3}{4}\right)z^3 + ...\right],\ z \in \mathbb{U} \tag{2.3}$$

and

$$\varpi(w) = \frac{q(w)-1}{q(w)+1} = \frac{1}{2}\left[q_1 w + \left(q_2 - \frac{q_1^2}{2}\right)w^2 + \left(q_3 - q_1 q_2 + \frac{q_1^3}{4}\right)w^3 + ...\right],\ w \in \mathbb{U}_{r_0}. \tag{2.4}$$

Changing the expressions of functions $\omega(z)$ and $\varpi(w)$ in (2.1) and (2.2) with expressions in (2.3) and (2.4), we obtain

$$(1-\beta)\frac{zD_q f(z)}{f(z)} + \beta\left(1 + \frac{zD_q^2 f(z)}{D_q f(z)}\right) = 1 + \frac{p_1}{2}z + \left(\frac{p_2}{2} - \frac{p_1^2}{8}\right)z^2 + \left(\frac{p_3}{2} - \frac{p_1 p_2}{4}\right)z^3 + ... \tag{2.5}$$

and

$$(1-\beta)\frac{wD_q g(w)}{g(w)} + \beta\left(1 + \frac{wD_q^2 g(w)}{D_q g(w)}\right) = 1 + \frac{q_1}{2}w + \left(\frac{q_2}{2} - \frac{q_1^2}{8}\right)w^2 + \left(\frac{q_3}{2} - \frac{q_1 q_2}{4}\right)w^3 + ... \tag{2.6}$$

If the operations and simplifications on the left side of (2.5) and (2.6) are made and the coefficients of the terms of the same degree are equalized, the following equations are obtained for $a_2$, $a_3$ and $a_4$

$$\left([2]_q - 1 + \beta\right)a_2 = \frac{p_1}{2}, \tag{2.7}$$

$$\begin{aligned}&\left\{[2]_q[3]_q\beta+\left([3]_q-1\right)(1-\beta)\right\}a_3-\left\{[2]_q^2\beta+\left([2]_q-1\right)(1-\beta)\right\}a_2^2 \\ =&\ \frac{p_2}{2}-\frac{p_1^2}{8},\end{aligned} \tag{2.8}$$

$$\begin{aligned}&\left\{[3]_q[4]_q\beta+\left([4]_q-1\right)(1-\beta)\right\}a_4 \\ &-\left\{\left([2]_q^2\beta-\beta+1\right)[3]_q+\left([3]_q\beta-\beta+1\right)[2]_q+2(\beta-1)\right\}a_2a_3 \\ &+\left\{\left([2]_q^2\beta-\beta+1\right)[2]_q+\beta-1\right\}a_2^3 \\ =&\ \frac{p_3}{2}-\frac{p_1p_2}{4}\end{aligned} \tag{2.9}$$

and

$$-\left([2]_q-1+\beta\right)a_2=\frac{q_1}{2}, \tag{2.10}$$

$$\begin{aligned}&-\left\{[2]_q[3]_q\beta+\left([3]_q-1\right)(1-\beta)\right\}a_3 \\ &+\left\{2\left[[2]_q[3]_q\beta+\left([3]_q-1\right)(1-\beta)\right]-\left[[2]_q^2\beta+\left([2]_q-1\right)(1-\beta)\right]\right\}a_2^2 \\ =&\ \frac{q_2}{2}-\frac{q_1^2}{8},\end{aligned} \tag{2.11}$$

$$\begin{aligned}&-\left\{[3]_q[4]_q\beta+\left([4]_q-1\right)(1-\beta)\right\}a_4 \\ &+\left\{\begin{array}{c}\left(5[4]_q-[2]_q\right)\left([3]_q\beta-\beta+1\right) \\ -\left([2]_q^2\beta-\beta+1\right)[3]_q+3(\beta-1)\end{array}\right\}a_2a_3- \\ &\left\{\begin{array}{c}\left(5[4]_q-2[2]_q\right)\left([3]_q\beta-\beta+1\right) \\ +\left([2]_q-2[3]_q\right)\left([2]_q^2\beta-\beta+1\right)+2(\beta-1)\end{array}\right\}a_2^3 \\ =&\ \frac{q_3}{2}-\frac{q_1q_2}{4}\end{aligned} \tag{2.12}$$

From equations (2.7) and (2.10), we write

$$\frac{p_1}{2\left([2]_q-1+\beta\right)}=a_2=-\frac{q_1}{2\left([2]_q-1+\beta\right)}\text{ and }p_1=-q_1. \tag{2.13}$$

From this and Lemma 1.9, the first result of the theorem is clear.
Subtracting (2.11) from (2.8) and considering the equality $p_1=-q_1$, we get

$$a_3=a_2^2+\frac{p_2-q_2}{4\left\{[2]_q[3]_q\beta+\left([3]_q-1\right)(1-\beta)\right\}};$$

that is,

$$a_3 = \frac{p_1^2}{4\left([2]_q - 1 + \beta\right)^2} + \frac{p_2 - q_2}{4\left\{[2]_q [3]_q \beta + \left([3]_q - 1\right)(1-\beta)\right\}}. \tag{2.14}$$

Also, subtracting the equation (2.12) from the equation (2.9), considering the equalities (2.13) and (2.14), we have

$$\begin{aligned} a_4 \;=\; & \frac{\left\{[3]_q + \left([3]_q - [2]_q\right)[2]_q\right\}[2]_q \beta + \left([3]_q - 1\right)(1-\beta)}{8\left\{[3]_q [4]_q \beta + \left([4]_q - 1\right)(1-\beta)\right\}\left([2]_q - 1 + \beta\right)^3} p_1^3 \\ & + \frac{5\left(p_2 - q_2\right)p_1}{16\left\{[2]_q [3]_q \beta + \left([3]_q - 1\right)(1-\beta)\right\}\left([2]_q - 1 + \beta\right)} \\ & + \frac{p_3 - q_3}{4\left\{[3]_q [4]_q \beta + \left([4]_q - 1\right)(1-\beta)\right\}} \\ & - \frac{\left(p_2 + q_2\right)p_1}{8\left\{[3]_q [4]_q \beta + \left([4]_q - 1\right)(1-\beta)\right\}}. \end{aligned} \tag{2.15}$$

Since $p_1 = -q_1$, according to Lemma 1.10, we can write

$$p_2 - q_2 = \frac{4 - p_1^2}{2}(x - y), \; p_2 + q_2 = p_1^2 + \frac{4 - p_1^2}{2}(x + y) \tag{2.16}$$

and

$$\begin{aligned} p_3 - q_3 \;=\; & \frac{p_1^3}{2} + \frac{\left(4 - p_1^2\right)p_1}{2}(x + y) - \frac{\left(4 - p_1^2\right)p_1}{4}\left(x^2 + y^2\right) \\ & + \frac{4 - p_1^2}{2}\left[\left(1 - |x|^2\right)z - \left(1 - |y|^2\right)w\right] \end{aligned} \tag{2.17}$$

for some $x, y, z, w$ with $|x| \leq 1, |y| \leq 1, |z| \leq 1, |w| \leq 1$.

