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Continuous wavelet transform on Triebel-Lizorkin spaces

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Abstract: The continuous wavelet transform in higher dimensions is used to prove the regularity of weak solutions $u \in L^p(\mathbb{R}^n)$ under $Qu = f$ where f belongs to the Triebel-Lizorkin space $F_p^{r,q}(\mathbb{R}^n)$ where $1 < p, q < \infty$, $0 < r < 1$, and where $Q = \sum_{|\beta| \leq m} c_\beta \partial^\beta$ is a linear partial differential operator of order $m > 0$ with positive constant coefficients c_β.

Key words: Admissible function, continuous wavelet transform, Triebel-Lizorkin spaces, weak solution, regularity, differential operators.

1. Introduction

According to [8], the continuous wavelet transform (CWT) for functions f in $L^2(\mathbb{R}^n)$ with respect to a function h in $L^2(\mathbb{R}^n)$ satisfying the admissibility condition

$$0 < C_h := \int_0^\infty |\eta(k)|^2 \frac{dk}{k} < \infty,$$

where $\widehat{h}(x) = \eta(|x|)$, is given by

$$(L_h f)(a,b) = \int_{\mathbb{R}^n} f(x) \overline{T_b J_a h(x)} dx = \int_{\mathbb{R}^n} f(x) \frac{1}{a^{\frac{n}{2}}} \overline{h\left(\frac{x-b}{a}\right)} dx,$$

where $(J_a h)(x) = \frac{1}{a^{\frac{n}{2}}} h(\frac{x}{a})$, $a > 0$ is the dilation operator and where $(T_b h)(x) = h(x-b)$, $x, b \in \mathbb{R}^n$ is the translation operator.

Moreover, the inversion (reconstruction) formula is given by

$$f = \frac{1}{C_h} \int_0^\infty \int_{\mathbb{R}^n} (L_h f)(a,b) T_b J_a h \frac{dbda}{a^{n+1}}, \tag{1.1}$$

where the convergence is in the weak sense. It was proved in [4] that with appropriate assumptions on $f \in L^2(\mathbb{R})$ that (1.1) holds in the pointwise sense.

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On the other hand, when we extend the CWT for functions in $L^p(\mathbb{R}^n)$, where $1 < p < \infty$, different types of convergence have been studied for the CWT in $L^p(\mathbb{R}^n)$, see [1, 5–7, 10], and [11]. For example, in [6] the dilation operator is defined so that it satisfies the L^1-normalization and the convergence for the inversion formula holds in the L^p sense, whereas the L^2-normalization for the dilation operator is given in [2].

The regularity of functions has been studied through the characterization of these functions in different spaces by means of the CWT. For instance, in [6], the CWT is used to determine if a function belongs to the Besov spaces in one dimension.

In this paper, we follow the notations given in [1], [10], and [2] with L^2-normalization for the dilation operator to define the continuous wavelet transform for functions in $L^p(\mathbb{R}^n)$, and we apply this transform to analyze the regularity of weak solutions $u \in L^p(\mathbb{R}^n)$ under the equation $Qu = f$ where f belongs to the Triebel-Lizorkin space $F_p^{r,q}(\mathbb{R}^n)$, with $1 < p, q < \infty$ and $0 < r < 1$, and where $Q = \sum_{|\beta| \leq m} c_\beta \partial^\beta$ is a linear partial differential operator of order m with positive coefficients c_β.

Next, we formalize these concepts for functions in $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

2. Preliminaries

The following definitions and results will be needed throughout the paper.

Definition 2.1 *For h in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, the dilation operator J_a and the translation operator T_b are defined respectively as:*

1) $(J_a h)(x) = a^{-\frac{n}{2}} h(a^{-1}x)$, where $a > 0$ and $x \in \mathbb{R}^n$.

2) $(T_b h)(x) = h(x - b)$, where $x, b \in \mathbb{R}^n$.

Definition 2.2 *A function h in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is said to be admissible if*

$$0 < C_h := \int_{\mathbb{R}^n} \left|\widehat{h}(k)\right|^2 \frac{1}{|k|^n} \, dk < \infty, \tag{2.1}$$

where

$$\widehat{h}(k) = \int_{\mathbb{R}^n} e^{-2\pi i k \cdot x} h(x) \, dx$$

is the Fourier transform of h.

In what follows, $C_0^\infty(\mathbb{R}^n)$ consists of those functions infinitely differentiable on $\mathbb{R}^n$ with compact support.

Lemma 2.3 *Suppose that $h \in C_0^\infty(\mathbb{R}^n)$ and $h \neq 0$. If for any multiindex $\alpha \in \mathbb{R}^n$,*

$$0 < C_{\partial^\alpha h} := \int_{\mathbb{R}^n} \left|\widehat{h}(k)\right|^2 |k|^{2|\alpha| - n} \, dk < \infty,$$

then $\partial^\alpha h$ is admissible.

Proof The proof follows from Definition 2.2 and from the fact that $\widehat{\partial^\alpha h}(k) = (2\pi i k)^\alpha \widehat{h}(k)$. □

Lemma 2.4 *If g, h are admissible and $g+h \neq 0$, then $g+h$ is admissible.*

Proof The proof comes from the fact that for any $1 < p < \infty$, we have $|g+h|^p \leq 2^p(|g|^p + |h|^p)$. □

Given the admissibility condition, we extend the wavelet transform to $L^p(\mathbb{R}^n)$, where $1 < p < \infty$.

