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On the behaviour of solutions to a kind of third order nonlinear neutral differential equation with delay

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Abstract: This paper presents a novel class of third order nonlinear nonautonomous neutral differential equation with delay. The third order neutral differential equation is cut down to a system of first order, a suitable complete Lyapunov-Krasovskii's functional is constructed and used, to obtain standard conditions on the nonlinear functions to ensure stability and uniform asymptotic stability of the trivial solutions, the existence of a unique periodic solution, uniform boundedness and uniform ultimate boundedness of solutions when the forcing term is nonzero. The obtained results are new and include many prominent results on neutral and nonneutral delay differential equations in literature. Finally, the practicability and reliability of the theoretical results are demonstrated.

Key words: Third order, nonlinear neutral differential equation, uniform stability, uniform ultimate boundedness

1. Introduction

Qualitative behaviour of ordinary and partial differential equations such as stability, instability, decay and boundedness, oscillation, asymptotic behaviour, convergence, and the existence of unique periodic solutions with or without delay, have been impressively examined by prominent researchers. Distinct techniques such as the generalized Riccati substitution (Cesarano and Bazighifan [14, 15], Chatzarakis *et al.* [16], Fite [25]); classical Riccati transformation technique (Wei [44]); linearization technique (Došlá and Liška [23]); the comparison theorem (or principle) (Trench [41]); Banach fixed point theorem (Stokes [39]); coincidence degree theory in (Dahiya, and Akinyele [17], Das and Misra [18]); the general theory of semigroup as found in (Avrin [9], Okoya and Ayeni [35], Okoya [36, 37]); and Lyapunov's second method in (Ademola and Arawomo [2, 3], Ademola *et al.* [4, 5], Kirane [30], Remili and Beldjerd [33], Remili and Oudjedi [34], Tejumola and Tchegnani [40], Tunç [42, 43]) to mention but a few, have been utilized by these authors. Outstanding results on the qualitative behaviour of solutions to both ordinary and partial differential equations with and without delay are discussed in these expository research works.

Functional differential equations are general applicable differential equations which include the classical ordinary and partial differential equations, its applications are found in technical problems, mechanical system under the action of dissipative, gyroscopic forces, hydraulic engineering applications (Arino *et al.* [8], Halanay and Răsvan [26], Kolmanovskii and Myshkis [31]); physical applications to control problems, feedback control

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2010 *AMS Mathematics Subject Classification:* 34K05, 34K12, 34K13, 34K20, 34K40 system, distribution of primes, the theory of growth in a species, examination of predator-prey models, electrical networks containing lossless transmission lines, variational problems (Arino *et al.* [8], Hale [27, 28], Hale and Lunel [29], Kolmanovskii and Myshkis [31]); study of materials which is viscous and elastic in nature, motion of a particle in a liquid and rigid body, relativistic dynamics and oscillation, study of nuclear rectors, distribute and neural networks (Kolmanovskii and Myshkis [31]); the circummutation of plants in sunflower (Hale and Lunel [29]); and model of fish population, decomposition theory, epidemic models, vaccinated state model (Arino *et al.* [8], Hale [28], Hale and Lunel [29]).

As a result of these untrammeled areas of applications in many subdivisions of human endeavour and things, authors in recent years, have developed various techniques to present background studies to system of functional differential equations (see Agarwal *et al.* [7], Arino *et al.* [8], Burton [11–13], Driver [24], Halanay and Răsvan [26], Hale [27, 28], Hale and Lunel [29] and Yoshizawa [45]). Therefore, the study of the asymptotic behaviour of solutions for nonlinear nonautonomous functional differential equations cannot be overemphasized. There are several estimable methods that have been developed by researchers to study the qualitative behaviour of solutions. Among these methods that provide concise and worthful information on the behaviour of solutions without solving the differential equation in question, is the direct method of Lyapunov. In this paper, we shall consider the nonlinear nonautonomous third order neutral differential equation with delay defined as

$$\begin{aligned}[r(t)(x''(t)+q(t)\Phi(x''(t-\tau_0)))]'+\varphi(t)f(x(t))x''(t)+\psi(t)g(x(t-\tau_1),x'(t-\tau_1))\\ +\mu(t)h(x(t-\tau_1))=p(t).\end{aligned} \tag{1.1}$$

Let $Z(t)=x''(t)+q(t)\Phi(x''(t-\tau_0))$. Equation (1.1) is equivalent to system of first order differential equations

$$\begin{aligned}&x'(t)=y(t),\quad y'(t)=z(t),\\ &r(t)Z'(t)=-r'(t)Z(t)-\varphi(t)f(x(t))z(t)-\psi(t)g(x(t),y(t))-\mu(t)h(x(t))+p(t)\\ &+\int_{t-\tau_1}^{t}\left[\psi(t)\left[g_x(x(s),y(s))y(s)+g_y(x(s),y(s))z(s)\right]+\mu(t)h'(x(s))y(s)\right]ds,\end{aligned} \tag{1.2}$$

where the functions $p(t),\ q(t),\ r(t),\ \varphi(t),\ \psi(t),\ \mu(t)\in C([t_x,\infty),\mathbb{R}),\ f(x(t)),\ h(x(t))\in C(\mathbb{R},\mathbb{R}),\ g(x(t),y(t))\in C(\mathbb{R}^2,\mathbb{R})$, with $|q(t)|\le q_1,\ q_1>0$ is a constant, and $q(0)=0=h(0)$. It is assumed that the derivatives $r'(t),\ \psi'(t),\ \varphi'(t),\ \mu'(t),\ \Phi'(z),\ h'(x(t)),\ g_x(x(t),y(t))$ and $g_y(x(t),y(t))$ exist and are continuous for all $t\ge t_0+\vartheta$ where $\vartheta=\max\{\tau_0,\tau_1\}$. Solution of equation (1.1) is defined as the continuous function $x:[t_x,\infty)\to\mathbb{R}$ so that $x(t)\in C^2([t_x,\infty),\mathbb{R})$ which satisfies (1.1) on $[t_x,\infty)$ and $r(t)Z(t)\in C^1([t_x,\infty),\mathbb{R})$. It is assumed end-to-end that every solution $x(t)$ of (1.1) is continuable to the right on some interval $[t_x,\infty)$.

Investigation on properties of solutions to functional differential equations with delay is dated back to Dahiya and Akinyele [17] where the oscillation theorems of $nth-$order functional differential equations with forcing terms were developed. Later, boundedness, oscillatory and non oscillatory properties of neutral differential equations with delay emerged see (Baculíková and Džurina [10], Das and Misra [18], Das [19, 20], Dorociaková [22], Mihalíková and Kostiková [32] and the references cited therein). Furthermore, papers on stability, boundedness, and existence of a unique periodic solution for third order differential equations with delay abound (see Ademola and Arawomo [3], Ademola *et al.* [4], Tejumola and Tchegnani [40], Tunç [42, 43] when $[r(t)(x''(t)+q(t)\Phi(x''(t-\tau_0)))]'=x'''(t)$, Remili and Beldjerd [33] for the case $[r(t)(x''(t)+q(t)\Phi(x''(t-\tau_0)))]'=$

$[g_1(x(t))x'(t)]''$, Remili and Oudjedi [34] when $[r(t)(x''(t)+q(t)\Phi(x''(t-\tau_0)))]' = [g_1(x''(t))x''(t)]'$, and the references cited therein to mention but a few).

