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Constant angle surfaces in the Lorentzian warped product manifold $I \times_f \mathbb{E}^2_1$

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Abstract: Let $I \times_f \mathbb{E}^2_1$ be a 3-dimensional Lorentzian warped product manifold with the metric $\tilde{g} = dt^2 + f^2(t)(dx^2 - dy^2)$, where I is an open interval, f is a strictly positive smooth function on I, and $\mathbb{E}^2_1$ is the Minkowski 2-plane. In this work, we give a classification of all space-like and time-like constant angle surfaces in $I \times_f \mathbb{E}^2_1$ with nonnull principal direction when the surface is time-like. In this classification, we obtain space-like and time-like surfaces with zero mean curvature, rotational surfaces, and surfaces with constant Gaussian curvature. Also, we have some results on constant angle surfaces of the anti-de Sitter space $\mathbb{H}^3_1(-1)$.

Key words: Constant angle surface, warped product, rotational surface, maximal surface, zero mean curvature, Gaussian curvature

1. Introduction

The study of constant angle surfaces whose normals make a constant angle with a fixed direction in the ambient space is a classical subject in the differential geometry, and it is important to classify such surfaces that satisfy certain geometrical properties such as being umbilical, constant mean curvature or constant Gaussian curvature. In recent years, many articles have appeared on constant angle surfaces, and several classification results were given in different ambient spaces, see [6–9, 11, 12, 14–17, 20, 21, 23]. These surfaces have an important role in the physics of interfaces in liquid crystals and of layered fluids, as studied by Cermelli and Di Scala in [6].

The concept of constant angle surfaces was extended to a Minkowski space in [12, 13, 15], and recently, space-like and time-like constant angle surfaces in the 3-dimensional Lorentzian Heisenberg group and Lorentzian Berger spheres were studied in [20, 21].

On the other hand, surfaces in warped product manifolds $I \times_f M$ have been investigated by several researchers, where I is an open interval and M is a surface, see [1–5, 9, 18], and the references therein. In [9], Dillen et al. classified constant angle surfaces in the 3-dimensional warped product $I \times_f \mathbb{E}^2$, and in this family they determined flat surfaces, rotational surfaces and minimal surfaces. In [10], the author and Turgay obtained a classification of all space-like and time-like constant angle surfaces in $-I \times_f \mathbb{E}^2$.

In this work, we continue to study constant angle surfaces in 3-dimensional Lorentzian warped product manifolds. Let $I \times_f \mathbb{E}^2_1$ be the 3-dimensional Lorentzian warped product manifold with the metric $\tilde{g} = dt^2 + f^2(t)(dx^2 - dy^2)$, where I is an open interval, f is a strictly positive smooth function on I, and $\mathbb{E}^2_1$ is

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the Minkowski 2-plane. We classify all space-like and time-like constant angle surfaces in $I \times_f \mathbb{E}^2_1$ with nonnull principal direction. In this classification, we get space-like and time-like surfaces with zero mean curvature, rotational surfaces, and surfaces with constant Gaussian curvature. Furthermore, we yield some results on constant angle surfaces of the anti-de Sitter space $\mathbb{H}^3_1(-1)$ with the metric $\tilde{g} = dt^2 + e^{2t}(dx^2 - dy^2)$.

2. Preliminaries

2.1. Basic formulae for the warped product manifold $I \times_f \mathbb{E}^2_1$

Let $\mathbb{E}^2_1$ be a 2-dimensional Minkowski space and $I \subseteq \mathbb{R}$ an open interval equipped with the metric dt^2. Let $I \times_f \mathbb{E}^2_1$ be the 3-dimensional Lorentzian warped product manifold equipped with the Lorentzian metric $\tilde{g} = dt^2 + f^2(t)(dx^2 - dy^2)$, where $f : I \to \mathbb{R}_+$ is a smooth function.

Let $\widetilde{\nabla}$ be the Levi-Civita connection of $(I \times_f \mathbb{E}^2_1, \tilde{g})$, and U, V, W the lifts of vector fields tangent to $\mathbb{E}^2_1$. Then, from [19] we have the connection $\widetilde{\nabla}$ on $I \times_f \mathbb{E}^2_1$ as

$$\widetilde{\nabla}_U V = D_U V - \frac{f'(t)}{f(t)}\tilde{g}(U,V)\partial_t, \tag{2.1a}$$

$$\widetilde{\nabla}_U \partial_t = \widetilde{\nabla}_{\partial_t} U = \frac{f'(t)}{f(t)}U, \tag{2.1b}$$

$$\widetilde{\nabla}_{\partial_t}\partial_t = 0, \tag{2.1c}$$

where D is the connection of $\mathbb{E}^2_1$. Then, the Riemannian curvature tensor $\widetilde{R}$ of $I \times_f \mathbb{E}^2_1$ is given by

$$\widetilde{R}(U,\partial_t)V = \frac{f''(t)}{f(t)}\tilde{g}(U,V)\partial_t, \tag{2.2a}$$

$$\widetilde{R}(U,\partial_t)\partial_t = -\frac{f''(t)}{f(t)}U, \tag{2.2b}$$

$$\widetilde{R}(U,V)\partial_t = 0, \tag{2.2c}$$

$$\widetilde{R}(U,V)W = -\frac{f'^2(t)}{f^2(t)}(\tilde{g}(V,W)U - \tilde{g}(U,W)V. \tag{2.2d}$$

For the orthonormal basis $\{\partial_t, \partial_x, \partial_y\}$ of $I \times_f \mathbb{E}^2_1$, we have the connection $\widetilde{\nabla}$ together with (2.1c) as

$$\widetilde{\nabla}_{\partial_x}\partial_x = -f(t)f'(t)\partial_t, \tag{2.3}$$

$$\widetilde{\nabla}_{\partial_y}\partial_y = f(t)f'(t)\partial_t, \tag{2.4}$$

$$\widetilde{\nabla}_{\partial_x}\partial_y = \widetilde{\nabla}_{\partial_y}\partial_x = 0, \tag{2.5}$$

$$\widetilde{\nabla}_{\partial_x}\partial_t = \widetilde{\nabla}_{\partial_t}\partial_x = \frac{f'(t)}{f(t)}\partial_x, \tag{2.6}$$

and thus the curvature tensor of $I \times_f \mathbb{E}^2_1$ is given by

$$\widetilde{R}(\partial_x, \partial_t)\partial_x = f(t)f''(t)\partial_t, \tag{2.7}$$

$$\widetilde{R}(\partial_y, \partial_t)\partial_y = -f(t)f''(t)\partial_t, \tag{2.8}$$

$$\widetilde{R}(\partial_x, \partial_t)\partial_t = -\frac{f''(t)}{f(t)}\partial_x, \tag{2.9}$$

$$\widetilde{R}(\partial_x, \partial_y)\partial_x = f'^2(t)\partial_y. \tag{2.10}$$

So, the sectional curvatures of $I \times_f \mathbb{E}^2_1$ are obtained as

$$K(\partial_x, \partial_y) = -\frac{\tilde{g}(\widetilde{R}(\partial_x, \partial_y)\partial_x, \partial_y)}{\tilde{g}(\partial_x, \partial_x)\tilde{g}(\partial_y, \partial_y) - (\tilde{g}(\partial_x, \partial_y))^2} = -\frac{f'^2(t)}{f^2(t)}, \tag{2.11}$$

$$K(\partial_x, \partial_t) = -\frac{\tilde{g}(\widetilde{R}(\partial_x, \partial_t)\partial_x, \partial_t)}{\tilde{g}(\partial_x, \partial_x)\tilde{g}(\partial_t, \partial_t) - (\tilde{g}(\partial_x, \partial_t))^2} = -\frac{f''(t)}{f(t)}, \tag{2.12}$$

$$K(\partial_y, \partial_t) = -\frac{\tilde{g}(\widetilde{R}(\partial_y, \partial_t)\partial_y, \partial_t)}{\tilde{g}(\partial_y, \partial_y)\tilde{g}(\partial_t, \partial_t) - (\tilde{g}(\partial_y, \partial_t))^2} = -\frac{f''(t)}{f(t)}. \tag{2.13}$$

Using the above sectional curvatures we can have the following proposition.

Proposition 2.1 *The warped product manifold $I \times_f \mathbb{E}^2_1$ with the metric $\tilde{g} = dt^2 + f^2(t)(dx^2 - dy^2)$ has a constant sectional curvature if and only if, up to a rigid motion of $I \times_f \mathbb{E}^2_1$, the warping function is $f(t) = e^{at}$, where $a \in \mathbb{R}$. Furthermore, when $a = 1$, $I \times_f \mathbb{E}^2_1$ with the warping function $f(t) = e^t$ is the anti-de Sitter space $\mathbb{H}^3_1(-1)$, and when $a = 0$, $I \times_f \mathbb{E}^2_1$ is the Minkowski $\mathbb{E}^3_1$.*

Let $\varphi : M_q \longrightarrow I \times_f \mathbb{E}^2_1$ be an isometric immersion of a surface M_q with index $q = 0, 1$, into the Lorentzian warped product manifold $(I \times_f \mathbb{E}^2_1, \tilde{g})$. It is said that φ is space-like (resp., time-like or light-like) if the induced metric g via φ is Riemannian (resp., Lorentzian or degenerate). It is equivalent to say that a normal vector ξ to M_q is time-like (resp., space-like or light-like). We take $q = 0$ for a space-like immersion and $q = 1$ for a time-like immersion.