Substituting the first equality (2.16) in (2.14), we write the following expression for the coefficient $a_3$

$$a_3 = \frac{p_1^2}{4\left([2]_q - 1 + \beta\right)^2} + \frac{4 - p_1^2}{8\left\{[2]_q [3]_q \beta + \left([3]_q - 1\right)(1-\beta)\right\}}(x - y).$$

Note that if we take $|p_1| = t$, we can write $\left|4 - p_1^2\right| = \left|4 - |p_1|^2\right| = \left|4 - t^2\right| = 4 - t^2$ (see, also Notation 1.13 at the end of the first section). That is, we may assume without restriction that $t \in [0, 2]$. In that case, using a triangle inequality and setting $|x| = \xi$ and $|y| = \eta$, we can write the following inequality for $|a_3|$

$$\begin{aligned} |a_3| \;\leq\; & \frac{t^2}{4\left([2]_q - 1 + \beta\right)^2} + \frac{4 - t^2}{8\left\{[2]_q [3]_q \beta + \left([3]_q - 1\right)(1-\beta)\right\}}(\xi + \eta), \\ (\xi, \eta) \;\in\; & [0, 1]^2. \end{aligned}$$

Now, let the function $F:\mathbb{R}^2\to\mathbb{R}$ defined as follows:

$$\begin{aligned} F(\xi,\eta) &= \frac{t^2}{4\left([2]_q-1+\beta\right)^2}+\frac{4-t^2}{8\left\{[2]_q[3]_q\beta+\left([3]_q-1\right)(1-\beta)\right\}}(\xi+\eta), \\ (\xi,\eta) &\in [0,1]^2. \end{aligned}$$

We need to maximize the function $F$ on the closed square $\Omega=\left\{(\xi,\eta):(\xi,\eta)\in[0,1]^2\right\}.$

It is clear that the function $F$ takes its maximum value at the boundary of the closed square $\Omega$.

Differentiating the function $F(\xi,\eta)$ respect to parameter $\xi$, we have

$$F_\xi(\xi,\eta)=\frac{4-t^2}{8\left\{[2]_q[3]_q\beta+\left([3]_q-1\right)(1-\beta)\right\}}.$$

Since $F_\xi(\xi,\eta)\geq 0,$ the function $F(\xi,\eta)$ is an increasing function respect to $\xi$ and maximum occurs at $\xi=1,$ so

$$\begin{aligned} \max\{F(\xi,\eta):\xi\in[0,1]\} &= F(1,\eta)=\frac{t^2}{4\left([2]_q-1+\beta\right)^2} \\ &\quad +\frac{4-t^2}{8\left\{[2]_q[3]_q\beta+\left([3]_q-1\right)(1-\beta)\right\}}(1+\eta) \end{aligned}$$

for each $\eta\in[0,1]$ and $t\in[0,2]$.

Now, differentiating the function $F(1,\eta)$, we have

$$F^{'}(1,\eta)=\frac{4-t^2}{8\left\{[2]_q[3]_q\beta+\left([3]_q-1\right)(1-\beta)\right\}}.$$

Since $F^{'}(1,\eta)\geq 0,$ the function $F(1,\eta)$ is an increasing function and maximum occurs at $\eta=1,$ so

$$\begin{aligned} \max\{F(1,\eta):\eta\in[0,1]\} &= F(1,1)=\frac{t^2}{4\left([2]_q-1+\beta\right)^2} \\ &\quad +\frac{4-t^2}{4\left\{[2]_q[3]_q\beta+\left([3]_q-1\right)(1-\beta)\right\}}, t\in[0,2]. \end{aligned}$$

Thus, we have

$$\begin{aligned} F(\xi,\eta) &\leq \max\{F(\xi,\eta):(\xi,\eta)\in\Omega\}=F(1,1) \\ &= \frac{t^2}{4\left([2]_q-1+\beta\right)^2}+\frac{4-t^2}{4\left\{[2]_q[3]_q\beta+\left([3]_q-1\right)(1-\beta)\right\}}. \end{aligned}$$

Since $|a_3| \leq F(\xi,\eta)$, we can write

$$|a_3| \leq a(q,\beta)t^2 + \frac{1}{[2]_q[3]_q\beta + \left([3]_q - 1\right)(1-\beta)}, t \in [0,2]$$

where

$$a(q,\beta) = \frac{1}{4}\left[\frac{1}{\left([2]_q - 1 + \beta\right)^2} - \frac{1}{[2]_q[3]_q\beta + \left([3]_q - 1\right)(1-\beta)}\right].$$

Now, let's find the maximum of the function $\chi : \mathbb{R} \to \mathbb{R}$ defined as follows

$$\chi(t) = a(q,\beta)t^2 + \frac{1}{[2]_q[3]_q\beta + \left([3]_q - 1\right)(1-\beta)}$$

in the interval $[0,2]$.