Recently, Domoshnitsky *et al.* [21] used Azbelev $W-$transform to study of exponential stability for third order neutral differential equations with delay. In addition, Oudjedi *et al.* [38] gave sufficient conditions for every solution to converge to zero when $h(t)=0$, boundedness and square integrability for a class of third order neutral delay differential equations defined as

$$[x(t)+\beta x(t-r)]''' + a(t)x''(t) + b(t)x'(t) + c(t)f(x(t-r)) = h(t).$$

If $r(t)\equiv 1$, $q(t)\equiv 1$, $f(x(t))\equiv 1$, and $g(x(t-\tau_1), x'(t-\tau_1)) \equiv x'(t)$, then equation (1.1) reduces to that considered in [38]. In [1] Ademola *et al.*, consider a more general third order neutral differential equation using a standard Lyapunov's functional to develop criteria guaranteeing uniform asymptotic stability when $p(t)=0$ and uniform ultimate boundedness of solutions of the equations

$$[x(t)+\beta x(t-\tau)]''' + a(t)x''(t) + b(t)g(x'(t-\tau)) + c(t)h(x(t-\tau)) = p(t),$$

where $\tau>0$ is a constant delay, β is a constant satisfying $0\le\beta\le 1$, the functions $a(t), b(t), c(t), g(x), h(x)$ are continuous in their respective arguments on $\mathbb{R}^+, \mathbb{R}^+, \mathbb{R}^+, \mathbb{R}, \mathbb{R}$ respectively with $\mathbb{R}^+=[0,\infty)$ and $\mathbb{R}=(-\infty,\infty)$. Besides, it is supposed that the derivatives $g'(x)$ and $h'(x)$ exist and are continuous for all x and $h(0)=0$. It is noted that when $r(t)\equiv 1$, $q(t)\equiv 1$ a positive constant, $\tau_0=\tau_1\equiv\tau$ and the function $p(t)$ in (1.1) and [1] have finite positive constants as bound, thus equation (1.1) includes the equation discussed in [1]. In 2021, Ademola [6] enumerates criteria for qualitative behaviour of solutions to a certain class of third order neutral functional differential equation

$$[x''(t)+\beta x''(t-\tau(t))]' + \varphi(t)x''(t) + f(t, x(t), x'(t-\tau(t))) + \psi(t)g(x(t), x(t-\tau(t))) = p(\cdot),$$

where $p(\cdot)=p(t,x(t),x(t-\tau(t)),x'(t),x'(t-\tau(t)))$, $|\beta|<1$, $\tau(t)\le\tau_1$, $\tau_1>0$ is a constant to be determined later, the derivative $\tau'(t)$ exists such that $\tau'(t)\le\mu$ for some constant $\mu\in(0,1)$, the functions φ,ψ,f and g are continuous in their respective arguments. The derivatives $\varphi',\psi',f_{x^*},g_x$ and g_{x^*} exist and are continuous with $g(0,x^*)=0$ for all x^*. We note that the neutral differential equations studied in [6] when compared with equation (1.1) is the case $r(t)\equiv 1$, $q(t)\equiv 1$, $\Phi(x''(t-\tau_0))\equiv x''(t-\tau(t))$, the functions $p(t)$ and $p(\cdot)$ have finite constants as bound.

In this paper, a more general class of third order neutral differential equation with delay is discussed using Lyapunov's second method. In addition, the function $g\in\mathbb{R}^2$ is completely delayed which, according to our observation from relevant literature, has never occurred in the study of neutral differential equations with delay. This work is actuated by the works in [1, 6, 38] and the references cited therein. The results of this paper are not only novel but include many salient existing results in the literature. Stability and boundedness results are presented in Sections 2 and 3 respectively while the last section presents examples to illustrate the theoretical results of Sections 2 and 3.

2. Stability results

This section states and proves results on the asymptotic stability and uniform asymptotic stability for equation (1.1) or its equivalent system (1.2) for $p(t)=0$, that is

$$\begin{aligned}[r(t)(x''(t)+q(t)\Phi(x''(t-\tau_0)))]' + \varphi(t)f(x(t))x''(t) + \psi(t)g(x(t-\tau_1), x'(t-\tau_1))\\ + \mu(t)h(x(t-\tau_1)) = 0.\end{aligned} \tag{2.1}$$

Our notations shall be $x = x(t),\ \Phi(z) = \Phi(z(t-\tau_0)),\ F(t) = f'(x(t))x'(t),\ X_t = x_t, y_t, z_t,$ and the subscript t stands for delay. Equation (2.1) is equivalent to

$$\begin{aligned} &x'(t) = y(t), \quad y'(t) = z(t), \\ &r(t)Z'(t) = -r'(t)Z(t) - \varphi(t)f(x(t))z(t) - \psi(t)g(x(t),y(t)) - \mu(t)h(x(t)) \\ &+ \int_{t-\tau_1}^{t} \Big[\psi(t)[g_x(x(s),y(s))y(s) + g_y(x(s),y(s))z(s)] + \mu(t)h'(x(s))y(s)\Big]ds. \end{aligned} \tag{2.2}$$

Next, a continuously differentiable functional employed in this paper is $V(t, X_t)$ defined as

$$V(t, X_t) = U(t, X_t)\exp\Big(-\frac{1}{\gamma}\int_{t_0}^{t}\beta(s)ds\Big), \tag{2.3}$$

where $U(t, X_t)$ is defined as

$$\begin{aligned} U(t,X_t) \ &= a\mu(t)\int_0^x h(s)ds + \mu(t)r(t)yh(x) + \frac{1}{2}r(t)[\mu(t)y^2 + r(t)Z^2] + \frac{1}{2}a\varphi(t)f(x)y^2 \\ &+ r(t)[\psi(t)yg(x,y) + a(x+y)Z] + \frac{1}{2}a\psi(t)x^2 + \int_{-\tau_1}^{0}\int_{t+s}^{t}[\lambda_1 y^2(u) + \lambda_2 z^2(u)]duds \\ &+ \int_{t-\tau_0}^{t}\lambda_3 z^2(s)ds, \end{aligned} \tag{2.4}$$

a, and γ are positive constants and the values of positive constants λ_i $(i = 1,2,3),$ will be determined later in this section.

Remark 2.1 *It is remarkable at this junction that the appearance of neutral and distinct delay terms in* (2.1) *added more difficulties to the construction of Lyapunov-Krasovskii's functional defined by* (2.3)*. Hence equation* (2.3) *includes and extends the main tools used in [1–6, 30, 33, 34, 40, 42, 43] and the cited references of these papers.*

Next, assumptions are presented as follows.

Assumption 2.2 In addition to the basic assumption on the functions defined above, let $b_i,\ c_i,\ d_1,\ h_i,\ L_j,\ q_1,\ r_i,$ $\beta_1,\ \eta,\ \mu_i,\ \sigma_i,\ \varphi_i,\ \psi_i,\ (i = 0,1),\ (j = 0,1,2)$ are positive constants such that for all $t \geq t_0$:

(i) $\mu_0 \leq \mu(t) \leq \mu_1,\ \varphi_0 \leq \varphi(t) \leq \varphi_1,\ \psi_0 \leq \psi(t) \leq \psi_1, r_0 \leq r(t) \leq r_1,\ -\sigma_0 \leq \mu'(t) \leq 0,$ $-\sigma_1 \leq \varphi'(t) \leq 0;$

(ii) $b_0 \leq f(x) \leq b_1,\ |\Phi(z)| \leq \eta|z|$ for all $z,\ |q(t)| \leq q_1;$

(iii) $h_0 \leq \dfrac{h(x)}{x} \leq h_1$ for all $x \neq 0,\ h'(x) \leq d_1$ and $|h'(x)| \leq L_0$ for all $x;$

(iv) $c_0 \leq \dfrac{g(x,y)}{y} \leq c_1$ for all $x, y \neq 0,\ g_x(x,y) \leq 0,\ |g_x(x,y)| \leq L_1,\ |g_y(x,y)| \leq L_2$ for all $x, y;$