Definition 2.5 *Let f be in $L^p(\mathbb{R}^n)$ with $1 < p < \infty$. Consider $a > 0$ and $b \in \mathbb{R}^n$. Let h be an admissible function in $L^1(\mathbb{R}^n)$. The wavelet transform of f with respect to h is defined as*

$$(L_h f)(a,b) = \int_{\mathbb{R}^n} f(x)\overline{T_b J_a h(x)}dx = \int_{\mathbb{R}^n} f(x)\overline{\frac{1}{a^{\frac{n}{2}}} h\left(\frac{x-b}{a}\right)}\,dx. \tag{2.2}$$

Note that the wavelet transform can be written as

$$(L_h f)(a,b) = \left[(J_a\overline{h})^{\sim} * f\right](b), \tag{2.3}$$

where $*$ means convolution and $h^{\sim}$ means $h^{\sim}(x) = h(-x)$.

Remark 2.6 *According to (2.3), and since $J_a h \in L^1(\mathbb{R}^n)$ and $f \in L^p(\mathbb{R}^n)$, it follows from Young's Inequality that $(J_a\overline{h})^{\sim} * f \in L^p(\mathbb{R}^n)$ and $\|(J_a\overline{h})^{\sim} * f\|_p \leq a^{\frac{n}{2}}\|J_a h\|_1\|f\|_p$. That is,*

$$\|(L_h f)(a,\cdot)\|_p \leq a^{\frac{n}{2}}\|h\|_1\|f\|_p. \tag{2.4}$$

In order to obtain a reconstruction formulae for the wavelet transform, we need the following result.

Lemma 2.7 *Suppose $h \in L^1(\mathbb{R}^n)$ is admissible. If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ where $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\int_{\mathbb{R}^n} f(x)\overline{g(x)}dx = \frac{1}{C_h}\int_0^{\infty}\int_{\mathbb{R}^n}(L_h f)(a,b)\overline{(L_h g)(a,b)}db\frac{da}{a^{n+1}}.$$

The integrals of the right hand side have to be taken in the sense of distributions.

Proof See [6]. □

Lemma 2.8 *Consider $f \in L^p(\mathbb{R}^n)$ with $1 < p < \infty$, and $h \in L^1(\mathbb{R}^n)$ admissible. Then*

$$f(x) = \frac{1}{C_h}\int_0^{\infty}\int_{\mathbb{R}^n}(L_h f)(a,b)\frac{1}{a^{\frac{n}{2}}}h\left(\frac{x-b}{a}\right)\,db\,\frac{da}{a^{n+1}}. \tag{2.5}$$

The equality holds in the L^p sense, and the integrals on the right-hand side have to be taken in the sense of distributions.

Proof See [6]. □

So now, we proceed with the study of some differential aspects of the admisible functions, the wavelet transform, and differential operators.

Lemma 2.9 *Let h be a differentiable function in an open set Ω in $\mathbb{R}^n$. Then for $b \in \mathbb{R}^n$, $a > 0$ and any multiindex α in $\mathbb{R}^n$,*

$$\partial_x^\alpha \left(h\left(\frac{x-b}{a}\right)\right) = \frac{1}{a^{|\alpha|}} \partial^\alpha h \left(\frac{x-b}{a}\right), \tag{2.6}$$

where $\frac{x-b}{a} \in \Omega$.

Proof The result comes from the chain rule. □

The subsequent result can be obtained by integration by parts. See also [3, Lemma 3.2].

Lemma 2.10 *Let h be in $C_0^\infty(\mathbb{R}^n)$ so that h is admissible, and let α be a multiindex in $\mathbb{R}^n$. Then for f and $\partial^\alpha f$ in $L^p(\mathbb{R}^n), 1 < p < \infty$,*

$$(L_h \partial^\alpha f)(a,b) = \frac{(-1)^{|\alpha|}}{a^{|\alpha|}} (L_{\partial^\alpha h} f)(a,b). \tag{2.7}$$

Corollary 2.11 *Suppose $h \in C_0^\infty(\mathbb{R}^n)$ is such that h and Qh are admissible, where $Q = \sum_{|\alpha|=m} c_\alpha \partial^\alpha$, is a linear operator with positive constant coefficients c_α of order $m \geq 0$. Then for f and Qf in $L^p(\mathbb{R}^n), 1 < p < \infty$,*

$$(L_h Q f)(a,b) \quad = \quad \frac{(-1)^m}{a^m} (L_{Qh} f)(a,b). \tag{2.8}$$

Proof It comes from Lemma 2.10. □

3. The continuous wavelet transform for functions in $F_p^{r,q}(\mathbb{R}^n)$

In this section, we introduce the Triebel-Lizorkin spaces by the first differences [9], and we analyze the boundedness for the continuous wavelet transform on these spaces.

Definition 3.1 *Given f in $L^p(\mathbb{R}^n)$, with $1 < p < \infty$ define the first difference of f by the formula*

$$(\Delta_c f)(x) = f(x+c) - f(x),$$

where $x, c \in \mathbb{R}^n$.