Differentiating the function $\chi(t)$, we have $\chi'(t) = 2a(q,\beta)t, t \in [0,2]$. Since $\chi'(t) \leq 0$ when $a(q,\beta) \leq 0$, the function $\chi(t)$ is a decreasing function and maximum occurs at $t = 0$, so

$$\max\{\chi(t) : t \in [0,2]\} = \chi(0) = \frac{1}{[2]_q[3]_q\beta + \left([3]_q - 1\right)(1-\beta)}$$

and $\chi'(t) \geq 0$ when $a(q,\beta) \geq 0$, the function $\chi(t)$ is an increasing function and maximum occurs at $t = 2$, so

$$\max\{\chi(t) : t \in [0,2]\} = \chi(2) = \frac{1}{\left([2]_q - 1 + \beta\right)^2}.$$

Thus, we obtain the following upper bound estimate for $|a_3|$

$$|a_3| \leq \max\left\{\frac{1}{\left([2]_q - 1 + \beta\right)^2}, \frac{1}{[2]_q[3]_q\beta + \left([3]_q - 1\right)(1-\beta)}\right\}.$$

From (2.15), using (2.16), (2.17) and triangle inequality, we obtain the following inequality for $|a_4|$

$$|a_4| \leq c_1(t) + c_2(t)(\xi + \eta) + c_3(t)\left(\xi^2 + \eta^2\right) := G(\xi,\eta),$$

where

$$\begin{aligned} c_1(t) &= \frac{\left\{[3]_q + \left([3]_q - [2]_q\right)[2]_q\right\}[2]_q\beta + \left([3]_q - 1\right)(1-\beta)}{8\left\{[3]_q[4]_q\beta + \left([4]_q - 1\right)(1-\beta)\right\}\left([2]_q - 1 + \beta\right)^3}t^3 \\ &+ \frac{4 - t^2}{4\left\{[3]_q[4]_q\beta + \left([4]_q - 1\right)(1-\beta)\right\}}, \end{aligned}$$

$$c_2(t) = \frac{5\left(4-t^2\right)t}{32\left\{[2]_q[3]_q\beta+\left([3]_q-1\right)(1-\beta)\right\}\left([2]_q-1+\beta\right)}$$
$$+\frac{\left(4-t^2\right)t}{16\left\{[3]_q[4]_q\beta+\left([4]_q-1\right)(1-\beta)\right\}},$$

$$c_3(t)=\frac{\left(4-t^2\right)(t-2)}{16\left\{[3]_q[4]_q\beta+\left([4]_q-1\right)(1-\beta)\right\}},$$

Now, we need to maximize the function $G$ on $\Omega$ for each $t\in[0,2]$.

Since the coefficients $c_1(t)$, $c_2(t)$ and $c_3(t)$ of the function $G$ depend on the parameter $t$, we must investigate the maximum of the function $G$ for different values of the parameter $t$.

For $t=0$, since $c_2(0)=0$,

$$c_1(0) = \frac{1}{[3]_q[4]_q\beta+\left([4]_q-1\right)(1-\beta)} \quad \text{and}$$
$$c_3(0) = \frac{-1}{2\left\{[3]_q[4]_q\beta+\left([4]_q-1\right)(1-\beta)\right\}},$$

we write

$$G(\xi,\eta)=\frac{1}{[3]_q[4]_q\beta+\left([4]_q-1\right)(1-\beta)}$$
$$-\frac{1}{2\left\{[3]_q[4]_q\beta+\left([4]_q-1\right)(1-\beta)\right\}}\left(\xi^2+\eta^2\right),(\xi,\eta)\in[0,1]^2.$$

From this, we have

$$G(\xi,\eta)\le\max\left\{G(\xi,\eta):(\xi,\eta)\in\Omega\right\}=G(0,0)=\frac{1}{[3]_q[4]_q\beta+\left([4]_q-1\right)(1-\beta)}.$$

Let $t=2$. Then, since $c_2(2)=c_3(2)=0$ and

$$c_1(2)=\frac{\left\{[3]_q+\left([3]_q-[2]_q\right)[2]_q\right\}[2]_q\beta+\left([3]_q-1\right)(1-\beta)}{\left\{[3]_q[4]_q\beta+\left([4]_q-1\right)(1-\beta)\right\}\left([2]_q-1+\beta\right)^3},$$

the function $G$ is constant as follows

$$G(\xi,\eta)=c_1(2)=\frac{\left\{[3]_q+\left([3]_q-[2]_q\right)[2]_q\right\}[2]_q\beta+\left([3]_q-1\right)(1-\beta)}{\left\{[3]_q[4]_q\beta+\left([4]_q-1\right)(1-\beta)\right\}\left([2]_q-1+\beta\right)^3}.$$

In the case $t \in (0,2)$, we can easily show that the function $G$ cannot have a maximum on the $\Omega$. Thus, we obtain

$$|a_4| \leq \max \left\{ \begin{array}{c} \frac{1}{[3]_q[4]_q\beta + \left([4]_q - 1\right)(1-\beta)}, \\ \frac{\left\{[3]_q + \left([3]_q - [2]_q\right)[2]_q\right\}[2]_q\beta + \left([3]_q - 1\right)(1-\beta)}{\left\{[3]_q[4]_q\beta + \left([4]_q - 1\right)(1-\beta)\right\}\left([2]_q - 1 + \beta\right)^3} \end{array} \right\}.$$

The results obtained in the theorem are sharp for $\beta \in [0,1]$. Really, the obtained results hold with equalities for the following function

$$f(z) = z + \frac{z^2}{[2]_q - 1 + \beta} + \frac{z^3}{\left([2]_q - 1 + \beta\right)^2}$$

$$+ \frac{\left\{[3]_q + \left([3]_q - [2]_q\right)[2]_q\right\}[2]_q\beta + \left([3]_q - 1\right)(1-\beta)}{\left\{[3]_q[4]_q\beta + \left([4]_q - 1\right)(1-\beta)\right\}\left([2]_q - 1 + \beta\right)^3} z^4, z \in U$$

for $\beta \in [0,1]$.

Thus, the proof of Theorem 2.1 is completed. □

In special cases $\beta = 0$ and $\beta = 1$, from the Theorem 2.1 we obtain the following results, respectively.

**Corollary 2.2** *Let the function $f$ given by (1.1) be in the class $\mathrm{S}^*_{q,\Sigma}(\varphi)$. Then,*

$$|a_2| \leq \frac{1}{[2]_q - 1}, \quad |a_3| \leq \frac{1}{\left([2]_q - 1\right)^2} \text{ and } |a_4| \leq \frac{[3]_q - 1}{\left([4]_q - 1\right)\left([2]_q - 1\right)^3}.$$

*The results obtained here are sharp. In particular, equalities hold for the function*

$$f(z) = z + \frac{z^2}{[2]_q - 1} + \frac{z^3}{\left([2]_q - 1\right)^2} + \frac{\left([3]_q - 1\right)z^4}{\left([4]_q - 1\right)\left([2]_q - 1\right)^3}, \; z \in U.$$

**Corollary 2.3** *Let the function $f$ given by (1.1) be in the class $C_{q,\Sigma}(\varphi)$. Then,*

$$|a_2| \leq \frac{1}{[2]_q}, \quad |a_3| \leq \frac{1}{[2]_q^2} \text{ and } |a_4| \leq \left\{ \begin{array}{ll} \frac{1}{[3]_q[4]_q}, & if \quad q \in (0, q_0], \\ \frac{[3]_q + \left([3]_q - [2]_q\right)[2]_q}{[2]_q^2[3]_q[4]_q}, & if \quad q \in [q_0, 1), \end{array} \right. .$$

*where* $q_0 = \frac{\sqrt{5}-1}{2}$ .