(v) $\max\Big\{d_1 r_1, \dfrac{d_1 r_1 \mu_1}{c_0\mu_0}, \dfrac{h_1 q_1 r_1 \eta}{h_0\mu_0}\Big\} < a < \min\Big\{b_0\varphi_0, b_1\varphi_1, c_1\psi_1, \dfrac{\psi_0}{4}, \dfrac{b_0\varphi_0}{4}\Big\};$

(vi) $\int\limits_{t_0}^{t}[|r'(s)| + |\psi'(s)| + |F(s)|]ds < \beta_1;$

(vii) $ah_0\mu_0 > h_1q_1r_1\eta,\ ac_0\psi_0 - d_1r_1\mu_1 > A_1,\ (b_0\varphi_0 - a)r_0 > A_2,$ where

$$A_1 := r_1[a + \mu_1 + \psi_1 L_2 + q_1\eta(c_1\psi_1 - a)];\ \text{and}\ A_2 := A_1 + q_1r_1\eta[h_1\mu_1 + c_1\psi_1 + b_1\varphi_1 - 2a].$$

Next, two stability theorems designed for this section are stated:

Theorem 2.3 *If assumptions (i) to (vii) hold, then the trivial solution of system (2.2) is asymptotically stable provided the inequality*

$$\tau_1 < \frac{1}{2}\min\left\{\frac{ah_0\mu_0 - h_1q_1r_1\eta}{A_3},\ \frac{ac_0\psi_0 - d_1r_1\mu_1 - A_1}{A_4},\ \frac{(b_0\varphi_0 - a)r_0 - A_2}{A_5}\right\} \tag{2.5}$$

holds, where

$$A_3 := a[L_0\mu_1 + (L_1 + L_2)\psi_1];\quad A_4 := [2a + r_1(1 + q_1\eta) + a\psi_1 L_2](L_0\mu_1 + L_1\psi_1);\ \textit{and}$$
$$A_5 := r_1(1 + q_1\eta)[L_0\mu_1 + (L_1 + L_2)\psi_1] + (2a + r_1(1 + q_1\eta))\psi_1 L_2.$$

Theorem 2.4 *If in addition to assumptions (i) to (vii), there exist positive constants D_0 and D_1 such that*

$$D_1 > D_0[|r'(t)| + |\psi'(t)| + |F(t)|]$$

for all $t \geq t_0$ then the trivial solution of system (2.2) is uniformly asymptotically stable if the inequality (2.5) holds.

Remark 2.5 *Theorems 2.3 and 2.4 are extensions to all stability results discussed in [1–4, 6, 21, 33, 34, 38] and the references cited therein.*

In the sequel, proofs of two lemmas that are crucial to the results are stated.

Lemma 2.6 *Under the conditions of Theorems 2.3 and 2.4 there exist positive constants D_2, D_3, D_4, D_5 and D_6 such that*

$$\begin{aligned} &D_2(x^2(t) + y^2(t) + Z^2(t)) \leq U(t, X_t) \leq D_3(x^2(t) + y^2(t) + Z^2(t)) \\ &+ D_4\int_{-\tau_1}^{0}\int_{t+s}^{t}[y^2(u) + z^2(u)]duds + D_5\int_{t-\tau_0}^{t} z^2(s)ds, \end{aligned} \tag{2.6}$$

for all $t \geq 0, x, y, Z,$ and

$$V(t, X_t) \geq D_6(x^2(t) + y^2(t) + Z^2(t)) \tag{2.7}$$

for all $t \geq 0, x, y$ and Z.

Proof Let (X_t) be any solution of system (2.2). Now since $h(0)=0,$ (2.4) can be redefined as

$$\begin{aligned} U(t,X_t) =& \frac{1}{2}\mu(t)r(t)(h(x)+y)^2+\mu(t)\int_0^x[a-r(t)h'(s)]h(s)ds+\frac{1}{8}[2ax+r(t)Z]^2 \\ &+\frac{a}{2}[(\varphi(t)f(x)-4a)y^2+(\psi(t)-4a)x^2]+r(t)\psi(t)\frac{g(x,y)}{y}y^2+\frac{1}{4}(r(t)Z)^2 \\ &+\frac{1}{8}[2ay+r(t)Z]^2+\int_{-\tau_1}^0\int_{t+s}^t[\lambda_1y^2(u)+\lambda_2z^2(u)]duds+\int_{t-\tau_0}^t\lambda_3z^2(s)ds. \end{aligned} \tag{2.8}$$

Since the integrals in the last two terms of (2.8) are nonnegative, applying assumptions (i) and (iii) we find that

$$\begin{aligned} \frac{1}{2}\mu(t)r(t)(h(x)+y)^2+&\frac{1}{8}\Big[[2ax+r(t)Z]^2+[2ay+r(t)Z]^2\Big] \\ &\geq \frac{1}{2}r_0\mu_0(h_0x+y)^2+\frac{1}{8}\Big[(2ax+r_0Z)^2+(2ay+r_0Z)^2\Big]\geq 0 \end{aligned}$$

for all $t\geq 0, x, y$ and Z. With this estimate and assumption (v), there exists a positive constant δ_0 such that

$$U(t,X_t)\geq \delta_0(x^2+y^2+Z^2), \tag{2.9}$$

for all $t\geq 0, x, y, Z$ where

$$\delta_0 := \frac{1}{2}\min\Big\{a(\psi_0-4a)+h_0\mu_0(a-d_1r_1),\ a(b_0\varphi_0-4a)+2c_0r_0\psi_0,\ \frac{1}{2}r_0^2\Big\}.$$

Moreover, since

$$\int_{t_0}^t\beta(s)ds<\beta_1$$

there exists a positive constant δ_1 such that

$$V(t,X_t)\geq \delta_1(x^2+y^2+Z^2), \tag{2.10}$$

for all $t\geq 0, x, y, Z$ where

$$\delta_1 := \delta_0\exp\Big(-\frac{\beta_1}{\gamma}\Big).$$

Furthermore, from the assumptions (i) to (iv), and the fact that $2|x_1x_2|\leq x_1^2+x_2^2$ for all $x_1, x_2\in\mathbb{R}$, there exist positive constants $\delta_i,\ (i=2,3,4)$ such that

$$U(t,X_t)\leq \delta_2(x^2+y^2+Z^2)+\delta_3\int_{-\tau_1}^0\int_{t+s}^t[y^2(u)+z^2(u)]duds+\delta_4\int_{t-\tau_0}^t z^2(s)ds, \tag{2.11}$$

for all $t\geq 0, x, y$ and Z, where

$$\delta_2 := \frac{1}{2}\max\Big\{\mu_1h_1(a+r_1)+a(r_1+\psi_1),\ r_1\mu_1(1+h_1)+ab_1\varphi_1+2c_1r_1\psi_1+ar_1,\ r_1(r_1+2a)\Big\},$$

$\delta_3 := \max\{\lambda_1,\ \lambda_2\},$ and $\delta_4 := \lambda_3.$

From estimates (2.9) and (2.11) inequality (2.6) of Lemma 2.6 holds with $D_2 \equiv \delta_0$, $D_3 \equiv \delta_2$, $D_4 \equiv \delta_3$ and $D_5 \equiv \delta_4$. Also, from estimate (2.10) inequality (2.7) of Lemma 2.6 holds with $D_6 \equiv \delta_1$. This completes the proof of Lemma 2.6. □

Lemma 2.7 *If assumptions (i) to (vii) and inequality (2.5) hold, then there exist positive constants D_7, D_8, and D_9 such that*

$$U'_{(2.2)}(t, X_t) \leq D_7[|r'(t)| + |\psi'(t)| + |F(t)|](x^2 + y^2) - D_8(x^2 + y^2 + z^2), \tag{2.12}$$

for all $t \geq 0, x, y, z$ and

$$V'_{(2.2)}(t, X_t) \leq -D_9(x^2 + y^2 + z^2), \tag{2.13}$$

for all $t \geq 0, x, y$ and z.