The Gauss and Weingarten formulae of the immersion are, respectively, given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.14}$$

and

$$\widetilde{\nabla}_X \xi = -AX, \tag{2.15}$$

for every X and Y tangent to M_q, where ξ is a local unit vector field normal to M_q with the signature $\varepsilon_\xi = \tilde{g}(\xi, \xi)$, ∇ is the induced connection, h is the second fundamental form, and A is the shape operator of M_q. It is well known that h and A are related by $\tilde{g}(h(X, Y), \xi) = g(AX, Y)$, and $h(X, Y) = \varepsilon_\xi g(AX, Y)\xi$. Also, the mean curvature vector H of M_q in $I \times_f \mathbb{E}^2_1$ is defined by

$$H = \frac{1}{2}\sum_{i=1}^{2} \varepsilon_i g(Ae_i, e_i), \tag{2.16}$$

where $\{e_1, e_2\}$ is a local orthonormal frame field on M_q with the signatures $\varepsilon_i = g(e_i, e_i)$, $i = 1, 2$.

2.2. Angles between vectors

Let $v=(v_1,v_2,v_3)$ be a vector in $T_{(t,x,y)}(I\times_f\mathbb{E}_1^2)$. Then, v is said to be space-like if $\tilde{g}(v,v)>0$ or $v=0$, light-like if $\tilde{g}(v,v)=0$ and $v\neq 0$, or time-like if $\tilde{g}(v,v)<0$. The norm of v is defined by $||v||=\sqrt{\nu\tilde{g}(v,v)}$, where ν is the signature of v. The vector v is said to be positive (resp., negative) if $v_1>0$ (resp., $v_1<0$).

The angle between two vectors in a Minkowski space was introduced in [22, 24].

Definition 2.2 *Let v and w be positive (negative) time-like vectors in a Minkowski space $\mathbb{E}_1^3$. Then, the unique non negative number θ such that*

$$\tilde{g}(v,w)=-||v||||w||\cosh\theta \tag{2.17}$$

is called the Lorentzian time-like angle.

Definition 2.3 *Let v be a space-like vector and w a positive time-like vector in $\mathbb{E}_1^3$. Then, the unique non negative number θ such that*

$$|\tilde{g}(v,w)|=||v||||w||\sinh\theta \tag{2.18}$$

is called the Lorentzian time-like angle.

A space-like vector v is Lorentzian orthogonal to a positive time-like vector w if and only if the Lorentzian time-like angle θ is zero.

Definition 2.4 *Let v and w be two space-like vectors in $\mathbb{E}_1^3$ that span a time-like vector subspace. Then, we have $|\tilde{g}(v,w)|>||v||||w||$, and the unique non negative number θ such that*

$$|\tilde{g}(v,w)|=||v||||w||\cosh\theta \tag{2.19}$$

is called the Lorentzian time-like angle.

Definition 2.5 *Let v and w be two space-like vectors in $\mathbb{E}_1^3$ that span a space-like vector subspace. Then, we have $|\tilde{g}(v,w)|\leq||v||||w||$, and the unique real number θ between 0 and π such that*

$$\tilde{g}(v,w)=||v||||w||\cos\theta \tag{2.20}$$

is called the Lorentzian space-like angle.

3. Constant Angle Surfaces in $I\times_f\mathbb{E}_1^2$

Let M_q be a surface in $I\times_f\mathbb{E}_1^2$ and $\xi=\xi_1\partial_t+\xi_2\partial_x+\xi_3\partial_y$ be a local unit vector field normal to M_q with the signature ε_ξ. At each point $p\in M_q$, it can be chosen a unit normal vector $\xi(p)$ such that $\xi_1(p)>0$ when $q=0$, and $\xi_1(p)\geq 0$ when $q=1$.

Now, we assume that the first component of ξ is nonnegative, i.e. $\xi_1\geq 0$. Then, we write ∂_t as

$$\partial_t=T+\Theta\xi \tag{3.1}$$

such that T is a nonnull time-like vector, where $\Theta=\varepsilon_\xi\tilde{g}(\partial_t,\xi)$ is called the angle function. We put $e_1=\frac{T}{||T||}$ and $\varepsilon_1=\tilde{g}(e_1,e_1)=\mp 1$, where $||T||^2=\varepsilon_1\tilde{g}(T,T)$. Thus, we have

$$\varepsilon_1||T||^2+\varepsilon_\xi\Theta^2=1, \tag{3.2}$$

where

$$\Theta = \begin{cases} \sinh\theta & \text{if} \quad \varepsilon_\xi = -1, \\ \cosh\theta & \text{if} \quad \varepsilon_\xi = 1, \varepsilon_1 = -1, \\ \cos\theta & \text{if} \quad \varepsilon_\xi = 1, \varepsilon_1 = 1, \end{cases}$$

with $\theta \geq 0$ because of (2.17)–(2.20).

A surface M_q is said to be a *constant angle surface* if the Lorentzian angle θ is constant on M_q.

Let R be the curvature tensor of M_q. By using (2.2), (2.14), (2.15) and (3.1) we obtain the Gauss and Codazzi equations, respectively, as follows:

$$\begin{aligned} R(X,Y)Z =& \varepsilon_\xi(g(AY,Z)AX - g(AX,Z)AY) \\ &- ((\ln f)' \circ \varphi)^2 \Big(g(Y,Z)X - g(X,Z)Y\Big) \\ &- ((\ln f)'' \circ \varphi)\big(g(Z,T)g(Y,T)X - g(Z,T)g(X,T)Y\big) \\ &+ ((\ln f)'' \circ \varphi)\Big(g(Y,T)g(X,Z) - g(X,T)g(Y,Z)\Big)T \end{aligned} \tag{3.3}$$

and

$$(\nabla_X A)Y - (\nabla_Y A)X = \varepsilon_\xi \Theta((\ln f)'' \circ \varphi)\Big(g(Y,T)X - g(X,T)Y\Big), \tag{3.4}$$

for X, Y and Z tangent to M_q.

Proposition 3.1 *Let X be a tangent vector to M_q. Then,*

$$\nabla_X T = \Theta AX + ((\ln f)' \circ \varphi)(X - g(X,T)T) \tag{3.5}$$

and

$$X(\Theta) = -\varepsilon_\xi g(AT,X) - \Theta((\ln f)' \circ \varphi)g(T,X). \tag{3.6}$$

Proof If X is tangent to M_q, then $\tilde{g}(X,\partial_t) = g(X,T)$, and also we can write $X = X^* + g(X,T)\partial_t$, where X^* is tangent to $\mathbb{E}^2_1$. By using (2.1b), (2.1c) and (3.1) we get

$$\widetilde{\nabla}_X \partial_t = ((\ln f)' \circ \varphi)(X - g(X,T)T) - \Theta((\ln f)' \circ \varphi)g(X,T)\xi.$$

On the other hand, by using the Gauss and Weingarten formulae and (3.1) we have

$$\widetilde{\nabla}_X \partial_t = \nabla_X T - \Theta AX + (\varepsilon_\xi g(X,AT) + X(\Theta))\xi.$$

Comparing the tangent and normal parts of the last two equations we acquire (3.5) and (3.6). □

From (3.6) we have the following proposition.

Proposition 3.2 *Let M_q be a constant angle surface in $I \times_f \mathbb{E}^2_1$. Then, the vector T is a principal direction of the shape operator, and the corresponding eigenvalue is $\kappa = -\varepsilon_\xi \Theta((\ln f)' \circ \varphi)$.*

Note that since the vector ∂_t is space-like, in the case of time-like surfaces, the principal direction T can be a null vector. In this work we will not consider this case.

From now on, we assume that Θ is constant. Let e_2 be a unit vector orthogonal to e_1 with signature $\varepsilon_2 = g(e_2, e_2) = \mp 1$. If M_q is a space-like surface, then e_2 is also a principal direction, and thus there is a function $\lambda \in C^\infty(M)$ such that $Ae_2 = \lambda e_2$.