*The results obtained here are sharp. In particular, equalities hold for the function*

$$f(z) = z + \frac{z^2}{[2]_q} + \frac{z^3}{[2]_q^2} + \frac{z^4}{[3]_q[4]_q}, \; z \in U$$

*for $q \in (0, q_0]$ and for the function*

$$f(z) = z + \frac{z^2}{[2]_q} + \frac{z^3}{[2]_q^2} + \frac{[3]_q + \left([3]_q - [2]_q\right)[2]_q}{[2]_q^2 [3]_q [4]_q} z^4, \ z \in U$$

*for* $q \in [q_0, 1)$.

Also, from the Theorem 2.1 the following result was obtained, when $q \to 1^-$.

**Theorem 2.4** *(see [24],Theorem 2.1) Let the function* $f$ *given by (1.1) be in the class* $\mathrm{M}_\Sigma(\varphi, \beta)$. *Then,*

$$|a_2| \leq \frac{1}{1+\beta}$$

$$|a_3| \leq \begin{cases} \frac{1}{(1+\beta)^2}, & if \quad \beta \in \left[0, 1+\sqrt{2}\right], \\ \frac{1}{2(1+2\beta)}, & if \qquad \beta \geq 1+\sqrt{2}, \end{cases}$$

$$|a_4| \leq \frac{1}{3(1+3\beta)} \begin{cases} \frac{2(1+4\beta)}{(1+\beta)^3}, & if \quad \beta \in [0, \beta_0], \\ 1, & if \qquad \beta \geq \beta_0, \end{cases}$$

*where* $\beta_0 = 1.3289$ *is the numerical solution of the equation* $\beta^3 + 3\beta^2 - 5\beta - 1 = 0$.

## 3. The second Hankel determinant and Fekete-Szegö inequality

In this section, we give an upper bound estimate for the second Hankel determinant and Fekete-Szegö inequality for the function belonging to the class $M_{q,\Sigma}(\varphi, \beta)$ defined by Definition 1.5.

Firstly, we prove the following theorem on the upper bound estimate of the second Hankel determinant.

**Theorem 3.1** *Let the function* $f$ *given by (1.1) be in the class* $M_{q,\Sigma}(\varphi, \beta), \beta \in [0,1]$. *Then,*

$$\left|a_2 a_4 - a_3^2\right| \leq \max\left\{A(q, \beta), \ B(q, \beta)\right\}. \tag{3.1}$$

*where*

$$A(q, \beta) = \frac{1}{\left\{[3]_q - 1 + \left[\left([2]_q - 1\right)[3]_q + 1\right]\beta\right\}^2},$$

$$B(q, \beta) = \frac{[4]_q - [3]_q + \left\{\left([3]_q + [2]_q\right)[2]_q^2 - \left([2]_q - 1\right)[3]_q - \left([3]_q + 1\right)[4]_q\right\}\beta}{\left\{[4]_q - 1 + \left[\left([3]_q - 1\right)[4]_q + 1\right]\beta\right\}\left([2]_q - 1 + \beta\right)^4}.$$

**Proof** Let $f \in M_{q,\Sigma}(\varphi, \beta), \ \beta \in [0,1]$. Then, from (2.13), (2.14) and (2.15), we write the following equality for $a_2 a_4 - a_3^2$

$$
\begin{aligned}
a_2a_4-a_3^2 \;=\;& \frac{\left\{\left([3]_q-[2]_q^2\right)[2]_q+\left([2]_q^2-[4]_q\right)[3]_q\right\}\beta+\left([3]_q-[4]_q\right)(1-\beta)}{16\left\{[3]_q[4]_q\beta+\left([4]_q-1\right)(1-\beta)\right\}\left([2]_q-1+\beta\right)^4}p_1^4 \\
&+\frac{(p_2-q_2)\,p_1^2}{32\left\{[2]_q[3]_q\beta+\left([3]_q-1\right)(1-\beta)\right\}\left([2]_q-1+\beta\right)^2} \\
&+\frac{(p_3-q_3)\,p_2}{8\left\{[3]_q[4]_q\beta+\left([4]_q-1\right)(1-\beta)\right\}\left([2]_q-1+\beta\right)} \\
&-\frac{(p_2+q_2)\,p_1^2}{16\left\{[3]_q[4]_q\beta+\left([4]_q-1\right)(1-\beta)\right\}\left([2]_q-1+\beta\right)} \\
&-\frac{(p_2-q_2)^2}{16\left\{[2]_q[3]_q\beta+\left([3]_q-1\right)(1-\beta)\right\}^2}.
\end{aligned}
$$

Using equalities (2.16) and (2.17), then triangle inequality and letting $|p_1|=t, |x|=\xi,\ |y|=\eta$, we obtain the following estimate for $\left|a_2a_4-a_3^2\right|$

$$
\left|a_2a_4-a_3^2\right| \le C_1(t)+C_2(t)(\xi+\eta)+C_3(t)\left(\xi^2+\eta^2\right)+C_4(t)(\xi+\eta)^2, \tag{3.2}
$$

where

$$
\begin{aligned}
C_1(t) \;=\;& \frac{\left\{\left([2]_q^2-[3]_q\right)[2]_q+\left([2]_q^2-[4]_q\right)[3]_q\right\}\beta+\left([4]_q-[3]_q\right)(1-\beta)}{16\left\{[3]_q[4]_q\beta+\left([4]_q-1\right)(1-\beta)\right\}\left([2]_q-1+\beta\right)^4}t^4 \\
&+\frac{\left(4-t^2\right)t}{8\left\{[3]_q[4]_q\beta+\left([4]_q-1\right)(1-\beta)\right\}\left([2]_q-1+\beta\right)} \ge 0,
\end{aligned}
$$

$$
C_2(t)=\frac{\left(4-t^2\right)t^2}{64\left([2]_q-1+\beta\right)}\left\{\begin{array}{c}\frac{1}{\left\{[2]_q[3]_q\beta+\left([3]_q-1\right)(1-\beta)\right\}\left([2]_q+\beta-1\right)} \\ +\frac{2}{[3]_q[4]_q\beta+\left([4]_q-1\right)(1-\beta)}\end{array}\right\} \ge 0,
$$

$$
C_3(t)=\frac{\left(4-t^2\right)(t-2)\,t}{32\left\{[3]_q[4]_q\beta+\left([4]_q-1\right)(1-\beta)\right\}\left([2]_q-1+\beta\right)} \le 0,
$$

$$
C_4(t)=\frac{\left(4-t^2\right)^2}{64\left\{[2]_q[3]_q\beta+\left([3]_q-1\right)(1-\beta)\right\}^2} \ge 0.
$$