Proof Let (X_t) be any solution of the system (2.2) and since $h(0) = 0$, the derivative of the functional $V(t, X_t)$ along the trajectory of (2.2) is given by

$$V'_{(2.2)}(t, X_t) = \left[U'_{(2.2)}(t, X_t) - \frac{1}{\gamma}\beta(t)U(t, X_t)\right] \exp\left(-\frac{1}{\gamma}\int_{t_0}^{t} \beta(s)ds\right), \tag{2.14}$$

where

$$U'_{(2.2)}(t, X_t) = \sum_{j=1}^{6} W_j + \lambda_1\tau_1 y^2 + (\tau_1\lambda_2 + \lambda_3)z^2 - \lambda_3 z^2(t - \tau_0) - \int_{t-\tau_1}^{t} [\lambda_1 y^2(\alpha) + \lambda_2 z^2(\alpha)]d\alpha, \tag{2.15}$$

$$\begin{aligned}
W_1 :=& \mu'(t)\left[\int_0^x [a - r(t)h'(s)]h(s)ds + \frac{1}{2}r(t)(h(x) + y)^2\right] + \frac{1}{2}r'(t)\left[\mu(t)\left(y^2 + 2\frac{h(x)}{x}xy\right)\right.\\
&\left. + r(t)y^2 + 2\psi(t)\frac{g(x,y)}{y}y^2\right] + \frac{1}{2}a\varphi(t)F(t)y^2;\\
W_2 :=& -\frac{1}{2}\left[a\mu(t)\frac{h(x)}{x}x^2 + [a\psi(t)\frac{g(x,y)}{y} - r(t)\mu(t)h'(x)]y^2 + r(t)(\varphi(t)f(x) - a)z^2\right];\\
W_3 :=& -W_{31} - W_{32};\\
W_{31} :=& \left[\frac{1}{4}a\mu(t)\frac{h(x)}{x}x^2 + a\psi(t)[\frac{g(x,y)}{y} - 1]xy + \frac{1}{2}[a\psi(t)\frac{g(x,y)}{y} - r(t)\mu(t)h'(x)]y^2\right]\\
W_{32} :=& \left[\frac{1}{4}a\mu(t)\frac{h(x)}{x}x^2 + a\varphi(t)f(x)xz + \frac{1}{2}r(t)(\varphi(t)f(x) - a)z^2\right];\\
W_4 :=& \psi'(t)[r(t)\frac{g(x,y)}{y}y^2 + \frac{1}{2}ax^2] + [\frac{1}{2}a\varphi'(t)f(x) + r(t)\psi(t)g_x(x,y)]y^2;\\
W_5 :=& -q(t)r(t)\left[\mu(t)\frac{h(x)}{x}x + \left(\psi(t)\frac{g(x,y)}{y} - a\right)y + (\varphi(t)f(x) - a)z\right]\Phi(z)\\
&+ r(t)[a\mu(t) + \psi(t)g_y(x,y)]yz; \text{ and}\\
W_6 :=& [a(x+y) + r(t)\big(z + q(t)\Phi(z)\big)]\left[\psi(t)\int_{t-\tau_1}^{t}[g_x(x(s),y(s))y(s) + g_y(x(s),y(s))z(s)]ds\right.\\
&\left. + \mu(t)\int_{t-\tau_1}^{t} h'(x(s))y(s)ds\right].
\end{aligned}$$

The results of applying assumptions (i), (iii) and (iv) are

$$\int_0^x [a - r(t)h'(s)]h(s)ds + \frac{1}{2}r(t)\left(\frac{h(x)}{x}x + y\right)^2 \geq \frac{1}{2}r_0(h_0x + y)^2 + (a - d_1r_1)h_0x^2 \geq 0,$$

for all $t \geq 0, x$ and y. Since $\mu'(t) \leq 0$ for all $t \geq 0$, it follows that

$$\mu'(t)\left[\int_0^x [a - r(t)h'(s)]h(s)ds + \frac{1}{2}r(t)\left(\frac{h(x)}{x}x + y\right)^2\right] \leq 0$$

for all $t \geq 0, x$ and y, so with the last inequality and assumptions (i), (iii) and (iv) there exist positive constants δ_5 and δ_6 such that

$$W_1 \leq \delta_5|r'(t)|(x^2 + y^2) + \delta_6|F(t)|y^2,$$

for all $t \geq 0, x$ and y, where

$$\delta_5 := \frac{1}{2}\max\{h_1\mu_1,\ h_1\mu_1 + r_1 + 2c_1\psi_1\} \text{ and } \delta_6 := \frac{1}{2}a\varphi_1\ .$$

Employing assumptions (i) to (iv), result in

$$W_2 \leq -\frac{1}{2}\left[ah_0\mu_0x^2 + [ac_0\psi_0 - d_1r_1\mu_1]y^2 + [(b_0\varphi_0 - a)r_0]z^2\right]$$

for all $t \geq 0, x, y$ and z.

Furthermore, $W_{31} = W_{31}(x, y)$ and $W_{32} = W_{32}(x, z)$ are symmetric quadratic functions which satisfy a 2×2 Hessian matrix. The discriminant $b^2 < ac$ yields inequalities

$$4\left[a\psi(t)\frac{g(x,y)}{y} - 1\right]^2 < 2a\mu(t)\frac{h(x)}{x}[a\psi(t)\frac{g(x,y)}{y} - r(t)\mu(t)h'(x)]$$

and

$$4[a\varphi(t)f(x)]^2 < 2ar(t)\mu(t)\frac{h(x)}{x}(\varphi(t)f(x) - a)$$

respectively. With these inequalities it is not difficult to see that $W_{31} \geq 0$, $W_{32} \geq 0$ for all $t \geq 0, x, y$ and z, so that

$$W_3 \leq 0$$

for all $t \geq 0, x, y$ and z.

Moreover, since $\varphi'(t) \leq 0$ for all $t \geq 0$ and $g_x(x, y) \leq 0$ for all x and y, it follows that

$$\frac{1}{2}a\varphi'(t)f(x) + r(t)\psi(t)g_x(x, y) \leq 0$$

for all $t \geq 0, x$ and y, so there exists a positive constant δ_7 such that

$$W_4 \leq \delta_7|\psi'(t)|(x^2 + y^2),$$

for all $t \geq 0, x$ and y, where $\delta_7 := \frac{1}{2}\max\{a, 2c_1r_1\}$. Next, apply assumptions (i) to (iv) to get

$$W_5 \leq \frac{1}{2}r_1\Big[h_1q_1\mu_1\eta x^2 + (a + \mu_1 + \psi_1L_2 + q_1\eta(c_1\psi_1 - a))y^2 + (a + \mu_1 + \psi_1L_2 + q_1\eta(b_1\varphi_1 - a))z^2 + q_1\eta[h_1\mu_1 + (c_1\psi_1 - a) + (b_1\varphi_1 - a)]z^2(t - \tau_0)\Big]$$

for all $t \geq 0, x, y$ and z.

Finally, on employing assumptions (i) to (iv), it follows that

$$W_6 \leq \frac{1}{2}[L_0\mu_1 + (L_1 + L_2)\psi_1]\tau_1[a(x^2 + y^2) + r_1z^2 + q_1r_1\eta z^2(t - \tau_0)] + \frac{1}{2}[2a + r_1(1 + q_1\eta)]\int_{t-\tau_1}^{t}[(L_0\mu_1 + L_1\psi_1)y^2(s) + \psi_1L_2z^2(s)]ds,$$

for all $t \geq 0, x, y$ and z.