Proposition 3.3 *Let M_q be a constant angle surface in $I \times_f \mathbb{E}_1^2$ with $\Theta \neq 0$ and a nonnull principal direction T when $q = 1$. Then, there is an orthonormal frame field $\{e_1, e_2\}$ with $\varepsilon_i = g(e_i, e_i) = \pm 1$ such that the shape operator A is diagonalizable, i.e.*

$$Ae_1 = -\varepsilon_\xi((\ln f)' \circ \varphi)\Theta e_1 \quad \textit{and} \quad Ae_2 = \lambda e_2, \tag{3.7}$$

for some $\lambda \in C^\infty(M_q)$, and the Levi-Civita connection ∇ is given by

$$\begin{array}{ll} \nabla_{e_1} e_1 = 0, & \nabla_{e_2} e_1 = \frac{1}{||T||}(\Theta\lambda + ((\ln f)' \circ \varphi))e_2, \\ \nabla_{e_1} e_2 = 0, & \nabla_{e_2} e_2 = -\varepsilon_1\varepsilon_2 \frac{1}{||T||}(\Theta\lambda + ((\ln f)' \circ \varphi))e_1. \end{array} \tag{3.8}$$

Proof Let M_q be a space-like constant angle surface in $I \times_f \mathbb{E}_1^2$ with $\Theta \neq 0$. From Proposition 3.2, $e_1 = \frac{T}{||T||}$ is the unit principal direction of A. As M_q is space-like, there is a unit space-like vector e_2 orthogonal to $e_1 = \frac{T}{||T||}$, and e_2 is also a principal direction. Thus, there is a function $\lambda \in C^\infty(M)$ such that $Ae_2 = \lambda e_2$.

Let M_q be a time-like surface with a nonnull principal direction. Assume that the shape operator A is not diagonalizable. Then, there is a unit vector e_2 orthogonal to e_1 with signature $\varepsilon_2 = g(e_2, e_2) = -\varepsilon_1$ such that $Ae_2 = \mu e_1 + \lambda e_2$. From (3.5) we have

$$\nabla_{e_2} e_1 = \frac{1}{||T||}\{\Theta\mu e_1 + (\Theta\lambda + ((\ln f)' \circ \varphi))e_2\}. \tag{3.9}$$

If we compare (3.9) to $\nabla_{e_2} e_1 = \varepsilon_2\omega_{12}(e_2)e_2$, we have $\mu = 0$ which implies that A is diagonalizable.

The Levi-Civita connection (3.8) is yielded from (3.5). □

We consider an orthonormal frame field $\{e_1, e_2\}$ with signatures $\varepsilon_i = g(e_i, e_i)$, $i = 1, 2$, as above. From (3.8) it is seen that $[e_1, e_2]$ is proportional to e_2. Thus, we can choose coordinates (u, v) on M_q such that $\partial_u = e_1$ and $\partial_v = \beta e_2$, for some smooth function $\beta(u, v)$. Therefore, the metric g on M_q takes the form

$$g = \varepsilon_1 du^2 + \varepsilon_2 \beta^2(u, v) dv^2. \tag{3.10}$$

So, the Levi-Civita connection according to the metric g is obtained by

$$\nabla_{\partial_u}\partial_u = 0, \ \nabla_{\partial_u}\partial_v = \nabla_{\partial_v}\partial_u = \frac{\beta_u}{\beta}\partial_v, \ \nabla_{\partial_v}\partial_v = -\varepsilon_1\varepsilon_2\beta\beta_u\partial_u + \frac{\beta_v}{\beta}\partial_v, \tag{3.11}$$

and β holds

$$\beta_u = \frac{\beta}{||T||}(\Theta\lambda + ((\ln f)' \circ \varphi)). \tag{3.12}$$

If we write

$$\varphi(u, v) = (t(u, v), x(u, v), y(u, v)),$$

then we get $t_u = \tilde{g}(\varphi_u, \partial_t) = \tilde{g}(e_1, \partial_t) = \tilde{g}(T/||T||, T + \Theta\xi) = \varepsilon_1||T||$ as $||T||^2 = \varepsilon_1\tilde{g}(T,T)$ and $t_v = \tilde{g}(\varphi_v, \partial_t) = \tilde{g}(\beta e_2, \partial_t) = \tilde{g}(\beta e_2, +T + \Theta\xi) = 0$. After a translation in u coordinate and integration, we have

$$t(u,v) = \varepsilon_1||T||u. \tag{3.13}$$

From (3.2) and (3.10) we yield

$$||T||^2 + f^2(\varepsilon_1||T||u)(x_u^2 - y_u^2) = \varepsilon_1, \tag{3.14a}$$

$$x_u x_v - y_u y_v = 0, \tag{3.14b}$$

$$f^2(\varepsilon_1||T||u)(x_v^2 - y_v^2) = \varepsilon_2\beta^2. \tag{3.14c}$$

Now, we define

$$\sigma(u) = ((\ln f) \circ \varphi)(u,v) = \ln f(\varepsilon_1||T||u). \tag{3.15}$$

Using (2.1), (3.14) and (3.15), by a straightforward computation we have

$$\widetilde{\nabla}_{\varphi_u}\varphi_u = \varphi_{uu} + 2\sigma'(u)\varphi_u - \sigma'(u)\Big(\frac{1}{||T||} + \varepsilon_1||T||\Big)\partial_t, \tag{3.16a}$$

$$\widetilde{\nabla}_{\varphi_u}\varphi_v = \varphi_{vu} + 2\sigma'(u)\varphi_v, \tag{3.16b}$$

$$\widetilde{\nabla}_{\varphi_v}\varphi_v = \varphi_{vv} - \varepsilon_1\varepsilon_2\frac{\sigma'(u)\beta^2}{||T||}\partial_t. \tag{3.16c}$$

On the other hand, using (3.1), (3.7), (3.11), (3.15) and the Gauss' formula we can state these covariant derivatives as follows:

$$\widetilde{\nabla}_{\varphi_u}\varphi_u = \sigma'(u)\varphi_u - \frac{\sigma'(u)}{||T||}\partial_t, \tag{3.17a}$$

$$\widetilde{\nabla}_{\varphi_u}\varphi_v = \frac{\beta_u}{\beta}\varphi_v, \tag{3.17b}$$

$$\widetilde{\nabla}_{\varphi_v}\varphi_v = -\varepsilon_2\Big(\varepsilon_1\beta\beta_u + \frac{\varepsilon_\xi\lambda\beta^2||T||}{\Theta}\Big)\varphi_u + \frac{\beta_v}{\beta}\varphi_v + \varepsilon_2\varepsilon_\xi\frac{\lambda\beta^2}{\Theta}\partial_t. \tag{3.17c}$$

Now, we are going to compare (3.16) and (3.17). From (3.16a) and (3.17a) we get $\varphi_{uu} + \sigma'(u)\varphi_u - \varepsilon_1||T||\sigma'(u)\partial_t = 0$. This equation holds for the t-component. For the x- and y-components we obtain $x_{uu} + \sigma'(u)x_u = 0$ and $y_{uu} + \sigma'(u)y_u = 0$ from which we obtain $x_u(u,v) = e^{-\sigma(u)}c_1(v)$ and $y_u(u,v) = e^{-\sigma(u)}c_2(v)$ for some functions $c_1(v)$ and $c_2(v)$. From (3.2) and (3.14a) we get $c_1^2(v) - c_2^2(v) = \varepsilon_1\varepsilon_\xi\Theta^2$. We put $p_1(v) = c_1(v)/\Theta$ and $p_2(v) = c_2(v)/\Theta$. Then,

$$\varphi_u(u,v) = (\varepsilon_1||T||, p_1(v)\Theta e^{-\sigma(u)}, p_2(v)\Theta e^{-\sigma(u)}), \quad p_1^2(v) - p_2^2(v) = \varepsilon_1\varepsilon_\xi. \tag{3.18}$$

From (3.16b) and (3.17b) we obtain that $\varphi_{vu} + \Big(\sigma'(u) - \frac{\beta_u}{\beta}\Big)\varphi_v = 0$. This equation holds also for the t-component. For the x- and y-components we have

$$x_{vu} + \Big(\sigma'(u) - \frac{\beta_u}{\beta}\Big)x_v = 0 \quad \text{and} \quad y_{vu} + \Big(\sigma'(u) - \frac{\beta_u}{\beta}\Big)y_v = 0,$$

from which we write

$$x_v(u,v) = q_1(v)\beta(u,v)e^{-\sigma(u)} \quad \text{and} \quad y_v(u,v) = q_2(v)\beta(u,v)e^{-\sigma(u)},$$

for some functions $q_1(v)$ and $q_2(v)$. From (3.14c) and (3.15) we get $q_1^2(v) - q_2^2(v) = \varepsilon_2$. Hence, we have

$$\varphi_v(u,v) = \beta(u,v)e^{-\sigma(u)}(0, q_1(v), q_2(v)), \quad q_1^2(v) - q_2^2(v) = \varepsilon_2. \tag{3.19}$$

The compatibility condition, i.e. $\varphi_{vu} = \varphi_{uv}$ implies that

$$(p_1', p_2') = \Big(\frac{\beta_u - \sigma'(u)\beta}{\Theta}\Big)(q_1, q_2). \tag{3.20}$$

Finally, Equations (3.16c) and (3.17c) yield

$$\varphi_{vv} + \varepsilon_2\Big(\varepsilon_1\beta\beta_u + \varepsilon_\xi\frac{\lambda\beta^2||T||}{\Theta}\Big)\varphi_u - \frac{\beta_v}{\beta}\varphi_v - \varepsilon_2\beta^2\Big(\frac{\sigma'}{||T||} + \frac{\lambda}{\Theta}\Big)\partial_t = 0. \tag{3.21}$$

Replacing (3.18) and (3.19) into (3.21), the x- and y-components give

$$(q_1'(v), q_2'(v)) = -\varepsilon_2(\varepsilon_1\Theta\beta_u + \varepsilon_\xi\lambda\beta||T||)(p_1(v), p_2(v)). \tag{3.22}$$

3.1. Space-like constant angle surfaces

Here we classify all space-like constant angle surfaces in $I \times_f \mathbb{E}_1^2$, and we give a corollary for space-like surfaces in the anti-de Sitter space $\mathbb{H}_1^3(-1)$.