Let the function $\Phi:\mathbb{R}^2\to\mathbb{R}$ defined as follows:

$$
\Phi(\xi,\eta)=C_1(t)+C_2(t)(\xi+\eta)+C_3(t)\left(\xi^2+\eta^2\right)+C_4(t)(\xi+\eta)^2,\ (\xi,\eta)\in[0,1]^2
$$

for each $t \in [0,2]$.

Now, we need to maximize the function $\Phi$ on $\Omega$ for each $t \in [0,2]$.

Since the coefficients of the function $\Phi$ depend on the parameter $t$, we must investigate the maximum for different values of the parameter $t$.

1. Let $t=0$. Since $C_1(0)=C_2(0)=C_3(0)=0$ and

$$C_4(0)=\frac{1}{4\left\{[2]_q[3]_q\beta+\left([3]_q-1\right)(1-\beta)\right\}^2},$$

the function $\Phi$ written as follows

$$\Phi(\xi,\eta)=\frac{(\xi+\eta)^2}{4\left\{[2]_q[3]_q\beta+\left([3]_q-1\right)(1-\beta)\right\}^2},(\xi,\eta)\in\Omega\ .$$

It is clear that the function $\Phi$ takes its maximum at the boundary of the closed square $\Omega$.

Now, differentiating the function $\Phi(\xi,\eta)$ respect to $\xi$, we have

$$\Phi_\xi(\xi,\eta)=\frac{\xi+\eta}{2\left\{[2]_q[3]_q\beta+\left([3]_q-1\right)(1-\beta)\right\}^2}$$

for each $\eta\in[0,1]$.

Since $\Phi_\xi(\xi,\eta)\geq 0$, the function $\Phi(\xi,\eta)$ is an increasing function respect to $\xi$ and maximum occurs at $\xi=1$. So

$$\max\{\Phi(\xi,\eta):\xi\in[0,1]\}=\Phi(1,\eta)=\frac{(1+\eta)^2}{4\left\{[2]_q[3]_q\beta+\left([3]_q-1\right)(1-\beta)\right\}^2},\eta\in[0,1]\,.$$

Differentiating the function $\Phi(1,\eta)$, we have

$$\Phi^{'}(1,\eta)=\frac{1+\eta}{2\left\{[2]_q[3]_q\beta+\left([3]_q-1\right)(1-\beta)\right\}^2}>0,\eta\in[0,1]\,.$$

Since $\Phi^{'}(1,\eta)>0$, the function $\Phi(1,\eta)$ is an increasing function and maximum occurs at $\eta=1$. Therefore,

$$\max\{\Phi(1,\eta):\eta\in[0,1]\}=\Phi(1,1)=\frac{1}{\left\{[2]_q[3]_q\beta+\left([3]_q-1\right)(1-\beta)\right\}^2}\ .$$

Thus, in the case $t=0$, we have

$$\begin{aligned}\Phi(\xi,\eta)&\leq\max\left\{\Phi(\xi,\eta):(\xi,\eta)\in[0,1]^2\right\}=\Phi(1,1)\\&=\frac{1}{\left\{[2]_q[3]_q\beta+\left([3]_q-1\right)(1-\beta)\right\}^2}\ .\end{aligned}$$

Since $\left|a_2a_4-a_3^2\right|\leq\Phi\left(\xi,\eta\right)$, we can write

$$\left|a_2a_4-a_3^2\right|\leq\frac{1}{\left\{[2]_q\,[3]_q\,\beta+\left([3]_q-1\right)(1-\beta)\right\}^2}.$$

2. Now let $t=2$. Since $C_2\left(2\right)=C_3\left(2\right)=C_4\left(2\right)=0$ and

$$C_1\left(2\right)=\frac{\left\{\left([2]_q^2-[3]_q\right)[2]_q+\left([2]_q^2-[4]_q\right)[3]_q\right\}\beta+\left([4]_q-[3]_q\right)(1-\beta)}{\left\{[3]_q\,[4]_q\,\beta+\left([4]_q-1\right)(1-\beta)\right\}\left([2]_q-1+\beta\right)^4},$$

the function $\Phi\left(\xi,\eta\right)$ is a constant as follows

$$\Phi\left(\xi,\eta\right)=C_1\left(2\right)=\frac{\left\{\left([2]_q^2-[3]_q\right)[2]_q+\left([2]_q^2-[4]_q\right)[3]_q\right\}\beta+\left([4]_q-[3]_q\right)(1-\beta)}{\left\{[3]_q\,[4]_q\,\beta+\left([4]_q-1\right)(1-\beta)\right\}\left([2]_q-1+\beta\right)^4}.$$

Thus, we have

$$\left|a_2a_4-a_3^2\right|\leq\frac{\left\{\left([2]_q^2-[3]_q\right)[2]_q+\left([2]_q^2-[4]_q\right)[3]_q\right\}\beta+\left([4]_q-[3]_q\right)(1-\beta)}{\left\{[3]_q\,[4]_q\,\beta+\left([4]_q-1\right)(1-\beta)\right\}\left([2]_q+\beta-1\right)^4}$$

in the case $t=2$ .

3. Finally, let $t\in(0,2)$. In this case, we must investigate the maximum of the function $\Phi$ taking into account the sign of $\Delta\left(\Phi\right)=\Phi_{\xi\xi}\left(\xi,\eta\right)\Phi_{\eta\eta}\left(\xi,\eta\right)-\left(\Phi_{\xi\eta}\left(\xi,\eta\right)\right)^2$.

We can easily see that $\Delta\left(\Phi\right)=4C_3\left(t\right)\left[C_3\left(t\right)+2C_4\left(t\right)\right]$. The sign of $\Delta\left(\Phi\right)$, we will investigate in two cases.