We now turn our attention to estimates $W_j,\ (j = 1, 2, 3, 4, 5)$ in equation (2.15) and on further simplification, we have

$$\begin{aligned}
U'_{(2.2)}(t, X_t) &\leq \delta_8[|r'(t)| + |\psi'(t)| + |F(t)|](x^2 + y^2)\\
&- \frac{1}{2}\Big\{ah_0\mu_0 - h_1q_1r_1 - a[L_0\mu_1 + (L_1 + L_2)\psi_1]\tau_1\Big\}x^2\\
&- \frac{1}{2}\Big\{ac_0\psi_0 - d_1r_1\mu_1 - r_1[a + \mu_1 + \psi_1L_2 + q_1\eta(c_1\psi_1 - a)] - [a(L_0\mu_1 + (L_1 + L_2)\psi_1) + 2\lambda_1]\tau_1\Big\}y^2\\
&- \frac{1}{2}\Big\{(b_0\varphi_0 - a)r_0 - r_1[a + \mu_1 + \psi_1L_2 + q_1\eta(c_1\psi_1 - a)] - 2\lambda_3 - [r_1(L_0\mu_1 + (L_1 + L_2)\psi_1) + 2\lambda_2]\tau_1\Big\}z^2\\
&- \frac{1}{2}\Big\{2\lambda_3 - q_1r_1\eta[h_1\mu_1 + c_1\psi_1 + b_1\varphi_1 - 2a + [L_0\mu_1 + (L_1 + L_2)\psi_1]\tau_1]\Big\}z^2(t - \tau_0)\\
&- \frac{1}{2}\Big\{2\lambda_1 - [2a + r_1(1 + q_1\eta)][L_0\mu_1 + L_1\psi_1]\Big\}\int_{t-\tau_1}^{t} y^2(\alpha)d\alpha\\
&- \frac{1}{2}\Big\{2\lambda_2 - \psi_1L_2[2a + r_1(1 + q_1\eta)]\Big\}\int_{t-\tau_1}^{t} z^2(\alpha)d\alpha,
\end{aligned} \tag{2.16}$$

where $\delta_8 := \max\{\delta_5,\ \delta_6,\ \delta_7\}$. Let

$$\lambda_1 := \frac{1}{2}[2a + r_1(1 + q_1\eta)][L_0\mu_1 + L_1\psi_1];\quad \lambda_2 := \frac{1}{2}[2a + r_1(1 + q_1\eta)]\psi_1L_2;\ \text{and}$$
$$\lambda_3 := \frac{1}{2}q_1r_1\eta[h_1\mu_1 + c_1\psi_1 + b_1\varphi_1 - 2a + [L_0\mu_1 + (L_1 + L_2)\psi_1]\tau_1].$$

On utilising $\lambda_i,\ (i = 1, 2, 3)$ in the estimate (2.16) gives

$$\begin{aligned}
U'_{(2.2)}(t, X_t) &\leq \delta_8[|r'(t)| + |\psi'(t)| + |F(t)|](x^2 + y^2) - \frac{1}{2}\Big\{ah_0\mu_0 - h_1q_1r_1\eta - A_3\tau_1\Big\}x^2\\
&- \frac{1}{2}\Big\{ac_0\psi_0 - d_1r_1\mu_1 - A_1 - A_4\tau_1\Big\}y^2 - \frac{1}{2}\Big\{(b_0\varphi_0 - a)r_0 - A_2 - A_5\tau_1\Big\}z^2,
\end{aligned} \tag{2.17}$$

for all $t \geq 0, x, y$ and z. In view of assumption (vii), there exists a positive constant δ_9 such that

$$U'_{(2.2)}(t, X_t) \leq \delta_8[|r'(t)| + |\psi'(t)| + |F(t)|](x^2 + y^2) - \delta_9(x^2 + y^2 + z^2) \tag{2.18}$$

for all $t \geq 0, x, y, z,$ where

$$\begin{aligned}\delta_9 := \max\Big\{ & ah_0\mu_0 - A_1 - \frac{1}{2}a\Big[L_0\mu_1 + (L_1 + L_2)\psi_1\Big]\tau_1, \\ & (ac_0\psi_0 - d_1r_1\mu_1) - A_2 - \frac{1}{2}\Big[2a + [a + r_1(1 + q_1\eta)](L_0\mu_1 + L_1\psi_1) + a\psi_1 L_2\Big]\tau_1, \\ & (b_0\varphi_0 - a)r_0 - A_3 - \frac{1}{2}\Big[r_1(1 + q_1\eta)(L_0\mu_1 + (L_1 + L_2)\psi_1) + (2a + r_1(1 + q_1\eta))\psi_1 L_2\Big]\tau_1\Big\}.\end{aligned}$$

Using assumption (vi), estimates (2.9), (2.18) in equation (2.14) with $\delta_8 = \dfrac{\delta_0}{\gamma}$, results in

$$V'_{(2.2)}(t, X_t) \leq -\delta_{10}(x^2 + y^2 + z^2) \tag{2.19}$$

for all $t \geq 0, x, y, z,$ where $\delta_{10} := \delta_9 \exp(-\beta_1/\gamma)$. Estimates (2.18) and (2.19) satisfy inequalities (2.12) and (2.13) of Lemma 2.7 respectively with $D_7 \equiv \delta_8$, $D_8 \equiv \delta_9$, and $D_9 \equiv \delta_{10}$. This completes the proof of Lemma 2.7. □

Proof of Theorem 2.3.
Let (X_t) be any solution of system (2.2). Estimates (2.10) and (2.19) established that the trivial solution of system (2.2) is asymptotically stable. This completes the proof of Theorem 2.3.

Proof of Theorem 2.4.
Let (X_t) be any solution of system (2.2). If $\delta_9 > \delta_8[|r'(t)| + |\psi'(t)| + |F(t)|]$, then there exists a positive constant δ_{11} such that the inequality (2.18) yields

$$U'_{(2.2)}(t, X_t) \leq -\delta_{11}(x^2 + y^2 + z^2) \tag{2.20}$$

for all $t \geq 0, x, y, z,$ where $\delta_{11} := \delta_9 - \delta_8[|r'(t)| + |\psi'(t)| + |F(t)|]$. In view of inequalities (2.9), (2.11) and (2.20), the trivial solution of system (2.2) is uniformly asymptotically stable. This completes the proof of Theorem 2.4.

3. Boundedness and existence theorems

In this section the boundedness and existence of a unique periodic solution of system (1.2) is discussed. Let (X_t) be a solution of (1.2), the following results emerge.

Theorem 3.1 *If in addition to the assumptions of Theorem 2.3, $|p(t)| \leq P_1$, $0 < P_1 < \infty$, then there exists a positive constant $D_{10} = D_{10}(\delta_1, x(t_0), y(t_0), Z(t_0))$ such that the solution of (1.2) satisfies*

$$|x(t)| \leq D_{10}, \ |y(t)| \leq D_{10}, \ |Z(t)| \leq D_{10} \tag{3.1}$$

for all $t \geq 0$ provided the inequality (2.5) holds.