Remark 3.2

$$\|(\Delta_c f)(\cdot)\|_p \to 0 \quad as \quad c \to 0. \tag{3.1}$$

Definition 3.3 *For $1 < p, q < \infty$, $c \in \mathbb{R}^n \setminus \{0\}$ and $0 < r < 1$, the Triebel-Lizorkin space $F_p^{r,q}(\mathbb{R}^n)$ is defined as the space of all functions $f \in L^p(\mathbb{R}^n)$ such that*

$$\left\| \left(\int_{\mathbb{R}^n} |(\Delta_c f)(\cdot)|^q \frac{dc}{|c|^{rq+n}} \right)^{\frac{1}{q}} \right\|_p < \infty.$$

Remark 3.4 *$F_p^{r,q}(\mathbb{R}^n)$ is a Banach space, where*

$$\|f\|_{F_p^{r,q}(\mathbb{R}^n)} = \|f\|_p + \left\| \left(\int_{\mathbb{R}^n} |(\Delta_c f)(\cdot)|^q \frac{dc}{|c|^{rq+n}} \right)^{\frac{1}{q}} \right\|_p < \infty.$$

Remark 3.5 *For simplicity, from now on we set*

$$\left|\left|\left| (\Delta_c f)(\cdot)\right|\right|\right|_{q,r} := \|(\Delta_c f)(\cdot)\|_{L^q(\mathbb{R}^n, \frac{dc}{|c|^{rq+n}})}.$$

That is,

$$\left|\left|\left| (\Delta_c f)(\cdot)\right|\right|\right|_{q,r} = \left(\int_{\mathbb{R}^n} |(\Delta_c f)(\cdot)|^q \frac{dc}{|c|^{rq+n}}\right)^{\frac{1}{q}}. \tag{3.2}$$

Hence,

$$\|f\|_{F_p^{r,q}(\mathbb{R}^n)} = \|f\|_p + \left\|\left|\left|\left| (\Delta_c f)(\cdot)\right|\right|\right|_{q,r}\right\|_p. \tag{3.3}$$

Definition 3.6 *Consider $r > 1$ where r is not an integer. Then r can be written as $r = [r] + t$, where $[r]$ is the integral part of r and $0 < t < 1$. Define the Triebel-Lizorkin space $F_p^{r,q}(\mathbb{R}^n)$ as the space of all function $f \in L^p(\mathbb{R}^n)$ such that for any multiindex $\beta \in \mathbb{R}^n$ with $|\beta| \le [r]$ we have $\partial^\beta f \in F_p^{t,q}(\mathbb{R}^n)$.*

Lemma 3.7 *For $a > 0$, $1 < p, q < \infty$, $0 < r < 1$, and h admissible in $L^1(\mathbb{R}^n)$,*

$$L_h(a, \cdot) : F_p^{r,q}(\mathbb{R}^n) \to F_p^{r,q}(\mathbb{R}^n)$$

is bounded. Moreover,

$$\|(L_h f)(a, \cdot)\|_{F_p^{r,q}(\mathbb{R}^n)} \le a^{\frac{n}{2}} \|h\|_1 \|f\|_{F_p^{r,q}(\mathbb{R}^n)}.$$

Proof From (3.3),

$$\|(L_h f)(a, \cdot)\|_{F_p^{r,q}(\mathbb{R}^n)} = \|(L_h f)(a, \cdot)\|_p + \left\|\left|\left|\left| (\Delta_c L_h f)(a, \cdot)\right|\right|\right|_{q,r}\right\|_p.$$

Firstly, note that by the Minkowski inequality for integrals,

$$\begin{aligned} \left|\left|\left| (\Delta_c L_h f)(a, \cdot)\right|\right|\right|_{q,r} &= \left|\left|\left| \int_{\mathbb{R}^n} (\Delta_c f)(ay + \cdot) a^{\frac{n}{2}} \overline{h(y)}\, dy \right|\right|\right|_{q,r} \\ &\le a^{\frac{n}{2}} \int_{\mathbb{R}^n} \left|\left|\left| (\Delta_c f)(ay + \cdot)\right|\right|\right|_{q,r} |h(y)|\, dy. \end{aligned} \tag{3.4}$$

That is,

$$\left|\left|\left| (\Delta_c L_h f)(a, \cdot)\right|\right|\right|_{q,r} \le a^{\frac{n}{2}} \int_{\mathbb{R}^n} \left|\left|\left| (\Delta_c f)(ay + \cdot)\right|\right|\right|_{q,r} |h(y)|\, dy. \tag{3.5}$$

Again by the Minkowski inequality for integrals,

$$\begin{aligned} \left\|\left|\left|\left| (\Delta_c L_h f)(a, \cdot)\right|\right|\right|_{q,r}\right\|_p &\le \left\| a^{\frac{n}{2}} \int_{\mathbb{R}^n} \left|\left|\left| (\Delta_c f)(ay + \cdot)\right|\right|\right|_{q,r} |h(y)|\, dy \right\|_p \\ &\le a^{\frac{n}{2}} \int_{\mathbb{R}^n} \left\|\left|\left|\left| (\Delta_c f)(\cdot)\right|\right|\right|_{q,r}\right\|_p |h(y)|\, dy \\ &= a^{\frac{n}{2}} \|h\|_1 \left\|\left|\left|\left| (\Delta_c f)(\cdot)\right|\right|\right|_{q,r}\right\|_p. \end{aligned} \tag{3.6}$$

That is,

$$\left\| \left|\left|\left| (\Delta_c L_h f)(a,\cdot) \right|\right|\right|_{q,r} \right\|_p \leq a^{\frac{n}{2}} \|h\|_1 \left\| \left|\left|\left| (\Delta_c f)(\cdot) \right|\right|\right|_{q,r} \right\|_p . \tag{3.7}$$

Thus, the proof comes from (2.4) and (3.7). □

Next result is used in the proof of the Theorem 4.1.