Theorem 3.4 *An immersion $\varphi : M_0 \longrightarrow I \times_f \mathbb{E}_1^2$ defines a space-like constant angle surface with a constant Lorentzian time-like angle $\theta \geq 0$ if and only if, up to rigid motions of $I \times_f \mathbb{E}_1^2$, one of the followings holds locally:*

i) There exist local coordinates (u,v) on M_0, with respect to which the immersion φ is given by

$$\varphi(u,v) = \Big(u\cosh\theta, \tanh\theta\sinh v\int^{u\cosh\theta}\frac{d\mu}{f(\mu)} + \int^{v}\alpha(\mu)\cosh\mu d\mu, \tag{3.23}$$

$$\tanh\theta\cosh v\int^{u\cosh\theta}\frac{d\mu}{f(\mu)} + \int^{v}\alpha(\mu)\sinh\mu d\mu\Big),$$

for some smooth function $\alpha(v)$.

ii) $\varphi(M_0)$ is an open part of the space-like cylinder

$$y - \tanh\theta\int^{t}\frac{d\mu}{f(\mu)} = 0. \tag{3.24}$$

This is a totally umbilical space-like surface with the principal curvature $\lambda = \sinh\theta f'(u\cosh\theta)/f(u\cosh\theta)$.

Proof We first show that the surfaces given in the theorem are constant angle surfaces.

For the case (i), the vectors

$$\varphi_u = \Big(\cosh\theta, \frac{\sinh\theta\sinh v}{f(u\cosh\theta)}, \frac{\sinh\theta\cosh v}{f(u\cosh\theta)}\Big),$$

$$\varphi_v = \Big(\tanh\theta\int^{u\cosh\theta}\frac{d\mu}{f(\mu)} + \alpha(v)\Big)(0,\cosh v,\sinh v)$$

form a basis for the tangent space of the surface.

Let $v=(v_1,v_2,v_3)$ and $w=(w_1,w_2,w_3)$ be two vectors in $T_{(t,x,y)}(I\times_f\mathbb{E}^2_1)$. Then, the vector defined by

$$v\times_f w = (f^2(t)(v_2w_3-v_3w_2),(v_3w_1-v_1w_3),(v_2w_1-v_1w_2)) \tag{3.25}$$

is orthogonal to v and w.

Hence, a unit normal vector to the surface is written as

$$\xi = \frac{\varphi_v\times_f\varphi_u}{||\varphi_v\times_f\varphi_u||} = \Big(\sinh\theta, \frac{\cosh\theta\sinh v}{f(u\cosh\theta)}, \frac{\cosh\theta\cosh v}{f(u\cosh\theta)}\Big)$$

with $\varepsilon_\xi = \tilde{g}(\xi,\xi) = -1$. Then, we get $\tilde{g}(\xi,\partial_t) = \sinh\theta\tilde{g}(\partial_t,\partial_t) = \sinh\theta$ which is a constant.

For the case (ii), we can parameterize the surface as

$$\varphi(u,v) = \Big(u,v,\tanh\theta\int^u\frac{d\mu}{f(\mu)}\Big).$$

Then, the vector $\xi = (\sinh\theta,0,\frac{\cosh\theta}{f(u)})$ is a unit normal, and $\tilde{g}(\xi,\partial_t)=\sinh\theta$ which is a constant.

Conversely, let $\varphi : M_0 \to I\times_f\mathbb{E}^2_1$ be a space-like constant angle surface with a constant Lorentzian time-like angle $\theta\in[0,\infty)$. Let (u,v) be local coordinates on M_0 as in (3.10). Then, we have Equations (3.10)–(3.22) for $\varepsilon_1=\varepsilon_2=1$ and $\varepsilon_\xi=-1$, also $||T||=\cosh\theta$ and $\Theta=\sinh\theta$.

If $\theta=0$, then the vector ∂_t is tangent to $\varphi(M_0)$ everywhere. It implies that $\varphi(M_0)$ is an open part of a space-like cylinder with rulings in the direction of ∂_t, or equivalently that there exist local coordinates (u,v) on M_0 such that $\varphi(u,v)=(u,\gamma_1(v),\gamma_2(v))$ for some smooth functions γ_1 and γ_2. If φ parametrizes a plane, this is the case (ii) of the theorem with $\theta=0$. If φ does not describe a plane, this is case (i) of the theorem with $\theta=0$.

Now we assume that $\theta>0$. Considering (3.20), we distinguish two cases depending if $p_1(v)$ and $p_2(v)$ are constant or not.

Case 1. $p_1(v)$ and $p_2(v)$ are constant. Then, from (3.20) we obtain that $\beta_u=\sigma'\beta$ from which we get $\beta(u,v)=\psi(v)f(u\cosh\theta)$. After changing in v-coordinate we can suppose that $\psi(v)=1$ such that $\beta(u,v)=f(u\cosh\theta)$. Then, from (3.12) we have $\lambda=\sinh\theta f'(u\cosh\theta)/f(u\cosh\theta)$. Using (3.7), it follows that M_0 is totally umbilical. Also, it is seen from (3.22) that q_1 and q_2 are constant. Integrating (3.18) and using (3.19), we obtain that

$$\varphi(u,v) = \Big(u\cosh\theta, p_1\sinh\theta\int^u e^{-\sigma(\tau)}d\tau + q_1v + a_1, p_2\sinh\theta\int^u e^{-\sigma(\tau)}d\tau + q_2v + a_2\Big), \tag{3.26}$$

for some constants a_1 and a_2 which can be taken zero after a translation in x and y. From (3.14b), (3.18) and (3.19), we have

$$p_1q_1 - p_2q_2 = 0, \quad p_2^2 - p_1^2 = 1 \text{ and } q_1^2 - q_2^2 = 1. \tag{3.27}$$

Considering (3.27), after a rotation around the t-axis which is an isometry of $I \times_f \mathbb{E}^2_1$, we may assume that $(p_1, p_2) = (0, 1)$ and $(q_1, q_2) = (1, 0)$. Hence, using (3.15) and after the substitution $\mu = \tau \cosh\theta$ in (3.26) we obtain

$$\varphi(u,v) = \Big(u\cosh\theta, v, \tanh\theta \int^{u\cosh\theta} \frac{d\mu}{f(\mu)}\Big), \tag{3.28}$$

which corresponds to case (ii) of the theorem.

Case 2. $p_1(v)$ and $p_2(v)$ are not constant. Then, from (3.27) we can assume that

$$(p_1(v), p_2(v)) = (\sinh v, \cosh v). \tag{3.29}$$

Considering (3.27), Equation (3.20) implies that

$$\beta_u - \sigma'(u)\beta = \pm\sinh\theta \tag{3.30}$$

and, without loss of generality, we can assume that the right hand side to be $\sinh\theta$. Integrating (3.30), we get

$$\beta(u,v)e^{-\sigma(u)} - \sinh\theta \int^u e^{-\sigma(\tau)}d\tau = \alpha(v), \tag{3.31}$$

for some function $\alpha(v)$. Moreover, (3.20) yields that

$$(q_1(v), q_2(v)) = (\cosh v, \sinh v). \tag{3.32}$$

Then, (3.18) and (3.19) reduce to

$$\varphi_u(u,v) = (\cosh\theta, e^{-\sigma(u)}\sinh\theta\sinh v, e^{-\sigma(u)}\sinh\theta\cosh v), \tag{3.33}$$

$$\varphi_v(u,v) = \beta(u,v)e^{-\sigma(u)}(0, \cosh v, \sinh v). \tag{3.34}$$

Integrating (3.33) we have

$$\varphi(u,v) = \Big(u\cosh\theta, \sinh\theta\sinh v \int^u e^{-\sigma(\tau)}d\tau + \gamma_1(v), \sinh\theta\cosh v \int^u e^{-\sigma(\tau)}d\tau + \gamma_2(v)\Big), \tag{3.35}$$

for some smooth functions $\gamma_1(v)$ and $\gamma_2(v)$. Taking derivative of (3.35) with respect to v and comparing it to (3.34), and also using (3.31), we obtain that

$$\gamma_1(v) = \int^v \alpha(\tau)\cosh\tau d\tau, \quad \gamma_2(v) = \int^v \alpha(\tau)\sinh\tau d\tau.$$

Therefore, considering (3.15) and using the substitution $\mu = \tau\cosh\theta$, we have (3.23) from (3.35). □

It follows from Proposition 2.1 that the warped product $I \times_f \mathbb{E}^2_1$ with the metric $\tilde{g} = dt^2 + f^2(t)(dx^2 - dy^2)$ is the anti-de Sitter space $\mathbb{H}^3_1(-1)$ when $f(t) = e^t$. Hence, we can conclude the following result from Theorem 3.4 for space-like constant angle surfaces in $\mathbb{H}^3_1(-1)$.