Proof Let (X_t) be a solution of (1.2), the derivative of $U(t,X_t)$ with respect to the independent variable t along the trajectory of (1.2) is defined as

$$U'_{(1.2)}(t,X_t)=U'_{(2.2)}(t,X_t)+[a(x+y)+q(t)Z]p(t). \tag{3.2}$$

Using inequality (2.18) and the fact that $|p(t)|\le P_1$, equation (3.2) becomes

$$U'_{(1.2)}(t,X_t)\le \delta_8[|r'(t)|+|\psi'(t)|+|F(t)|](x^2+y^2)-\delta_9(x^2+y^2+z^2)+\delta_{12}(x^2+y^2+Z^2+3) \tag{3.3}$$

for all $t\ge 0, x, y, z,$ where $\delta_{12}:=P_1\max\{a,r_1\}$. Employing estimate (3.3) and the fact that $\delta_9(x^2+y^2+z^2)\ge 0$ for all $t\ge 0, x, y$ and z, equation (2.14) becomes

$$V'_{(1.2)}(t,X_t)\le \delta_{13}(x^2+y^2+Z^2)+\delta_{14} \tag{3.4}$$

for all $t\ge 0, x, y, z,$ where $\delta_{13}:=\delta_{12}\exp(-\delta_0^{-1}\beta_1\delta_8)$ and $\delta_{14}:=3\delta_{12}\exp(-\delta_0^{-1}\beta_1\delta_8)$. Engaging estimate (2.10) in (3.4), results in

$$V'_{(1.2)}(t)\le \delta_{15}V+\delta_{14} \tag{3.5}$$

where $\delta_{15}:=\delta_1^{-1}\delta_{13}$. Solving the differential inequality (3.5) using integrating factor $\exp(-\delta_{15}t)$, to get

$$V(t)\le V(t_0)e^{\delta_{15}(t-t_0)}+\delta_{14}\delta_{15}^{-1}\bigg(e^{\delta_{15}(t-t_0)}-1\bigg). \tag{3.6}$$

Repeat estimate (2.10) and the fact that $t\ge t_0$, there exists a positive constant $\delta_{16}=\delta_{16}(\delta_1 V(t_0))$, such that

$$|x(t)|\le\delta_{16},\ |y(t)|\le\delta_{16},\ |Z(t)|\le\delta_{16},\quad \forall\ t\ge t_0.$$

This established estimate (3.1) with $D_{10}\equiv\delta_{16}$. This completes the proof of Theorem 3.1. □

Theorem 3.2 *In addition to the assumptions of Theorem 3.1, suppose that* $q(0)=0$, *then the solution* (X_t) *of* (1.2) *is uniformly bounded and uniformly ultimately bounded if the inequality* (2.5) *holds.*

Proof Let (X_t) be any solution of (1.2). In view of estimate (2.9) and assumptions of Theorem 3.2, it follows that $U(t,X_t)=0$ if and only if $x^2+y^2+z^2=0$, and $U(t,X_t)>0$ if and only if $x^2+y^2+z^2\ne 0$, and that

$$U(t,X_t)\to+\infty \text{ as } x^2+y^2+z^2\to\infty. \tag{3.7}$$

Next, the derivative of the functional $U(t,X_t)$, defined by (2.4), along the trajectory of the system (1.2) is defined in equation (3.2) i.e.

$$U'_{(1.2)}(t,X_t)=U'_{(2.2)}(t,X_t)+[a(x+y)+q(t)Z]p(t).$$

Estimate (2.20) results in

$$U'_{(1.2)}(t,X_t)\le -\delta_{11}(x^2+y^2+z^2)+\delta_{17}[|x|+|y|+|Z|],$$

where $\delta_{17} := P_1 \max\{a, q_1(1+q_1)\}$. Recall $(|x|+|y|+|Z|)^2 \leq 3(x^2+y^2+z^2)$ so there exist positive constants δ_{18} and δ_{19} such that

$$U'_{(1.2)}(t, X_t) \leq -\delta_{18}(x^2+y^2+z^2), \tag{3.8}$$

for all $t \geq 0, x, y,$ and z, provided that $(x^2+y^2+z^2)^{1/2} \geq \delta_{19}$ where $\delta_{18} := \frac{1}{2}\delta_{11}$ and $\delta_{19} := 2\sqrt{3}\delta_{11}^{-1}\delta_{17}$. Hence, from estimates (2.9), (2.11), (3.7), and (3.8) the solution (X_t) is uniformly bounded and uniformly ultimately bounded. This completes the proof of Theorem 3.2. □

Theorem 3.3 *Suppose that all assumptions of Theorem 3.2 hold true, then there exists a periodic solution of system (1.2) of period say ω, provided the inequality (2.5) holds true.*

Proof Let (X_t) be any solution of (1.2), since the solution (X_t) is uniformly bounded and uniformly ultimately bounded by Theorem 3.2, it follows that a periodic solution exists of period say ω. This completes the proof of Theorem 3.3. □

Remark 3.4 *Our results in Theorems 3.1, 3.2i and 3.3 are generalization of boundedness and existence results considered in [1–6, 10, 15–19, 21, 23, 32–34, 38, 40, 42–44] and the references cited in these papers.*

4. Examples and discussion

This section presents examples on the theoretical results obtained in Sections 2 and 3.

Example 4.1 *Consider the following third order neutral differential equation with delay defined as*

$$\begin{aligned}&\left[\frac{2t+21}{10+t}\left(x''+(100+\cos t)^{-1}\Phi(x''(t-\tau_0))\right)\right]'+\left(\frac{221+60t}{11+3t}\right)\left(\frac{101+20x^2}{10+2x^2}\right)x''\\&+\left(\frac{281+56t}{10+2t}\right)\left[\frac{51x'(t-\tau_1)+5x'(t-\tau_1)A(x(t-\tau_1),x'(t-\tau_1))}{10+A(x(t-\tau_1),x'(t-\tau_1))}\right]\\&+\left(\frac{9+2t^2}{4+t^2}\right)\left[\frac{31x'(t-\tau_1)+9x^3(t-\tau_1)}{10+3x^2(t-\tau_1)}\right]=0,\end{aligned} \tag{4.1}$$

where

$$A(x(t-\tau_1), x'(t-\tau_1)) := \exp\left(\frac{1+x(t-\tau_1)x'(t-\tau_1)+x'^2(t-\tau_1)}{5+x'^2(t-\tau_1)}\right).$$

Equation (4.1) can be rewritten as

$$\begin{aligned}&x'=y, \quad y'=z\\&\left(\frac{21+2t}{10+t}\right)Z'=-\frac{Z}{(10+t)^2}-\left(\frac{221+60t}{11+3t}\right)\left(\frac{101+20x^2}{10+2x^2}\right)z-\left(\frac{9+2t^2}{4+t^2}\right)\left[\frac{31x+9x^3}{10+3x^2}\right]\\&-\left(\frac{281+56t}{10+2t}\right)\left[\frac{51y+5yA(x,y)}{10+A(x,y)}\right]+\left(\frac{9+2t^2}{4+t^2}\right)\int_{t-\tau_1}^{t}\left[3+\frac{10-3x^2(s)}{(10+3x^2(s))^2}\right]y(s)ds\\&-\left(\frac{281+56t}{10+2t}\right)\int_{t-\tau_1}^{t}\left[\frac{y^2(s)A(x(s),y(s))}{[10+A(x(s),y(s))]^2(5+y^2(s))}\right]y(s)ds\\&+\left(\frac{281+56t}{10+2t}\right)\int_{t-\tau_1}^{t}\left[5+\frac{1}{10+A(x(s),y(s))}-\frac{y(s)A(x(s),y(s))B(x(s),y(s))}{[10+A(x(s),y(s))]^2}\right]z(s)ds,\end{aligned} \tag{4.2}$$

where

$$B(x,y) := \frac{x+2y}{5+y^2} - \frac{2y(1+xy+y^2)}{(5+y^2)^2}.$$

The following relations emanated from systems (2.2) and (4.2):
(i) The function

$$r(t) := \frac{21+2t}{10+t} = 2 + \frac{1}{10+t}.$$

Since $0 < \dfrac{1}{10+t} \leq 0.1$ *for all* $t \geq 0$, *it follows that* $2 = r_0 \leq r(t) \leq r_1 = 2.1$ *and*

$$|r'(t)| = \frac{1}{(10+t)^2} \leq 0.1 \tag{4.3}$$

for all $t \geq 0$;
(ii) The function

$$\varphi(t) := \frac{221+60t}{11+3t} = 20 + \frac{1}{11+3t},$$

the maximum value of $\dfrac{1}{11+3t}$ *when* $t = 0$ *is* 0.09, *so that* $20 = \varphi_0 \leq \varphi(t) \leq \varphi_1 = 20.09$ *for all* $t \geq 0$. *Also*