3.1. Let $C_3\left(t\right)+2C_4\left(t\right)\leq0$ for same $t\in(0,2)$. In this case, since $\Phi_{\xi\eta}\left(\xi,\eta\right)=\Phi_{\eta\xi}\left(\xi,\eta\right)=2C_4\left(t\right)\geq0$ and $\Delta\left(\Phi\right)\geq0$, from the elementary calculus the function $\Phi$ ( have a minimum) cannot have a maximum on the square $\Omega$.

3.2. Now let $C_3\left(t\right)+2C_4\left(t\right)\geq0$ for some $t\in(0,2)$. In this case, since $\Delta\left(\Phi\right)\leq0$, the function $\Phi$ cannot have a maximum on the square $\Omega$.

Thus, as a result of all three cases, we write

$$\left|a_2a_4-a_3^2\right|\leq\max\left\{\begin{array}{c}\frac{1}{\left\{[2]_q[3]_q\beta+\left([3]_q-1\right)(1-\beta)\right\}^2}\\ \frac{\left\{\left([2]_q^2-[3]_q\right)[2]_q+\left([2]_q^2-[4]_q\right)[3]_q\right\}\beta+\left([4]_q-[3]_q\right)(1-\beta)}{\left\{[3]_q[4]_q\beta+\left([4]_q-1\right)(1-\beta)\right\}\left([2]_q-1+\beta\right)^4}\end{array}\right\}.$$

Thus, the proof of Theorem 3.1 is completed. □

In special values of the parameters, from Theorem 3.1 we obtain the following results.

**Corollary 3.2** *Let the function $f$ given by (1.1) be in the class $S^*_{q,\Sigma}\left(\varphi\right)$. Then,*

$$\left|a_2a_4-a_3^2\right|\leq\frac{[4]_q-[3]_q}{\left([4]_q-1\right)\left([2]_q-1\right)^4}.$$

**Corollary 3.3** *Let the function* $f$ *given by (1.1) be in the class* $C_{q,\Sigma}(\varphi)$. *Then,*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{[2]_q^2\,[3]_q^2}.$$

Also, from Theorem 3.1, we obtain the following theorem when $q \to 1^-$.

**Theorem 3.4** *(see [24],Theorem 3.1) Let the function* $f$ *given by (1.1) be in the class* $\mathrm{M}_\Sigma(\varphi,\beta)$. *Then,*

$$\left|a_2a_4 - a_3^2\right| \leq \begin{cases} \frac{1}{3(1+3\beta)(1+\beta)^3}, & if \quad \beta \in [0,\beta_1] \\ \frac{1}{4(1+2\beta)^2} & if \quad \beta \geq \beta_1, \end{cases}$$

*where* $\beta_1 = 0.16357$ *is numerical solution of equation* $9\beta^4 + 30\beta^3 + 20\beta^2 + 2\beta - 1 = 0$.

Now, we give the following theorem on the Fekete-Szegö inequality.

**Theorem 3.5** *Let the function* $f$ *given by (1.1) be in the class* $M_{q,\Sigma}(\varphi,\beta)$ *and* $\mu \in \mathbb{C}$. *Then,*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{l(q,\beta)}{\left([2]_q-1+\beta\right)^2} & if \quad |1-\mu| \leq l(q,\beta), \\ \frac{|1-\mu|}{\left([2]_q-1+\beta\right)^2} & if \quad |1-\mu| \geq l(q,\beta), \end{cases} \tag{3.3}$$

*where*

$$l(q,\beta) = \frac{\left([2]_q - 1 + \beta\right)^2}{[3]_q - 1 + \left[\left([2]_q - 1\right)[3]_q + 1\right]\beta}.$$

*The result obtained here is sharp.*

**Proof** Let $f \in M_{q,\Sigma}(\varphi,\beta), \beta \geq 0$ and $\mu \in \mathbb{C}$. Then, from the equalities (2.13), (2.14) and (2.16) the expression $a_3 - \mu a_2^2$ written as

$$a_3 - \mu a_2^2 = (1-\mu)\frac{p_1^2}{\left([2]_q - 1 + \beta\right)^2} + \frac{4 - p_1^2}{8\left\{[2]_q\,[3]_q\,\beta + \left([3]_q - 1\right)(1-\beta)\right\}}(x - y) \tag{3.4}$$

*for* some $x, y$ with $|x| \leq 1, |y| \leq 1$.

Using triangle inequality to the equality (3.4) and setting $|x| = \xi, |y| = \eta,\ |p_1| = t$, we obtain the following estimate for the upper bound of $\left|a_3 - \mu a_2^2\right|$

$$\begin{aligned} \left|a_3 - \mu a_2^2\right| \quad &\leq \quad \frac{|1-\mu|\,t^2}{4\left([2]_q - 1 + \beta\right)^2} \\ &+ \frac{4 - t^2}{8\left\{[2]_q\,[3]_q\,\beta + \left([3]_q - 1\right)(1-\beta)\right\}}(\xi + \eta), \quad (\xi,\eta) \in \Omega \end{aligned} \tag{3.5}$$

for each $t \in [0,2]$.

Let us define the function $\psi : \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$\begin{aligned}\psi(\xi,\eta) &= \frac{|1-\mu| t^2}{4\left([2]_q - 1 + \beta\right)^2} + \frac{4-t^2}{8\left\{[2]_q [3]_q \beta + \left([3]_q - 1\right)(1-\beta)\right\}}(\xi+\eta), \\ (\xi,\eta) &\in \Omega\end{aligned}$$

for each $t \in [0,2]$.

Let's find the maximum of the function $\psi$ on the closed square $\Omega$.

It is clear that the function $\psi$ takes its maximum value at the boundary of the closed square $\Omega$ .

Differentiating the function $\psi(\xi,\eta)$ respect to $\xi$, we have

$$\psi_\xi(\xi,\eta) = \frac{4-t^2}{8\left\{[2]_q [3]_q \beta + \left([3]_q - 1\right)(1-\beta)\right\}}$$

for each $t \in [0,2]$.

Since $\psi_\xi(\xi,\eta) \geq 0$, the function $\psi(\xi,\eta)$ is an increasing function respect to $\xi$ and maximum occurs at $\xi = 1$.