Corollary 3.5 *An immersion $\varphi : M_0 \to \mathbb{H}^3_1(-1)$ defines a space-like constant angle surface with a constant Lorentzian time-like angle $\theta \geq 0$ if and only if, up to rigid motions of $I \times_f \mathbb{E}^2_1$, one of the followings holds locally:*

i) There exist local coordinates (u,v) *on* M_0*, with respect to which the immersion* φ *is given by*

$$\varphi(u,v)=\Big(u\cosh\theta,-e^{-u\cosh\theta}\tanh\theta\sinh v+\int^{v}\alpha(\mu)\cosh\mu\,d\mu,\qquad(3.36)$$
$$-e^{-u\cosh\theta}\tanh\theta\cosh v+\int^{v}\alpha(\mu)\sinh\mu\,d\mu\Big),$$

for some smooth function $\alpha(v)$*.*

ii) $\varphi(M_0)$ *is an open part of the space-like cylinder*

$$y+e^{-t}\tanh\theta=0.\qquad(3.37)$$

This is a totally umbilical space-like surface with the principal curvature $\lambda=\sinh\theta$*.*

3.2. Time-like constant angle surfaces

Here we classify all time-like constant angle surfaces in $I\times_f\mathbb{E}^2_1$, and we give a corollary for time-like surfaces in the anti-de Sitter space $\mathbb{H}^3_1(-1)$.

Theorem 3.6 *An immersion* $\varphi: M_1\longrightarrow I\times_f\mathbb{E}^2_1$ *defines a time-like constant angle surface with a constant Lorentzian angle* θ *and a nonnull principal direction if and only if, up to rigid motions of* $I\times_f\mathbb{E}^2_1$*, one of the followings holds locally:*

i) There exist local coordinates (u,v) *on* M_1*, with respect to which the immersion* φ *is given by*

$$\varphi(u,v)=\Big(u\sin\theta,\cot\theta\cosh v\int^{u\sin\theta}\frac{d\mu}{f(\mu)}+\int^{v}\alpha(\mu)\sinh\mu d\mu,\qquad(3.38)$$
$$\cot\theta\sinh v\int^{u\sin\theta}\frac{d\mu}{f(\mu)}+\int^{v}\alpha(\mu)\cosh\mu d\mu\Big),$$

for some smooth function $\alpha(v)$*, where the Lorentzian space-like angle* $\theta\in(0,\pi/2)\cup(\pi/2,\pi)$*.*

ii) There exist local coordinates (u,v) *on* M_1*, with respect to which the immersion* φ *is given by*

$$\varphi(u,v)=\Big(-u\sinh\theta,-\coth\theta\sinh v\int^{-u\sinh\theta}\frac{d\mu}{f(\mu)}+\int^{v}\alpha(\mu)\cosh\mu d\mu,\qquad(3.39)$$
$$-\coth\theta\cosh v\int^{-u\sinh\theta}\frac{d\mu}{f(\mu)}+\int^{v}\alpha(\mu)\sinh\mu d\mu\Big),$$

for some smooth function $\alpha(v)$*, where the Lorentzian time-like angle* $\theta>0$*.*

iii) $\varphi(M_1)$ *is an open part of the time-like cylinder*

$$x-\cot\theta\int^{t}\frac{d\tau}{f(\tau)}=0,\qquad(3.40)$$

where the Lorentzian space-like angle $\theta\in(0,\pi/2)\cup(\pi/2,\pi)$*. This is a totally umbilical surface with the principal curvature* $\lambda=-\cos\theta f'(u\sin\theta)/f(u\sin\theta)$*.*

iv) $\varphi(M_1)$ is an open part of the time-like cylinder

$$y + \coth\theta \int^t \frac{d\tau}{f(\tau)} = 0, \tag{3.41}$$

where the Lorentzian time-like angle $\theta > 0$. This is a totally umbilical time-like surface with the principal curvature $\lambda = -\cosh\theta f'(-u\sinh\theta)/f(-u\sinh\theta)$.

v) $\varphi(M_1)$ is an open part of the time-like surface $t = t_0$, for some real number t_0 and $\theta = 0$.

Proof As in the proof of Theorem 3.4, it can be shown that the surfaces defined by (3.38), (3.39), (3.40) and (3.41) are constant angle surfaces.

Conversely, let $\varphi : M_1 \to I \times_f \mathbb{E}^2_1$ be a time-like constant angle surface with a constant Lorentzian angle θ and the nonnull principal direction T. As the normal vector ξ and the vector field ∂_t are space-like, the vector T, i.e. e_1 is space-like or time-like.

Case 1. T is a space-like vector. Let (u, v) be local coordinates on M_1 as in (3.10). Then, we have Equations (3.10)–(3.22) for $\varepsilon_1 = -\varepsilon_2 = 1$ and $\varepsilon_\xi = 1$, also $||T|| = \sin\theta$ and $\Theta = \cos\theta$ for $0 \leq \theta < \pi$.

If $\theta = 0$, then $\varphi(M_1)$ is a surface of type (v) given in the theorem. If $\theta = \pi/2$, then the vector ∂_t is tangent to $\varphi(M_1)$ everywhere. This means that $\varphi(M_1)$ is an open part of a time-like cylinder with rulings in the direction of ∂_t, or equivalently that there exist local coordinates (u, v) on M_1 such that $\varphi(u, v) = (u, \gamma_1(v), \gamma_2(v))$ for some smooth functions γ_1 and γ_2. If φ parameterizes a time-like plane, this is the case (iii) of the theorem with $\theta = \pi/2$. If φ does not describe a plane, this is case (i) of the theorem with $\theta = \pi/2$.

Now we assume that $\theta \in (0, \pi/2) \cup (\pi/2, \pi)$. Considering (3.20), we distinguish two cases depending if $p_1(v)$ and $p_2(v)$ are constant or not.

Subcase 1.1. $p_1(v)$ and $p_2(v)$ are constant. Then, from (3.20) we obtain that $\beta_u = \sigma'\beta$ from which we get $\beta(u, v) = \psi(v) f(u\sin\theta)$. After changing in v-coordinate we can suppose that $\psi(v) = 1$ such that $\beta(u, v) = f(u\sin\theta)$. Then, from (3.12) we have $\lambda = -\cos\theta f'(u\sin\theta)/f(u\sin\theta)$. Using (3.7), it follows that M_1 is totally umbilical. Also, it is seen from (3.22) that q_1 and q_2 are constants. Integrating (3.18) and using (3.19) we obtain that

$$\varphi(u, v) = \Big(u\sin\theta, p_1\cos\theta \int^u e^{-\sigma(\tau)} d\tau + q_1 v + a_1, p_2\cos\theta \int^u e^{-\sigma(\tau)} d\tau + q_2 v + a_2\Big), \tag{3.42}$$

for some constants a_1 and a_2 which can be taken zero after a translation in x and y. From (3.14b), (3.18) and (3.19), we have

$$p_1 q_1 - p_2 q_2 = 0, \quad p_1^2 - p_2^2 = 1 \text{ and } q_2^2 - q_1^2 = 1. \tag{3.43}$$

Considering (3.43), after a rotation around the t-axis which is an isometry of $I \times_f \mathbb{E}^2_1$, we may assume that $(p_1, p_2) = (1, 0)$ and $(q_1, q_2) = (0, 1)$. Hence, using (3.15) and after the substitution $\mu = \tau\cosh\theta$ in (3.42) we obtain

$$\varphi(u, v) = \Big(u\sin\theta, \cot\theta \int^{u\sin\theta} \frac{d\tau}{f(\tau)}, v\Big), \tag{3.44}$$

which corresponds to the case (iii) of the theorem.