Corollary 3.8 *Let $a > 0$, $1 < p, q < \infty$, $0 < r < 1$. If $f \in F_p^{r,q}(\mathbb{R}^n)$, h is admissible in $L^1(\mathbb{R}^n)$ and $\left\| \left|\left|\left| (\Delta_c f)(\cdot) \right|\right|\right|_{q,r} \right\|_p = \mathcal{O}(a^r)$, then*

$$\left\| \left|\left|\left| (\Delta_c L_h f)(a,\cdot) \right|\right|\right|_{q,r} \right\|_p = \mathcal{O}(a^{\frac{n}{2}+r}). \tag{3.8}$$

Proof The proof comes from (3.7). □

Lemma 3.9 *Suppose $1 < p, q < \infty$, and $0 < r < 1$. Suppose also that h is admissible in $L^1(\mathbb{R}^n)$ such that $h \in F_1^{r,q}(\mathbb{R}^n)$. If $f \in L^p(\mathbb{R}^n)$ and $\left\| \left|\left|\left| (L_h \Delta_c f)(a,\cdot) \right|\right|\right|_{q,r} \right\|_p = \mathcal{O}(a^{\frac{n}{2}+r})$ as $a \to 0$, then $f \in F_p^{r,q}(\mathbb{R}^n)$.*

Proof See appendix A. □

4. Main theorem

Now we are able to state the following result.

Theorem 4.1 *Suppose $1 < p, q < \infty$, $0 < r < 1$, and let $Q = \sum_{|\beta| \leq m} c_\beta \partial^\beta$ be a partial differential operator of order m with positive coefficients c_β. Let $T = \sum_{|\beta| < m} c_\beta \partial^\beta$ and define $Q_m := \sum_{|\beta| = m} c_\beta \partial^\beta$. Suppose also h is admissible in $C_0^\infty(\mathbb{R}^n)$ so that $Q^* h \in F_1^{r,q}(\mathbb{R}^n)$, and $f \in L^p(\mathbb{R}^n)$ with $\left\| \left|\left|\left| (\Delta_c f)(\cdot) \right|\right|\right|_{q,r} \right\|_p = \mathcal{O}(a^r)$ as $a \to 0$. If $u \in W^{m,p}(\mathbb{R}^n)$ is a weak solution of $Qu = f$ and $f \in F_p^{r,q}(\mathbb{R}^n)$, then $u \in F_p^{m+r,q}(\mathbb{R}^n)$.*

Proof

1) We prove first that $u \in F_p^{|\gamma|+r,q}(\mathbb{R}^n)$ for each multiindex $\gamma \in \mathbb{R}^n$ with $|\gamma| < m$.

Since $\left\| \left|\left|\left| (\Delta_c f)(\cdot) \right|\right|\right|_{q,r} \right\|_p = \mathcal{O}(a^r)$, then from (3.8),

$$\left\| \left|\left|\left| (\Delta_c L_h f)(a,\cdot) \right|\right|\right|_{q,r} \right\|_p = \mathcal{O}(a^{\frac{n}{2}+r}).$$

On the other hand, since u is a weak solution of $Qu = f$, it follows that for $T_b J_a h \in C_0^\infty(\mathbb{R}^n)$, we have from (2.8), that $(L_h f)(a,b) = \frac{(-1)^{|\beta|}}{a^{|\beta|}} (L_{Qh} u)(a,b)$. Hence, for $|\gamma| < |\beta| \leq m$ and for $S := \sum_{|\beta| \leq m} c_\beta \partial^{\beta-\gamma}$ we have that for $a \to 0$,

$$\left\| \left|\left|\left| (\Delta_c L_{Sh} \partial^\gamma u)(a,\cdot) \right|\right|\right|_{q,r} \right\|_p = \mathcal{O}(a^{\frac{n}{2}+|\beta-\gamma|+r}) = \mathcal{O}(a^{\frac{n}{2}+r}).$$

Then by Lemma 3.9, we conclude that $\partial^\gamma u \in F_p^{r,q}(\mathbb{R}^n)$. This means that for each $|\gamma| < m$, we have $u \in F_p^{|\gamma|+r,q}(\mathbb{R}^n)$.

2) Now we prove that $u \in F_p^{|\gamma|+r,q}(\mathbb{R}^n)$ for each multiindex $\gamma \in \mathbb{R}^n$ with $|\gamma| = m$.