Therefore,

$$\max\{\psi(\xi,\eta) : \xi \in [0,1]\} = \psi(1,\eta)$$

$$= \frac{|1-\mu| t^2}{4\left([2]_q - 1 + \beta\right)^2} + \frac{4-t^2}{8\left\{[2]_q [3]_q \beta + \left([3]_q - 1\right)(1-\beta)\right\}}(1+\eta)$$

for each $\eta \in [0,1]$ and $t \in [0,2]$.

Now, differentiating the function $\psi(1,\eta)$, we have

$$\psi'(\xi,\eta) = \frac{4-t^2}{8\left\{[2]_q [3]_q \beta + \left([3]_q - 1\right)(1-\beta)\right\}}$$

for each $t \in [0,2]$.

Since $\psi'(1,\eta) \geq 0$, the function $\psi(1,\eta)$ is an increasing function and maximum occurs at $\eta = 1$, so

$$\max\{\psi(1,\eta) : \eta \in [0,1]\} = \psi(1,1)$$

$$= \frac{|1-\mu| t^2}{4\left([2]_q - 1 + \beta\right)^2} + \frac{4-t^2}{4\left\{[2]_q [3]_q \beta + \left([3]_q - 1\right)(1-\beta)\right\}}, t \in [0,2].$$

Thus, we have

$$\begin{aligned}\psi(\xi,\eta) &\leq \max\{(\xi,\eta) : (\xi,\eta) \in \Omega\} = \psi(1,1) \\ &= \frac{|1-\mu| t^2}{4\left([2]_q - 1 + \beta\right)^2} + \frac{4-t^2}{4\left\{[2]_q [3]_q \beta + \left([3]_q - 1\right)(1-\beta)\right\}}.\end{aligned}$$

Since $\left|a_3-\mu a_2^2\right| \leq \psi\left(\xi,\eta\right)$ , we can write the following estimate

$$\left|a_3-\mu a_2^2\right| \leq \frac{\left|1-\mu\right|-l\left(q,\beta\right)}{4\left(\left[2\right]_q-1+\beta\right)^2}t^2+\frac{l\left(q,\beta\right)}{\left(\left[2\right]_q-1+\beta\right)^2},$$

where

$$l\left(q,\beta\right)=\frac{\left(\left[2\right]_q-1+\beta\right)^2}{\left[2\right]_q\left[3\right]_q\beta+\left(\left[3\right]_q-1\right)\left(1-\beta\right)}.$$

In that case, if we find the maximum of the function $\varphi:[0,2]\to\mathbb{R}$ defined as follows

$$\varphi\left(t\right)=\frac{\left|1-\mu\right|-l\left(q,\beta\right)}{4\left(\left[2\right]_q-1+\beta\right)^2}t^2+\frac{l\left(q,\beta\right)}{\left(\left[2\right]_q-1+\beta\right)^2}$$

the proof of the theorem is completed.

Differentating the function $\varphi\left(t\right)$, we have

$$\varphi^{'}\left(t\right)=\frac{\left|1-\mu\right|-l\left(q,\beta\right)}{2\left(\left[2\right]_q-1+\beta\right)^2}t,\ t\in\left[0,2\right].$$

Since $\varphi^{'}\left(t\right)\leq 0$, the function $\varphi\left(t\right)$ is a decreasing function, if $\left|1-\mu\right|\leq l\left(q,\beta\right)$ and maximum occurs at $t=0$, so

$$\max\left\{\varphi\left(t\right):t\in\left[0,2\right]\right\}=\varphi\left(0\right)=\frac{l\left(q,\beta\right)}{\left(\left[2\right]_q-1+\beta\right)^2}$$

and $\varphi^{'}\left(t\right)\geq 0$, the function $\varphi\left(t\right)$ is an increasing function, if $\left|1-\mu\right|\geq l\left(q,\beta\right)$ and maximum occurs at $t=2$, so

$$\max\left\{\varphi\left(t\right):t\in\left[0,2\right]\right\}=\varphi\left(2\right)=\frac{\left|1-\mu\right|}{\left(\left[2\right]_q-1+\beta\right)^2}.$$

Thus, as a result, we obtain

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{l(q,\beta)}{\left([2]_q-1+\beta\right)^2} & \text{if} \quad \left|1-\mu\right|\leq l\left(q,\beta\right), \\ \frac{|1-\mu|}{\left([2]_q-1+\beta\right)^2} & \text{if} \quad \left|1-\mu\right|\geq l\left(q,\beta\right). \end{cases}$$

The result obtained here is sharp for $\left|1-\mu\right|\geq l\left(q,\beta\right)$ , if we choose the function $f$ as follows:

$$f(z)=z+\frac{z^2}{\left[2\right]_q-1+\beta}+\frac{z^3}{\left(\left[2\right]_q-1+\beta\right)^2},\ z\in U\ .$$

Thus, the proof of Theorem 3.5 is completed. □

In the cases $\beta = 0$ and $\beta = 1$, from the Theorem 3.5 we obtain the following results, respectively.

**Corollary 3.6** *Let the function $f$ given by (1.1) be in the class $S^*_{q,\Sigma}(\varphi)$ and $\mu \in \mathbb{C}$. Then,*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{1}{[3]_q - 1} & \text{if} \quad |1-\mu| \le l_0(q), \\ \frac{|1-\mu|}{\left([2]_q - 1\right)^2} & \text{if} \quad |1-\mu| \ge l_0(q). \end{cases}$$

*where $l_0(q) = \frac{\left([2]_q-1\right)^2}{[3]_q-1}$. The result obtained here is sharp for the function*

$$f(z) = z + \frac{z^2}{[2]_q - 1} + \frac{z^3}{\left([2]_q - 1\right)^2}, \; z \in U$$

*for $|1-\mu| \ge l_0(q)$ .*

**Corollary 3.7** *Let the function $f$ given by (1.1) be in the class $C_{q,\Sigma}(\varphi)$ and $\mu \in \mathbb{C}$. Then,*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{1}{[2]_q [3]_q} & \text{if} \quad |1-\mu| \le l_1(q), \\ \frac{|1-\mu|}{[2]_q^2} & \text{if} \quad |1-\mu| \ge l_1(q), \end{cases}$$

*where $l_1(q) = \frac{[2]_q}{[3]_q}$ . The result obtained here is sharp for the function*

$$f(z) = z + \frac{z^2}{[2]_q} + \frac{z^3}{[2]_q^2}, \; z \in U$$

*for $|1-\mu| \ge l_1(q)$ .*

Also, from the Theorem 3.5 we obtain the following theorem, when $q \to 1^-$.