Subcase 1.2. $p_1(v)$ and $p_2(v)$ are not constant. Then, from (3.18) we can assume that

$$(p_1(v), p_2(v)) = (\cosh v, \sinh v). \tag{3.45}$$

Considering (3.19), Equation (3.20) implies that

$$\beta_u - \sigma'(u)\beta = \pm\cos\theta \tag{3.46}$$

and, without loss of generality, we can assume that the right hand side to be $\cos\theta$. Integrating (3.46) we get

$$\beta(u,v)e^{-\sigma(u)} - \cos\theta \int^u e^{-\sigma(\tau)}d\tau = \alpha(v), \tag{3.47}$$

for some function $\alpha(v)$. Moreover, (3.20) yields that

$$(q_1(v), q_2(v)) = (\sinh v, \cosh v). \tag{3.48}$$

Then, (3.18) and (3.19) reduce to

$$\varphi_u(u,v) = (\sin\theta, e^{-\sigma(u)}\cos\theta\cosh v, e^{-\sigma(u)}\cos\theta\sinh v), \tag{3.49}$$

$$\varphi_v(u,v) = \beta(u,v)e^{-\sigma(u)}(0, \sinh v, \cosh v). \tag{3.50}$$

Integrating (3.49) we have

$$\varphi(u,v) = \Big(u\sin\theta, \cos\theta\cosh v\int^u e^{-\sigma(\tau)}d\tau + \gamma_1(v), \cos\theta\sinh v\int^u e^{-\sigma(\tau)}d\tau + \gamma_2(v)\Big), \tag{3.51}$$

for some smooth functions $\gamma_1(v)$ and $\gamma_2(v)$. Taking derivative of (3.51) with respect to v and comparing it to (3.50), and also using (3.47) we obtain that

$$\gamma_1(v) = \int^v \alpha(\tau)\sinh\tau d\tau, \quad \gamma_2(v) = \int^v \alpha(\tau)\cosh\tau d\tau.$$

Therefore, considering (3.15) and using the substitution $\mu = \tau\sin\theta$, we have (3.38) from (3.51).

Case 2. T is a time-like vector. Let (u,v) be local coordinates on M_1 as in (3.10). Then, we have Equations (3.10)–(3.22) for $-\varepsilon_1 = \varepsilon_2 = 1$ and $\varepsilon_\xi = 1$, also $||T|| = \sinh\theta$ and $\Theta = \cosh\theta$ for $\theta \geq 0$. If $\theta = 0$, then $\varphi(M_1)$ is a surface of type (v) given in the theorem.

Since the rest of proof is similar to subcases 1.1 and 1.2, we omit the proof of (iii) and (iv). □

Corollary 3.7 *An immersion $\varphi: M_1 \to (\mathbb{H}^3_1(-1), \tilde{g} = dt^2 + e^{2t}(dx^2 - dy^2))$ defines a time-like constant angle surface with a constant Lorentzian angle θ and a nonnull principal direction if and only if, up to rigid motions of $I \times_f \mathbb{E}^2_1$, one of the followings holds locally:*

i) There exist local coordinates (u,v) on M_1, with respect to which the immersion φ is given by

$$\varphi(u,v) = \Big(u\sin\theta, -e^{-u\sin\theta}\cot\theta\cosh v + \int^v \alpha(\mu)\sinh\mu d\mu, \tag{3.52}$$

$$-e^{-u\sin\theta}\cot\theta\sinh v + \int^v \alpha(\mu)\cosh\mu d\mu\Big),$$

for some smooth function $\alpha(v)$, *where the Lorentzian space-like angle* $\theta \in (0, \pi/2) \cup (\pi/2, \pi)$.

ii) There exist local coordinates (u, v) *on* M_1, *with respect to which the immersion* φ *is given by*

$$\varphi(u,v) = \Big(- u\sinh\theta, e^{u\sinh\theta}\coth\theta\cosh v + \int^v \alpha(\mu)\sinh\mu d\mu, \qquad (3.53)$$

$$e^{u\sinh\theta}\coth\theta\sinh v + \int^v \alpha(\mu)\cosh\mu d\mu\Big),$$

for some smooth function $\alpha(v)$, *where the Lorentzian time-like angle* $\theta > 0$.

iii) $\varphi(M_1)$ *is an open part of the time-like cylinder*

$$x + e^{-t}\cot\theta = 0, \qquad (3.54)$$

where the Lorentzian space-like angle $\theta \in (0, \pi/2) \cup (\pi/2, \pi)$. *This is a totally umbilical surface with the principal curvature* $\lambda = -\cos\theta$.

iv) $\varphi(M_1)$ *is an open part of the time-like cylinder*

$$x - e^{-t}\coth\theta = 0, \qquad (3.55)$$

where the Lorentzian time-like angle $\theta > 0$. *This is a totally umbilical surface with the principal curvature* $\lambda = -\cosh\theta$.

v) $\varphi(M_1)$ *is an open part of the surface* $t = t_0$, *for some real number* t_0 *and* $\theta = 0$.

Note that in the case (i) of Theorem 3.4, and (i) and (ii) of Theorem 3.6, the function λ is given from (3.12) and (3.22) by

$$\lambda = \frac{||T||}{\beta} - \varepsilon_\xi \Theta \frac{f'(\varepsilon_1||T||u)}{f(\varepsilon_1||T||u)}. \qquad (3.56)$$

4. Constant angle rotational surfaces in $I \times_f \mathbb{E}^2_1$

In this section, we study constant angle rotational surfaces in the Lorentzian warped product manifold $I \times_f \mathbb{E}^2_1$ with the Lorentzian metric $\tilde{g} = dt^2 + f^2(t)(dx^2 - dy^2)$.

The rotation about the t-axis given by

$$R_v : I \times_f \mathbb{E}^2_1 \longrightarrow I \times_f \mathbb{E}^2_1 : (t, x, y) \longrightarrow (t, x\cosh v + y\sinh v, x\sinh v + y\cosh v)$$

is a one-parameter group of isometries which left the t-axis pointwise invariant. Let $\gamma = \gamma(u)$, $u \in J$, be a smooth curve in $I \times_f \mathbb{E}^2_1$ defined on an open interval J. Then, a rotational surface about the t-axis in $I \times_f \mathbb{E}^2_1$ with a profile curve γ is defined by $\varphi(u, v) = R_v(\gamma(u))$. Without loss of generality, we may assume that the profile curve lies in a nondegenerate plane of $I \times_f \mathbb{E}^2_1$. Thus, we consider two cases.

Case 1. Profile curve γ lies in a space-like plane. Let γ be a smooth curve defined an open interval J in the totally geodesic plane of $I \times_f \mathbb{E}^2_1$ containing t- and x-axes. Assume that $\gamma(u) = (a(u), b(u), 0)$ is an arc length parametrized curve, that is,

$$a'^2(u) + f^2(a(u))b'^2(u) = 1, \qquad (4.1)$$

which means that the profile curve is space-like. Note that $a : J \to I$ and f is defined on I. Then, the rotational surface about the t-axis in $I \times_f \mathbb{E}^2_1$ is given by

$$\varphi(u, v) = (a(u), b(u)\cosh v, b(u)\sinh v). \tag{4.2}$$

A unit normal vector field on the surface is obtained as

$$\xi = \frac{\varphi_u \times_f \varphi_v}{||\varphi_u \times_f \varphi_v||} = \Big(b'(u)f(a(u)), -\frac{a'(u)}{f(a(u))}\cosh v, -\frac{a'(u)}{f(a(u))}\sinh v\Big)$$

with $\varepsilon_\xi = 1$, that is, the rotational surface is time-like. So, the plane spanned by $\{e_1, \xi\}$ is space-like and $\varepsilon_2 = -1$. Considering ξ, we have $\tilde{g}(\xi, \partial_t) = b'(u)f(a(u))$. Also, from (3.1) we have $b'(u)f(a(u)) = \varepsilon_\xi\Theta = \Theta$ as $\varepsilon_\xi = 1$. Thus, the time-like rotational surface (4.2) is a constant angle surface if and only if

$$b'(u)f(a(u)) = \cos\theta. \tag{4.3}$$

Using (4.1), we have $a(u) = u\sin\theta + c$ for some real constant c. After a change of the arc length parameter u of γ, we may assume that $a(u) = u\sin\theta$ without loss of generality. Then, it follows from (4.3) that

$$b(u) = \cos\theta \int^u \frac{d\tau}{f(\tau\sin\theta)} = \cot\theta \int^{u\sin\theta} \frac{d\mu}{f(\mu)}, \tag{4.4}$$

where $\theta \in (0, \pi/2) \cup (\pi/2, \pi)$.

If $\theta = 0$, we have $a(u) = a_0 \in I$ and $b(u) = (1/f(a_0))u + b_1$, where a_0 and b_1 are constant. Thus, the time-like rotational surface is planar.

If $\theta = \pi/2$, we have $a(u) = u$ and $b(u) = b_0 > 0$ without loss of generality, where b_0 is a constant. Then, the time-like rotational constant angle surface is a rotational cylinder which is a special case of (i) of Theorem 3.6 when $\alpha(v) = \alpha_0$ is a constant.