For this purpose, we use induction over the number of terms in Q_m. Suppose then that the cardinality of Q_m is s.

i) Suppose first that $Q_m = c_{(m,0,0,\ldots,0)}\partial_1^m$. Since $Qu = Tu + Q_m u$, then $c_{(m,0,0,\ldots,0)}\partial_1^m u = f - Tu$.

Since $f \in F_p^{r,q}(\mathbb{R}^n)$, and since for $|\beta| < m$ we have from part 1) that $\partial^\beta f \in F_p^{r,q}(\mathbb{R}^n)$, then from (3.8),

$$\begin{aligned}
\left\| \left|\left|\left| (\Delta_c L_h \partial_1^m u)(a,\cdot) \right|\right|\right|_{q,r} \right\|_p &\leq \frac{1}{c_{(m,0,0,\ldots,0)}} \left\| \left|\left|\left| (\Delta_c L_h f)(a,\cdot) \right|\right|\right|_{q,r} \right\|_p \\
&+ \frac{1}{c_{(m,0,0,\ldots,0)}} \sum_{|\beta|<m} c_\beta \left\| \left|\left|\left| (\Delta_c L_h \partial^\beta u)(a,\cdot) \right|\right|\right|_{q,r} \right\|_p \\
&\leq \frac{1}{c_{(m,0,0,\ldots,0)}} \left(\mathcal{O}(a^{\frac{n}{2}+r}) + \sum_{|\beta|<m} c_\beta \mathcal{O}(a^{\frac{n}{2}+r}) \right) \\
&= \mathcal{O}(a^{\frac{n}{2}+r}).
\end{aligned}$$

Then by Lemma 3.9,

$$\partial_1^m u \in F_p^{r,q}(\mathbb{R}^n). \tag{4.1}$$

ii) Suppose now that $Q_m = c_{(m,0,0,\ldots,0)}\partial_1^m + c_{(m-1,1,0,\ldots,0)}\partial_1^{m-1}\partial_2$. Then

$$c_{(m-1,1,0,\ldots,0)}\partial_1^{m-1}\partial_2 u = f - \sum_{|\beta|<m} c_\beta \partial^\beta - c_{(m,0,0,\ldots,0)}\partial_1^m u.$$

Since $f \in F_p^{r,q}(\mathbb{R}^n)$, from part 1) for $|\beta| < m$, $\partial^\beta f \in F_p^{r,q}(\mathbb{R}^n)$, and from (4.1), $c_{(m,0,0,\ldots,0)}\partial_1^m u \in F_p^{r,q}(\mathbb{R}^n)$, then from (3.8),

$$\begin{aligned}
&\left\| \left|\left|\left| (\Delta_c L_h \partial_1^{m-1}\partial_2 u)(a,\cdot) \right|\right|\right|_{q,r} \right\|_p \\
&\leq \frac{1}{c_{(m-1,1,0,\ldots,0)}} \left\| \left|\left|\left| (\Delta_c L_h f)(a,\cdot) \right|\right|\right|_{q,r} \right\|_p \\
&\quad + \frac{1}{c_{(m-1,1,0,\ldots,)}} \sum_{|\beta|<m} c_\beta \left\| \left|\left|\left| (\Delta_c L_h \partial^\beta u)(a,\cdot) \right|\right|\right|_{q,r} \right\|_p \\
&\quad + \frac{1}{c_{(m-1,1,0,\ldots,0)}} \left\| \left|\left|\left| (\Delta_c L_h c_{(m,0,0,\ldots,0)}\partial_1^m u)(a,\cdot) \right|\right|\right|_{q,r} \right\|_p \\
&= \mathcal{O}(a^{\frac{n}{2}+r}).
\end{aligned}$$

Then by Lemma 3.9,

$$\partial_1^{m-1}\partial_2 u \in F_p^{r,q}(\mathbb{R}^n). \tag{4.2}$$

Repeating this process s-times, we get that for any $\gamma \in \mathbb{R}^n$ with $|\gamma| = m$ we have $\partial^\gamma u \in F_p^{r,q}(\mathbb{R}^n)$. Hence, from 1) and 2), $u \in F_p^{m+r,q}(\mathbb{R}^n)$. This proves Theorem 4.1. □

5. Futher results

Lemma 5.1 *Suppose $f \in F_p^{r,q}(\mathbb{R}^n)$ where $1 < p, q < \infty$ and $0 < r < 1$. Suppose also h is admissible in $L^1(\mathbb{R}^n)$. If $\widehat{h}(0) = 1$, then*

$$\lim_{a\to 0} \frac{1}{a^{\frac{n}{2}}}(L_h f)(a,\cdot) = f(\cdot) \quad in \quad F_p^{r,q}(\mathbb{R}^n).$$

Proof Note that from (3.3),

$$\left\| \frac{1}{a^{\frac{n}{2}}}(L_h f)(a,\cdot) - f(\cdot)\right\|_{F_p^{r,q}(\mathbb{R}^n)}$$
$$= \left\| \frac{1}{a^{\frac{n}{2}}}(L_h f)(a,\cdot) - f(\cdot)\right\|_p + \left\| \left\| \left\| \Delta_c \left(\frac{1}{a^{\frac{n}{2}}}(L_h f)(a,\cdot) - f(\cdot)\right)\right\|\right\|_{q,r}\right\|_p .$$