$$\varphi'(t) = \frac{-3}{(11+3t)^2} \leq 0$$

for all $t \geq 0$. *Since* $\lim_{t\to\infty} \varphi'(t) = 0$ *it follows that* $-0.27 = -\sigma_1 \leq \varphi'(t) \leq 0$;
(iii) The function

$$f(x) := \frac{101+20x^2}{10+2x^2} = 10 + \frac{1}{10+2x^2}.$$

It can be shown that $b_0 = 10 \leq f(x) \leq b_1 = 10.1$ *for all* x. *In addition,*

$$f'(x) = \frac{-4x}{(10+2x^2)^2}.$$

The function $F(t) = f'(x)x' = f'(x)y$ *assuming, in this case, that* $|y| < 1$ *leads to*

$$|F(t)| = |f'(x)y| \leq \frac{4(1+x^2)}{(10+2x^2)^2} \leq 0.04 \tag{4.4}$$

for all x;
(iv) The function

$$\psi(t) := \frac{281+56t}{10+2t} = 28 + \frac{1}{10+2t}.$$

Since $\dfrac{1}{10+2t} \leq 0.1$ *for all* $t \geq 0$, *then it is logical to have* $28 = \psi_0 \leq \psi(t) \leq \psi_1 = 28.1$, *for all* $t \geq 0$. *Furthermore,*

$$|\psi'(t)| = \frac{2}{(10+2t)^2} \leq 0.02 \tag{4.5}$$

for all $t \geq 0$;
(v) The function

$$g(x,y) := \frac{51y + 5yA(x,y)}{10 + A(x,y)} = 5y + \frac{y}{10 + A(x,y)},$$

since $\dfrac{1}{10 + A(x,y)} \leq 0.089$ *for all* x, y *then it is reasonable to conclude* $5 = c_0 \leq \dfrac{g(x,y)}{y} \leq c_1 = 5.089$ *for all* x *and* $y \neq 0$. *The facts can be traced in Figure 1 for all* $x, y \in [-4, 4]$. *These bounds on the function* $\dfrac{g(x,y)}{y}$ *are also true as* $x, y \to \infty$.

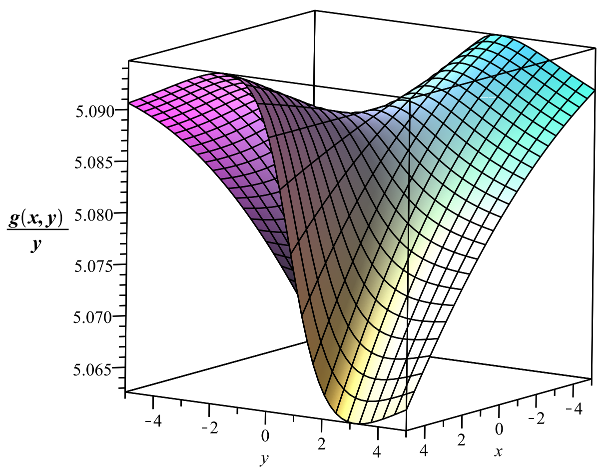

Figure 1. The behaviour of function $\dfrac{g(x,y)}{y}$ for $x, y \in [-4, 4]$

Next, the derivative of the function $g(x,y)$ *with respect to* x *is given by*

$$g_x(x,y) = -\frac{y^2 A(x,y)}{(5 + y^2)(10 + A(x,y))^2} \leq 0$$

for all x *and* y. *See Figure 2 pictorially confirms this inequality. Moreover,*

$$|g_x(x,y)| = \frac{y^2 A(x,y)}{(5 + y^2)(10 + A(x,y))^2} \leq L_1 = 0.25$$

for all x, y. *This inequality is depicted in Figure 3 for all* $x, y \in [-1000, 1000]$. *What is more, the derivative of the function* $g(x,y)$ *with respect to* y *is*

$$g_y(x,y) = 5 + \frac{1}{10 + A(x,y)} - \frac{yA(x,y)B(x,y)}{(10 + A(x,y))^2}.$$

Since

$$\lim_{x\to\infty}\left(\lim_{y\to\infty}|g_y(x,y)|\right) = 5.1 = \lim_{y\to\infty}\left(\lim_{x\to\infty}|g_y(x,y)|\right),$$

it follows that

$$|g_y(x,y)| \leq L_2 = 5.1$$

for all x, y*. This estimate is confirmed in Figure 4 for* $x, y \in [-40, 40]$*. The bound* $L_2 = 5.1$ *holds true for smaller or larger value of* x *and* y*.*

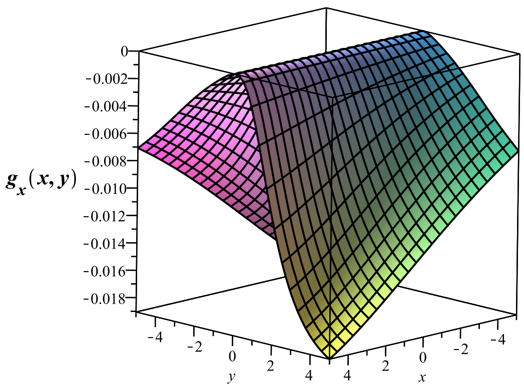

Figure 2. The function $g_x(x,y)$ for all $x, y \in [-4, 4]$.

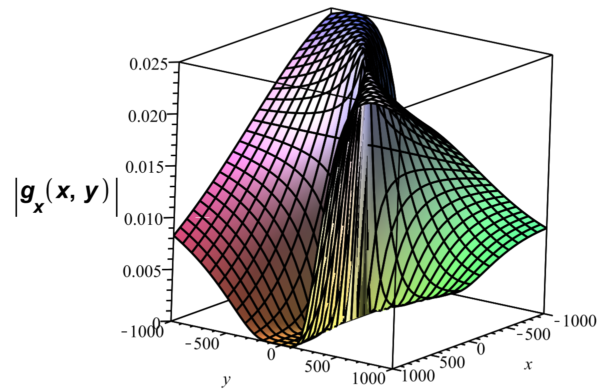

Figure 3. The behaviour of $|g_x(x,y)|$ for all $x, y \in [-1000, 1000]$

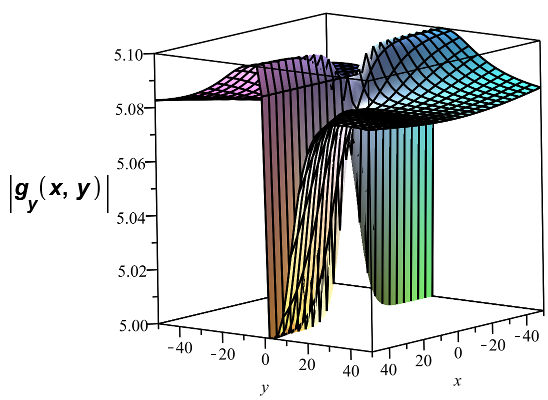

Figure 4. The behaviour of $|g_y(x,y)|$ for $x, y \in [-40, 40]$.

(vi) The function

$$\mu(t) := \frac{9 + 2t^2}{4 + t^2} = 2 + \frac{1}{4 + t^2},$$

since $\dfrac{1}{4 + t^2} \leq 0.25$ for all $t \geq 0$, it follows that $2 = \mu_0 \leq \mu(t) \leq \mu_1 = 2.25$. Also, the derivative of μ with respect to t is defined as

$$\mu'(t) = \frac{-2t}{(4 + t^2)^2} \leq 0$$

for all $t \geq 0$. Since $\lim\limits_{t \to \infty} \mu'(t) = 0$, then the maximum value of $\mu'(t)$ for $t > 0$ is attained when $t = 1$, thus $-0.08 = -\sigma_0 \leq \mu'(t) \leq 0$ for all $t \geq 0$.