Therefore, the time-like rotational constant angle surface with a space-like profile curve in $I \times_f \mathbb{E}^2_1$ defined by (4.2) is determined. This time-like rotational constant angle surface is a special case of Theorem 3.6 when $\alpha(v) = 0$ or when $\theta = \pi/2$ and $\alpha(v) = \alpha_0$.

Case 2. Profile curve γ lies in a time-like plane. Let γ be a smooth curve defined on an open interval J in the totally geodesic plane of $I \times_f \mathbb{E}^2_1$ containing t- and y-axes. Assume that $\gamma(u) = (a(u), 0, b(u))$ is an arc length parametrized curve, that is,

$$a'^2(u) - f^2(a(u))b'^2(u) = \varepsilon_1. \tag{4.5}$$

Note that $a : J \to I$ and f is defined on I.

Then, the rotational surface with this profile curve about the t-axis in $I \times_f \mathbb{E}^2_1$ is given by

$$\varphi(u, v) = (a(u), b(u)\sinh v, b(u)\cosh v). \tag{4.6}$$

A unit vector field normal to the surface is obtained as

$$\xi = \frac{\varphi_u \times_f \varphi_v}{||\varphi_u \times_f \varphi_v||} = \Big(- b'(u)f(a(u)), -\frac{a'(u)}{f(a(u))}\sinh v, -\frac{a'(u)}{f(a(u))}\cosh v\Big)$$

with $\varepsilon_\xi = -\varepsilon_1$ from which we have $\tilde{g}(\xi, \partial_t) = -b'(u)f(a(u))$. Considering (3.1) we have $b'(u)f(a(u)) = -\varepsilon_\xi \Theta = \varepsilon_1 \Theta$. Thus, the rotational surface (4.6) is a constant angle surface if and only if

$$\begin{array}{lll} b'(u)f(a(u)) = \sinh\theta & \text{if} & \varepsilon_1 = 1, \\ b'(u)f(a(u)) = -\cosh\theta & \text{if} & \varepsilon_1 = -1. \end{array} \tag{4.7}$$

On the other hand, using (3.1), (4.5), and $a'(u) = \tilde{g}(\varphi_u, \partial_t) = \tilde{g}(\varphi_u, T) = \varepsilon_1 ||T||$ we have $a(u) = u\cosh\theta + c$ for $\varepsilon_1 = 1$ or $a(u) = -u\sinh\theta + c$ for $\varepsilon_1 = -1$, and for some real constant c. Without loss of generality we may assume that $c = 0$. Then, it follows from (4.7) that

$$\begin{array}{lll} b(u) = \sinh\theta \displaystyle\int^u \frac{d\mu}{f(\mu\cosh\theta)} = \tanh\theta \int^{u\cosh\theta} \frac{d\mu}{f(\mu)} & \text{if} & \varepsilon_1 = 1, \\ b(u) = \cosh\theta \displaystyle\int^u \frac{d\mu}{f(-\mu\sinh\theta)} = -\coth\theta \int^{-u\sinh\theta} \frac{d\mu}{f(\mu)} & \text{if} & \varepsilon_1 = -1, \end{array} \tag{4.8}$$

for $\theta > 0$.

If $\varepsilon_1 = 1$ and $\theta = 0$, we have $a(u) = u$ and $b(u) = b_0 > 0$ without loss of generality, where b_0 is a constant. Then, the space-like rotational constant angle surface is a rotational cylinder which is a special case of (i) of Theorem 3.4 when $\alpha(v) = \alpha_0$ is a constant.

If $\varepsilon_1 = -1$ and $\theta = 0$ we have $a(u) = a_0 \in I$ and $b(u) = -(1/f(a_0))u + b_1$, where a_0 and b_1 are constant. Thus, the time-like rotational surface is planar.

Therefore, the space-like and time-like rotational constant angle surfaces of $I \times_f \mathbb{E}^2_1$ with profile curves lying in time-like planes defined by (4.6) are determined.

For $\varepsilon = 1$ the space-like rotational constant angle surface defined by (4.6) is a spacial case (i) of Theorem 3.4 when $\alpha(v) = \alpha_0$ is a constant; for $\varepsilon = -1$ the time-like rotational constant angle surface defined by (4.6) is a special case (ii) of Theorem 3.6 when $\alpha(v) = 0$.

So, considering Theorem 3.4, Theorem 3.6 and the above results, we obtain the followings:

Theorem 4.1 *A space-like rotational surface φ in $I \times_f \mathbb{E}^2_1$ defined by (4.6) is a constant angle surface with a constant Lorentzian angle $\theta \geq 0$ if and only if it is congruent to the space-like surface given by*

$$\varphi(u,v) = \Big(u\cosh\theta, (b_0 + \tanh\theta \int^{u\cosh\theta} \frac{d\mu}{f(\mu)})\sinh v, (b_0 + \tanh\theta \int^{u\cosh\theta} \frac{d\mu}{f(\mu)})\cosh v\Big).$$

Corollary 4.2 *A space-like rotational surface φ defined by (4.2) in the anti-de Sitter space $(\mathbb{H}^3_1(-1), \tilde{g} = dt^2 + e^{2t}(dx^2 - dy^2))$ is a space-like constant angle surface with a constant Lorentzian angle $\theta \geq 0$ if and only if it is congruent to the space-like surface given by*

$$\varphi(u,v) = \Big(u\cosh\theta, (b_0 - e^{-u\cosh u}\tanh\theta)\sinh v, (b_0 - e^{-u\cosh u}\tanh\theta)\cosh v\Big),$$

where b_0 is a constant.

Theorem 4.3 *Nonplanar time-like rotational surfaces in $I \times_f \mathbb{E}^2_1$ defined by (4.2) and (4.6) are constant angle surfaces with a constant Lorentzian angle θ if and only if they are, respectively, congruent to the following surfaces:*

i) The time-like surface given by

$$\varphi(u,v)=\Big(u\sin\theta,(b_0+\cot\theta\int^{u\sin\theta}\frac{d\mu}{f(\mu)})\cosh v,(b_0+\cot\theta\int^{u\sin\theta}\frac{d\mu}{f(\mu)})\sinh v\Big),$$

where b_0 *is a constant and the Lorentzian space-like angle* $\theta\in(0,\pi)$*.*

ii) The time-like surface given by

$$\varphi(u,v)=\Big(-u\sinh\theta,-\coth\theta\sinh v\int^{-u\sinh\theta}\frac{d\mu}{f(\mu)},-\coth\theta\cosh v\int^{-u\sinh\theta}\frac{d\mu}{f(\mu)}\Big),$$

where the Lorentzian time-like angle $\theta>0$*.*

Corollary 4.4 *Nonplanar time-like rotational surfaces* φ *in the anti-de Sitter space* $(\mathbb{H}_1^3(-1),\tilde{g}=dt^2+e^{2t}(dx^2-dy^2))$ *defined by (4.2) and (4.6) are space-like constant angle surfaces with a constant Lorentzian angle* θ *if and only if they are, respectively, congruent to the following surfaces:*

(i) The time-like surface given by

$$\varphi(u,v)=\Big(u\sin\theta,(b_0-\cot\theta e^{-u\sin\theta})\cosh v,(b_0-\cot\theta e^{-u\sin\theta})\sinh v\Big),$$

where b_0 *is a constant and the Lorentzian space-like angle* $\theta\in(0,\pi)$*.*

(ii) The time-like surface given by

$$\varphi(u,v)=\Big(-u\sinh\theta,\coth\theta\sinh v e^{u\sinh\theta},\coth\theta\cosh v e^{u\sinh\theta}\Big),$$

where the Lorentzian time-like angle $\theta>0$*.*

5. Constant angle surfaces in $I\times_f\mathbb{E}_1^2$ with constant Gaussian curvature

We consider surfaces in $I\times_f\mathbb{E}_1^2$ of type (ii) of Theorem 3.4 and of types (iii) and (iv) of Theorem 3.6 to give examples of space-like and time-like constant angle surfaces with constant Gaussian curvature. Using the Gauss equation (3.3) and (3.7) we get the Gaussian curvature as

$$\begin{aligned}K=&\varepsilon_1\varepsilon_2(\varepsilon_\xi\Theta^2-1)\Big(\frac{f'(\varepsilon_1\varepsilon_t||T||u)}{f(\varepsilon_1||T||u)}\Big)^2-\varepsilon_2||T||^2\Big(\frac{f''(\varepsilon_1||T||u)}{f(\varepsilon_1||T||u)}-\Big(\frac{f'(\varepsilon_1||T||u)}{f(\varepsilon_1||T||u)}\Big)^2\Big)\\=&-\varepsilon_2||T||^2\frac{f''(\varepsilon_1||T||u)}{f(\varepsilon_1||T||u)}.\end{aligned}$$

Thus, we have the followings:

1) The space-like constant angle surface (ii) of Theorem 3.4 has constant Gaussian curvature K if and only if $f(t)=a\cosh(\frac{\sqrt{-K}}{\cosh\theta}t+b)$ when $K<0$, or $f(t)=a\cos(\frac{\sqrt{K}}{\cosh\theta}t+b)$ when $K>0$, or $f(t)=at+b$ when $K=0$.