On the one hand, since $\widehat{h}(0) = 1$, then

$$\frac{1}{a^{\frac{n}{2}}}(L_h f)(a,b) - f(b) = \left[\int_{\mathbb{R}^n} f(ay+b)\overline{h(y)}\,dy\right] - f(b)\int_{\mathbb{R}^n} \overline{h(y)}\,dy$$
$$= \int_{\mathbb{R}^n} [f(ay+b) - f(b)]\overline{h(y)}\,dy = \int_{\mathbb{R}^n} [(T_{-ay}f)(b) - f(b)]\overline{h(y)}\,dy.$$

Hence, by the Minkowski inequality for integrals,

$$\begin{aligned}\left\| \frac{1}{a}(L_h f)(a,\cdot) - f(\cdot)\right\|_p &= \left\| \int_{\mathbb{R}^n} [(T_{-ay}f)(\cdot) - f(\cdot)]\overline{h(y)}\,dy\right\|_p \\ &\le \int_{\mathbb{R}^n} \|(T_{-ay}f)(\cdot) - f(\cdot)\|_p |h(y)|\,dy.\end{aligned}$$

That is,

$$\left\| \frac{1}{a^{\frac{n}{2}}}(L_h f)(a,\cdot) - f(\cdot)\right\|_p \le \int_{\mathbb{R}^n} \|(T_{-ay}f)(\cdot) - f(\cdot)\|_p |h(y)|\,dy. \tag{5.1}$$

Note that since $h \in L^1(\mathbb{R}^n)$ and $\|(T_{-ay}f)(\cdot) - f(\cdot)\|_p$ is bounded by $2\|f\|_p$ and from (3.1) tends to zero as $a \to 0$ for each $y \in \mathbb{R}^n$, then by the dominated convergence theorem

$$\lim_{a\to 0} \|(T_{-ay}f)(\cdot) - f(\cdot)\|_p = 0.$$

Hence,

$$\lim_{a\to 0} \left\| \frac{1}{a^{\frac{n}{2}}}(L_h f)(a,\cdot) - f(\cdot)\right\|_p = 0. \tag{5.2}$$

On the other hand, by (5.1) and the Minkowski inequality for integrals,

$$\left|\left|\left| \Delta_c\Big(\frac{1}{a^{\frac{n}{2}}}(L_h f)(a,\cdot) - f(\cdot)\Big) \right|\right|\right|_{q,r} = \left|\left|\left| \frac{1}{a^{\frac{n}{2}}}(L_h \Delta_c f)(a,\cdot) - \Delta_c f(\cdot) \right|\right|\right|_{q,r}$$

$$\leq \left|\left|\left| \int_{\mathbb{R}^n} |(T_{-ay}\Delta_c f)(\cdot) - (\Delta_c f)(\cdot)|\, |h(y)|\, dy \right|\right|\right|_{q,r}$$

$$\leq \int_{\mathbb{R}^n} |||\, (T_{-ay}\Delta_c f)(\cdot) - (\Delta_c f)(\cdot)\, |||_{q,r}\, |h(y)|\, dy.$$

Again by the Minkowski inequality for integrals,

$$\left\| \left|\left|\left| \Delta_c \left(\frac{1}{a^{\frac{n}{2}}}(L_h f)(a,\cdot) - f(\cdot)\right) \right|\right|\right|_{q,r} \right\|_p$$

$$\leq \int_{\mathbb{R}^n} \left\| |||\, (T_{-ay}\Delta_c f)(\cdot) - (\Delta_c f)(\cdot)\, |||_{q,r} \right\|_p |h(y)|\, dy.$$

Note that since $h \in L^1(\mathbb{R}^n)$ and $\left\| |||\, (\Delta_c f)(\cdot)\, |||_{q,r} \right\|_p < \infty$, then

$$\left\| |||\, (T_{-ay}\Delta_c f)(\cdot) - (\Delta_c f)(\cdot)\, |||_{q,r} \right\|_p |h(y)|$$

$$\leq 2 \left\| |||\, (\Delta_c f)(\cdot)\, |||_{q,r} \right\|_p |h(y)| \in L^1(\mathbb{R}^n).$$

Hence,

$$\left\| |||\, (T_{-ay}\Delta_c f)(\cdot) - (\Delta_c f)(\cdot)\, |||_{q,r} \right\|_p \to 0 \quad as \quad a \to 0.$$

That is,

$$\lim_{a\to 0} \left\| \left|\left|\left| \Delta_c \left(\frac{1}{a^{\frac{n}{2}}}(L_h f)(a,\cdot) - f(\cdot)\right) \right|\right|\right|_{q,r} \right\|_p = 0. \tag{5.3}$$

Therefore, the proof comes from (5.2) and (5.3). □

As a consequence of Lemma 3.7, we have the following result.