**Theorem 3.8** *(see [24], Theorem 3.4) Let the function $f$ given by (1.1) be in the class $M_\Sigma(\varphi, \beta)$ and $\mu \in \mathbb{C}$. Then,*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{1}{2(1+2\beta)} & if \quad |1-\mu| \le \frac{(1+\beta)^2}{2(1+2\beta)}, \\ \frac{|1-\mu|}{(1+\beta)^2} & if \quad |1-\mu| \ge \frac{(1+\beta)^2}{2(1+2\beta)}. \end{cases}$$

In the case $\mu \in \mathbb{R}$, the Theorem 3.5 is given as follows.

**Theorem 3.9** *Let the function $f$ given by (1.1) be in the class $M_{q,\Sigma}(\varphi, \beta)$ and $\mu \in \mathbb{R}$. Then,*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{1-\mu}{\left([2]_q - 1 + \beta\right)^2} & \text{if} \quad \mu \le 1 - l(q,\beta), \\ \frac{l(q,\beta)}{\left([2]_q - 1 + \beta\right)^2} & \text{if} \quad 1 - l(q,\beta) \le \mu \le 1 + l(q,\beta), \\ \frac{\mu-1}{\left([2]_q - 1 + \beta\right)^2} & \text{if} \quad 1 + l(q,\beta) \le \mu. \end{cases} \tag{3.6}$$

*where* $l(q,\beta)=\frac{\left([2]_q-1+\beta\right)^2}{[3]_q-1+\left[\left([2]_q-1\right)[3]_q+1\right]\beta}$

**Proof** Let $f\in M_{q,\Sigma}(\varphi,\beta)$, $\beta\geq 0$ and $\mu\in\mathbb{R}$. Since in the case $\mu\in\mathbb{R}$ inequalities $|1-\mu|\geq l(q,\beta)$ and $|1-\mu|\leq l(q,\beta)$ are equivalent to the inequalities

$$\mu\leq 1-l(q,\beta) \text{ or } \mu\geq 1+l(q,\beta)$$

and

$$1-l(q,\beta)\leq \mu\leq 1+l(q,\beta),$$

respectively, from the Theorem 3.5 we obtain the result of the theorem.
This completes the proof of Theorem 3.9. □

In the cases $\beta=0$ and $\beta=1$, from the Theorem 3.9 we obtain the following results, respectively.

**Corollary 3.10** *Let the function $f$ given by (1.1) be in the class $S^*_{q,\Sigma}(\varphi)$ and $\mu\in\mathbb{R}$. Then,*

$$\left|a_3-\mu a_2^2\right|\leq\begin{cases}\frac{1-\mu}{\left([2]_q-1\right)^2} & \text{if} \quad \mu\leq 1-l_0(q),\\ \frac{1}{[3]_q-1} & \text{if} \quad 1-l_0(q)\leq \mu\leq 1+l_0(q),\\ \frac{\mu-1}{\left([2]_q-1\right)^2} & \text{if} \quad 1+l_0(q)\leq \mu.\end{cases}$$

*where* $l_0(q)=\frac{\left([2]_q-1\right)^2}{[3]_q-1}$.

**Corollary 3.11** *Let the function $f$ given by (1.1) be in the class $C_{q,\Sigma}(\varphi)$ and $\mu\in\mathbb{R}$. Then, we have*

$$\left|a_3-\mu a_2^2\right|\leq\begin{cases}\frac{1-\mu}{[2]_q^2} & \text{if} \quad \mu\leq 1-l_1(q),\\ \frac{1}{[2]_q[3]_q} & \text{if} \quad 1-l_1(q)\leq \mu\leq 1+l_1(q),\\ \frac{\mu-1}{[2]_q^2} & \text{if} \quad 1+l_1(q)\leq \mu,\end{cases}$$

*where* $l_1(q)=\frac{[2]_q}{[3]_q}$.

Furthermore, from the Theorem 3.9 we obtain the following results for $\mu=1$.

**Corollary 3.12** *Let the function $f$ given by (1.1) be in the class $\mathrm{M}_{q,\Sigma}(\varphi,\beta)$. Then, we have*

$$\left|a_3-a_2^2\right|\leq\frac{1}{[2]_q[3]_q\beta+\left([3]_q-1\right)(1-\beta)}.$$

**Corollary 3.13** *Let the function $f$ given by (1.1) be in the class $S^*_{q,\Sigma}(\varphi)$. Then, we have*

$$\left|a_3-a_2^2\right|\leq\frac{1}{[3]_q-1}.$$

**Corollary 3.14** *Let the function $f$ given by (1.1) be in the class $C_{q,\Sigma}(\varphi)$. Then, we have*

$$\left|a_3 - a_2^2\right| \leq \frac{1}{[2]_q\,[3]_q}.$$

In the case $\mu = 0$, from the Theorem 3.9 we obtain the following results, which confirm the second inequality obtained in Theorem 2.1

**Corollary 3.15** *Let the function $f$ given by (1.1) be in the class $\mathrm{M}_{q,\Sigma}(\varphi,\beta)$. Then, we have*

$$|a_3| \leq \begin{cases} \frac{1}{\left([2]_q-1+\beta\right)^2} & \text{if} \quad l(q,\beta) \leq 1, \\ \frac{l(q,\beta)}{\left([2]_q-1+\beta\right)^2} & \text{if} \quad l(q,\beta) \geq 1\ , \end{cases}$$

*where* $l(q,\beta) = \frac{\left([2]_q-1+\beta\right)^2}{[2]_q[3]_q\beta+\left([3]_q-1\right)(1-\beta)}$ .

From the Theorem 3.9, we obtained the following theorem, when $q \to 1^-$.

**Theorem 3.16** *(see [24],Theorem 3.7) Let the function $f$ given by (1.1) be in the class $\mathrm{M}_{\Sigma}(\varphi,\beta)$ and $\mu \in \mathbb{R}$. Then, we have*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{1-\mu}{(1+\beta)^2} & \text{if} \quad \mu \leq 1 - l(\beta), \\ \frac{1}{2(1+2\beta)} & \text{if} \quad 1 - l(\beta) \leq \ \mu \leq 1 + l(\beta), \\ \frac{\mu-1}{(1+\beta)^2} & \text{if} \quad 1 + l(\beta) \leq \ \mu, \end{cases}$$

*where* $l(\beta) = \frac{(1+\beta)^2}{2(1+2\beta)}$.

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