(vii) The function

$$h(x) := \frac{31x + 9x^3}{10 + 3x^2} = 3x + \frac{x}{10 + 3x^2}.$$

Clearly $h(0) = 0$ and $\frac{h(x)}{x} = 3 + \frac{1}{10 + 3x^2}$, since $\frac{1}{10 + 3x^2} \le 0.1$ for all x, it follows that $3 = h_0 \le \frac{h(x)}{x} \le h_1 = 3.1$ for all $x \neq 0$. Also

$$h'(x) = 3 + \frac{1}{10 + 3x^2} - \frac{6x^2}{(10 + 3x^2)^2}.$$

Since $\frac{6x^2}{(10 + 3x^2)^2} \ge 0$ for all x then $h'(x) \le d_1 = 3.1$ for all x, and $|h'(x)| \le L_0 = 3.1$ for all x.

(viii) The function

$$\Phi(z) := \frac{z}{10 + e^{2z}}.$$

Noting that $\frac{1}{10 + e^{2z}} \le 0.1$ for all z, it follows that $|\Phi(z)| \le \eta|z|$ so that $\eta = 0.1$.

(ix) Function

$$q(t) := \frac{1}{100 + \cos t}.$$

It is not difficult to show that $|q(t)| \le q_1 = 0.01$ for all $t \ge 0$.

Remark 4.2 *The following assumptions hold for the trivial solution of (4.2) to be asymptotically stable:*

(i) $\begin{cases} 2 = \mu_0 \le \mu(t) \le \mu_1 = 2.25,\ 20 = \varphi_0 \le \varphi(t) \le \varphi_1 = 20.09,\ 28 = \psi_0 \le \psi(t) \le \psi_1 = 28.1, \\ 2 = r_0 \le r(t) \le r_1 = 2.1,\ -0.08 = -\sigma_0 \le \mu'(t) \le 0,\ -0.27 = -\sigma_1 \le \varphi'(t) \le 0 \ \textit{for all}\ t \ge 0; \end{cases}$

(ii) $10 = b_0 \le f(x) \le b_1 = 10.1,\ |\Phi(z)| \le \eta|z|,\ \eta = 0.1,\ |q(t)| \le q_1 = 0.01;$

(iii) $h(0) = 0, 3 = h_0 \le \frac{h(x)}{x} \le h_1 = 3.1$ *for* $x \neq 0,\ h'(x) \le d_1 = 3.1,\ |h'(x)| \le L_0 = 3.1;$

(iv) $5 = c_0 \le \frac{g(x,y)}{y} \le c_1 = 5.089,\ g_x(x,y) \le 0, |g_x(x,y)| \le L_1 = 0.25, |g_y(x,y)| \le L_2 = 5.1;$

(v) $\max\{6.51, 1.46, 0.0005\} < a < \min\{200.202, 143, 7, 50\} \ \Rightarrow 6.51 < a < 7$ *and* $a = 6.51001$ *is chosen;*

(vi) Estimates (4.3), (4.4), and (4.5) produce $\int_{t_0}^{t} [|r'(s)| + |\psi'(s)| + |F(s)|]ds \le \beta_1 = 0.07;$

(vii) $ah_0\mu_0 - h_1 q_1 r_1 \eta = 39.057 > 0,\ ac_0\psi_0 - d_1 r_1 \mu_1 - A_1 = 571.219 > 0,\ (b_0\varphi_0 - a)r_0 - A_2 = 60.732 > 0;$

(viii) estimate (2.5) becomes

$$\tau_1 < \frac{1}{2}\min\{0.039,\ 0.077,\ 0.024\} = 0.012.$$

Let $\tau_1 = 0.01199$

The assumptions of Theorem 2.3 hold, thus by Theorem 2.3 the trivial solution of (4.2) is asymptotically stable.

Remark 4.3 *The assumption that* $\delta_9 - \delta_8[|r'(t)| + |\psi'(t)| + |F(t)|] = 515.699 > 0$ *hold true, then the trivial solution of* (4.2) *is uniformly asymptotically stable.*

Example 4.4 *Consider the following third order neutral differential equation with delay defined as*

$$\begin{aligned}&\left[\frac{2t+21}{10+t}\Big(x'' + (100+\cos t)^{-1}\Phi(x''(t-\tau_0))\Big)\right]' + \left(\frac{221+60t}{11+3t}\right)\left(\frac{101+20x^2}{10+2x^2}\right)x'' \\ &+ \left(\frac{281+56t}{10+2t}\right)\left[\frac{51x'(t-\tau_1) + 5x'(t-\tau_1)A(x(t-\tau_1), x'(t-\tau_1))}{10 + A(x(t-\tau_1), x'(t-\tau_1))}\right] \\ &+ \left(\frac{9+2t^2}{4+t^2}\right)\left[\frac{31x'(t-\tau_1) + 9x^3(t-\tau_1)}{10+3x^2(t-\tau_1)}\right] = 5 + \frac{1}{2}\sin 2t,\end{aligned} \tag{4.6}$$

Equation (4.6) *as system of first order*

$$\begin{aligned}&x' = y, \quad y' = z \\ &\left(\frac{21+2t}{10+t}\right)Z' = -\frac{Z}{(10+t)^2} - \left(\frac{221+60t}{11+3t}\right)\left(\frac{101+20x^2}{10+2x^2}\right)z - \left(\frac{9+2t^2}{4+t^2}\right)\left[\frac{31x+9x^3}{10+3x^2}\right] \\ &- \left(\frac{281+56t}{10+2t}\right)\left[\frac{51y+5yA(x,y)}{10+A(x,y)}\right] + \left(\frac{9+2t^2}{4+t^2}\right)\int_{t-\tau_1}^{t}\left[3 + \frac{10-3x^2(s)}{(10+3x^2(s))^2}\right]y(s)ds \\ &- \left(\frac{281+56t}{10+2t}\right)\int_{t-\tau_1}^{t}\left[\frac{y^2(s)A(x(s),y(s))}{[10+A(x(s),y(s))]^2(5+y^2(s))}\right]y(s)ds + 5 + \frac{1}{2}\sin 2t \\ &+ \left(\frac{281+56t}{10+2t}\right)\int_{t-\tau_1}^{t}\left[5 + \frac{1}{10+A(x(s),y(s))} - \frac{y(s)A(x(s),y(s))B(x(s),y(s))}{[10+A(x(s),y(s))]^2}\right]z(s)ds,\end{aligned} \tag{4.7}$$

Next, comparing equations (1.2) *and* (4.7) *the function below results*

$$p(t) := 5 + \frac{1}{2}\sin 2t.$$

Since $\frac{1}{2}\sin 2t \le 0.5$ *for all* $t \ge 0$, *then* $|p(t)| \le 5.5$ *for all* $t \ge 0$, *the following result is produced.*

Remark 4.5 *In addition to assumptions (i) to (viii) of Remark 4.2, suppose*

(ix) $|p(t)| \le P_1 = 5.5 < \infty$ *for all* $t \ge 0$,

then the conclusions of Theorems 3.1, 3.2, and 3.3 follow immediately for equation (4.7).

5. Conclusion

This paper discussed stability, uniform stability, the existence of a unique periodic solution, boundedness and uniform ultimate boundedness of solutions to a novel class of third order nonlinear nonautonomous neutral functional differential equations with delay. The direct method of Lyapunov is adopted, by constructing a complete Lyapunov-Krasovskii functional to obtain stability and boundedness results. The behaviour of solutions of equation (1.1), if the constant delays are replaced with variable delay and the function f is defined in more general space, is still unresolved.

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