Corollary 5.2 *If $f_1, f_2 \in F_p^{r,q}(\mathbb{R}^n)$ and h_1, h_2 are admissible in $L^1(\mathbb{R}^n)$, then*

$$\|(L_{h_1} f_1)(a,\cdot) - (L_{h_2} f_2)(a,\cdot)\|_{F_p^{r,q}(\mathbb{R}^n)}$$

$$\leq \quad a^{\frac{n}{2}} \|h_1 - h_2\|_1 \|f_1\|_{F_p^{r,q}(\mathbb{R}^n)} + a^{\frac{n}{2}} \|h_2\|_1 \|f_1 - f_2\|_{F_p^{r,q}(\mathbb{R}^n)}.$$

A. Proof of Lemma 3.9

Proof From the reconstruction formula given in (2.5) for $f \in L^p(\mathbb{R}^n)$, $f(x) = I_1(x) + I_2(x)$, where

$$I_1(x) = \frac{1}{C_h} \int_0^1 \int_{\mathbb{R}^n} (L_h f)(a,b) \frac{1}{a^{\frac{n}{2}}} h\left(\frac{x-b}{a}\right) db \frac{da}{a^{n+1}},$$

and

$$I_2(x) = \frac{1}{C_h}\int_1^\infty \int_{\mathbb{R}^n} (L_h f)(a,b)\frac{1}{a^{\frac{n}{2}}} h\left(\frac{x-b}{a}\right) db \frac{da}{a^{n+1}},$$

1) We prove first that $I_1 \in F_p^{r,q}(\mathbb{R}^n)$. If $y = \frac{x-b}{a}$, then I_1 can be written as

$$I_1(x) = \frac{1}{C_h}\int_0^1 \int_{\mathbb{R}^n} (L_h f)(a, x-ay)h(y)dy\frac{da}{a^{\frac{n}{2}+1}}.$$

Then for $c \in \mathbb{R}^n \setminus \{0\}$,

$$(\Delta_c I_1)(x) = \frac{1}{C_h}\int_0^1 \int_{\mathbb{R}^n} (L_h \Delta_c f)(a, x-ay)h(y)dy\frac{da}{a^{\frac{n}{2}+1}}.$$

Hence, by the Minkowski inequality for integrals,

$$\begin{aligned} \big|\big|\big|(\Delta_c I_1)(\cdot)\big|\big|\big|_{q,r} &= \big|\big|\big| \frac{1}{C_h}\int_0^1 \int_{\mathbb{R}^n} (L_h \Delta_c f)(a, \cdot - ay)h(y)dy\frac{da}{a^{\frac{n}{2}+1}} \big|\big|\big|_{q,r} \\ &\leq \frac{1}{C_h}\int_0^1 \int_{\mathbb{R}^n} \big|\big|\big| (L_h \Delta_c f)(a, \cdot - ay) \big|\big|\big|_{q,r} |h(y)|\, dy\frac{da}{a^{\frac{n}{2}+1}}. \end{aligned} \tag{A.1}$$

Again, by the Minkowski inequality for integrals,

$$\begin{aligned} &\Big\|\big|\big|\big| (\Delta_c I_1)(\cdot) \big|\big|\big|_{q,r} \Big\|_p \\ &\leq \left\| \frac{1}{C_h}\int_0^1 \int_{\mathbb{R}^n} \big|\big|\big| (L_h \Delta_c f)(a, \cdot - ay) \big|\big|\big|_{q,r} |h(y)|\, dy\frac{da}{a^{\frac{n}{2}+1}} \right\|_p \\ &\leq \frac{1}{C_h}\int_0^1 \int_{\mathbb{R}^n} \Big\|\big|\big|\big| (L_h \Delta_c f)(a, \cdot) \big|\big|\big|_{q,r} \Big\|_p |h(y)|\, dy\frac{da}{a^{\frac{n}{2}+1}} \\ &\leq \frac{1}{C_h}\int_0^1 \int_{\mathbb{R}^n} \mathcal{K} a^{\frac{n}{2}+r} |h(y)|\, dy\frac{da}{a^{\frac{n}{2}+1}} \\ &= \frac{1}{C_h}\mathcal{K}\|h\|_1 \left(\int_0^1 a^{r-1}\, da\right) = \frac{1}{C_h}\mathcal{K}\|h\|_1 \frac{1}{r} < \infty, \end{aligned} \tag{A.2}$$

where $\mathcal{K}$ is a positive constant. Thus, $I_1 \in F_p^{r,q}(\mathbb{R}^n)$.

2) Next, we prove that $I_2 \in F_p^{r,q}(\mathbb{R}^n)$.

Note that since $(\Delta_c I_2)(x) = I_2(x+c) - I_2(x)$, then

$$\begin{aligned} (\Delta_c I_2)(x) &= \frac{1}{C_h}\int_1^\infty \int_{\mathbb{R}^n} (L_h f)(a,b) \left(\Delta_{\frac{c}{a}} h\right)\left(\frac{x-b}{a}\right)\, db\frac{da}{a^{\frac{n}{2}+n+1}} \\ &= \frac{1}{C_h}\int_1^\infty \int_{\mathbb{R}^n} (L_h f)(a, x-ay) \left(\Delta_{\frac{c}{a}} h\right)(y)\, dy\frac{da}{a^{\frac{n}{2}+1}}. \end{aligned} \tag{A.3}$$

Hence, by the Minkowski inequality for integrals,

$$\begin{aligned}
\left|\left|\left| (\Delta_c I_2)(\cdot) \right|\right|\right|_{q,r} &= \left(\int_{\mathbb{R}^n} |(\Delta_c I_2)(\cdot)|^q \frac{dc}{|c|^{rq+n}} \right)^{\frac{1}{q}} \\
&\leq \left(\int_{\mathbb{R}^n} \left(\frac{1}{C_h} \int_1^\infty \int_{\mathbb{R}^n} |(L_h f)(a, \cdot - ay)| \, |(\Delta_{\frac{c}{a}} h)(y)| \; dy \frac{da}{a^{\frac{n}{2}+1}} \right)^q \frac{dc}{|c|^{rq+n}} \right)^{\frac{1}{q}} \\
&\leq \frac{1}{C_h} \int_1^\infty \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |(L_h f)(a, \cdot - ay)|^q \, |(\Delta_{\frac{c}{a}} h)(y)|^q \frac{dc}{|c|^{rq+n}} \right)^{\frac{1}{q}} dy \frac{da}{a^{\frac{n}{2}+1}} \\
&= \frac{1}{C_h} \int_1^\infty \int_{\mathbb{R}^n} |(L_h f)(a, \cdot - ay)| \left(\int_{\mathbb{R}^n} |(\Delta_{\frac{c}{a}} h)(y)|^q \frac{dc}{|c|^{rq+n}} \right)^{\frac{1}{q}} dy \frac{da}{a^{\frac{n}{2}+1}}. \qquad \text{(A.4)}
\end{aligned}$$

Let $z = \frac{c}{a}$, then

$$\begin{aligned}
&\left|\left|\left| (\Delta_c I_2)(\cdot) \right|\right|\right|_{q,r} \\
&\leq \frac{1}{C_h} \int_1^\infty \int_{\mathbb{R}^n} |(L_h f)(a, \cdot - ay)| \left(\int_{\mathbb{R}^n} |(\Delta_z h)(y)|^q \frac{dz}{|z|^{rq+n}} \right)^{\frac{1}{q}} dy \frac{da}{a^{\frac{n}{2}+1+r}} \\
&= \frac{1}{C_h} \int_1^\infty \int_{\mathbb{R}^n} |(L_h f)(a, \cdot - ay)| \left|\left|\left|(\Delta_z h)(y)\right|\right|\right|_{q,r} dy \frac{da}{a^{\frac{n}{2}+1+r}}. \qquad \text{(A.5)}
\end{aligned}$$

Now, from (2.4), from hypothesis and again by the Minkowski inequality for integrals,

$$\begin{aligned}
\left\| \left|\left|\left| (\Delta_c I_2)(\cdot) \right|\right|\right|_{q,r} \right\|_p &= \left(\int_{\mathbb{R}^n} \left|\left|\left| (\Delta_c I_2)(x) \right|\right|\right|_{q,r}^p dx \right)^{\frac{1}{p}} \\
&\leq \left(\int_{\mathbb{R}^n} \left| \frac{1}{C_h} \int_1^\infty \int_{\mathbb{R}^n} |(L_h f)(a, x - ay)| \left|\left|\left|(\Delta_z h)(y)\right|\right|\right|_{q,r} \; dy \frac{da}{a^{\frac{n}{2}+1}} \right|^p dx \right)^{\frac{1}{p}} \\
&\leq \frac{1}{C_h} \int_1^\infty \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |(L_h f)(a, x - ay)|^p \; \left|\left|\left|(\Delta_z h)(y)\right|\right|\right|_{q,r}^p dx \right)^{\frac{1}{p}} \; dy \frac{da}{a^{\frac{n}{2}+1}} \\
&= \frac{1}{C_h} \int_1^\infty \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |(L_h f)(a, w)|^p \, dw \right)^{\frac{1}{p}} \; \left|\left|\left|(\Delta_z h)(y)\right|\right|\right|_{q,r} \; dy \frac{da}{a^{\frac{n}{2}+1}}. \qquad \text{(A.6)}
\end{aligned}$$

That is,

$$\begin{aligned}
\left\| \left|\left|\left| (\Delta_c I_2)(\cdot) \right|\right|\right|_{q,r} \right\|_p &\leq \frac{1}{C_h} \int_1^\infty \int_{\mathbb{R}^n} \|(L_h f)(a, \cdot)\|_p \left|\left|\left|(\Delta_z h)(y)\right|\right|\right|_{q,r} \; dy \frac{da}{a^{\frac{n}{2}+1}} \\
&\leq \frac{1}{C_h} \left(\int_{\mathbb{R}^n} \left|\left|\left|(\Delta_z h)(y)\right|\right|\right|_{q,r} dy \right) \left(\int_1^\infty a^{\frac{n}{2}} \|h\|_1 \|f\|_p \frac{da}{a^{\frac{n}{2}+1+r}} \right) \\
&= \frac{1}{C_h} \|h\|_1 \|f\|_p \left\| \left|\left|\left|(\Delta_z h)(\cdot)\right|\right|\right|_{q,r} \right\|_1 \; \int_1^\infty \frac{da}{a^{1+r}} < \infty. \qquad \text{(A.7)}
\end{aligned}$$

This proves that $I_2 \in F_p^{r,q}(\mathbb{R}^n)$.

Hence, from 1) and 2), $f \in F_p^{r,q}(\mathbb{R}^n)$.

□

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