2) The time-like constant angle surface (iii) of Theorem 3.6 has constant Gaussian curvature K if and only if $f(t)=a\cos(\frac{\sqrt{-K}}{\sin\theta}t+b)$ when $K<0$, or $f(t)=a\cosh(\frac{\sqrt{K}}{\sin\theta}t+b)$ when $K>0$, or $f(t)=at+b$ when $K=0$.

3) The time-like constant angle surface (iv) of Theorem 3.6 has constant Gaussian curvature K if and only if $f(t)=a\cosh(\frac{\sqrt{-K}}{\sinh\theta}t+b)$ when $K<0$, or $f(t)=a\cos(\frac{\sqrt{K}}{\sinh\theta}t+b)$ when $K>0$, or $f(t)=at+b$ when $K=0$, where $a\neq 0,b$ are constants.

6. Constant angle surfaces in $I \times_f \mathbb{E}^2_1$ with zero mean curvature

In this section we classify constant angle surfaces in $I \times_f \mathbb{E}^2_1$ with zero mean curvature. Let φ be a time-like constant angle immersion of type (v) in Theorem 3.6. Then, ∂_t is a normal to the surface $\varphi(M_1) = \{t_0\} \times_f \mathbb{E}^2_1$, and it follows from (2.1b) that the surface $\varphi(M_0)$ is totally umbilical and the shape operator $A = \frac{f'(t_0)}{f(t_0)} I_1$, where I_1 is the identity transformation on $\mathbb{E}^2_1$. Thus, such a surface has zero mean curvature if and only if $f'(t_0) = 0$, that is, it is also totally geodesic.

Now, let the space-like constant angle surface be of type (ii) of Theorem 3.4. Then, it is maximal only if it is totally geodesic. Since the principal curvatures are of the form $\lambda = \sinh\theta f'(u\cosh\theta)/f(u\cosh\theta)$, we have either $\theta = 0$, i.e. the surface is a warped product of the interval I and a space-like straight line in $\mathbb{E}^2_1$, or $f' = 0$, that is, the ambient space is a direct product, and M_0 is a space-like plane in $\mathbb{E}^3_1$.

Similarly, we assume that the time-like constant angle surface is of type (iii) or (iv) of Theorem 3.6. Then, it has zero mean curvature only if it is totally geodesic. When we consider the principal curvatures of the surface we have $f' = 0$, i.e. the ambient space is a direct product, and M_1 is a time-like plane in $\mathbb{E}^3_1$.

Finally, we assume that the space-like constant angle surface is of type (i) of the Theorem 3.4, or the time-like constant angle surface is of type (i) or (ii) of the Theorem 3.6. Then, from (3.7) and (3.56) it follows that $H = 0$ if and only if

$$2\Theta\beta(u,v)\sigma'(u) = -\varepsilon_2 ||T||^2, \tag{6.1}$$

which implies that β depends only on u. From (3.30) and (3.46), we can write $\beta_u - \sigma'(u)\beta = \Theta$. Using this and differentiating (6.1) we get

$$\Big(\frac{1}{\sigma'}\Big)' = \frac{\varepsilon_1 - \varepsilon_2\Theta^2}{||T||^2}. \tag{6.2}$$

Integrating (6.2) and using (3.15) we obtain that

$$f(t) = b\Big(\frac{\varepsilon_1}{||T||} t + c\Big)^m, \tag{6.3}$$

where $m = \frac{||T||^2}{\varepsilon_1 - \varepsilon_2\Theta^2}$, and $b \neq 0$ and c are constants. Without loss of generality we can assume that $c = 0$. Hence we have

$$\int^{\varepsilon_1 ||T|| u} \frac{d\tau}{f(\tau)} = \frac{\varepsilon_1 ||T||}{b(1-m)u^{m-1}}. \tag{6.4}$$

From (3.56) and (6.1) we obtain that $\beta(u) = -\varepsilon_2 \frac{||T||^2 u}{2m\Theta^2}$ and $\lambda(u) = \varepsilon_1\varepsilon_\xi \frac{m\Theta}{||T||u}$. Also, considering $\varepsilon_1\varepsilon_2\varepsilon_\xi = -1$ and $\sigma'(u) = \frac{m}{u}$, it can be shown that

$$\beta(u,v)e^{-\sigma(u)} - \Theta \int^u e^{-\sigma(\tau)} d\tau = 0. \tag{6.5}$$

Therefore we have $\alpha(v) = 0$ in (3.31) and (3.47).

Case 1. M_q is space-like in $I \times_f \mathbb{E}^2_1$. Then, we have $\varepsilon_1 = \varepsilon_2 = 1$ and $\varepsilon_\xi = -1$, and hence $\Theta = \sinh\theta$ and $||T|| = \cosh\theta$. Therefore, from (3.23) and (6.4) we get

$$\varphi(u,v)=\Big(u\cosh\theta,\frac{\sinh\theta\sinh v}{b(1-m)u^{m-1}},\frac{\sinh\theta\cosh v}{b(1-m)u^{m-1}}\Big),\tag{6.6}$$

where $m=\frac{\cosh^2\theta}{1-\sinh^2\theta}$ and $u>0$ when $\theta\in(0,\ln(1+\sqrt{2}))$, and $u\in\mathbb{R}$ when $\theta>\ln(1+\sqrt{2})$.

Case 2. M_q is time-like in $I\times_f\mathbb{E}^2_1$. Then, we have two cases as $\varepsilon_1=\varepsilon_\xi=1$, $\varepsilon_2=-1$, or $\varepsilon_1=-1$, $\varepsilon_2=\varepsilon_\xi=1$.

Subcase 2.1. $\varepsilon_1=\varepsilon_\xi=1$ and $\varepsilon_2=-1$. Then, $\Theta=\cos\theta$ and $||T||=\sin\theta$. Therefore, from (3.38) and (6.4) we get

$$\varphi(u,v)=\Big(u\sin\theta,\frac{u^{1-m}\cos\theta\cosh v}{b(1-m)},\frac{u^{1-m}\cos\theta\sinh v}{b(1-m)}\Big),\tag{6.7}$$

where $m=\frac{\sin^2\theta}{1+\cos^2\theta}\in(0,1)$.

Subcase 2.2. $\varepsilon_1=-1$ and $\varepsilon_2=\varepsilon_\xi=-1$. Then, $\Theta=\cosh\theta$ and $||T||=\sinh\theta$. Therefore, from (3.39) and (6.4) we get

$$\varphi(u,v)=\Big(-u\sinh\theta,\frac{u^{1-m}\cosh\theta\sinh v}{b(1-m)},\frac{u^{1-m}\cosh\theta\cosh v}{b(1-m)}\Big),\tag{6.8}$$

where $m=-\frac{\sinh^2\theta}{1+\cosh^2\theta}\in(-1,0)$.

All these surfaces are rotational.

Finally, we state the followings by considering Theorems 3.4 and 3.6.

Theorem 6.1 *Let $\varphi:M_0\longrightarrow I\times_f\mathbb{E}^2_1$ be a space-like immersion, and assume that it is not totally geodesic. Then, φ is constant angle and maximal if and only if the warping function is $f(t)=bt^m$ and φ is a space-like rotational immersion given by*

$$\varphi(u,v)=\Big(u\cosh\theta,\frac{\sinh\theta\sinh v}{b(1-m)u^{m-1}},\frac{\sinh\theta\cosh v}{b(1-m)u^{m-1}}\Big),$$

where $m=\frac{\cosh^2\theta}{1-\sinh^2\theta}$ and $u>0$ when $\theta\in(0,\ln(1+\sqrt{2}))$, and $u\in\mathbb{R}$ when $\theta>\ln(1+\sqrt{2})$.

Theorem 6.2 *Let $\varphi:M_0\longrightarrow I\times_f\mathbb{E}^2_1$ be a space-like immersion, and assume that it is not totally geodesic. Then, φ is constant angle and has zero mean curvature if and only if the warping function $f(t)=bt^m$ and φ is one of the following time-like rotational immersions:*

i)

$$\varphi(u,v)=\Big(u\sin\theta,\frac{u^{1-m}\cos\theta\cosh v}{b(1-m)},\frac{u^{1-m}\cos\theta\sinh v}{b(1-m)}\Big),$$

where $m=\frac{\sin^2\theta}{1+\cos^2\theta}\in(0,1)$;

ii)

$$\varphi(u,v)=\Big(-u\sinh\theta,\frac{u^{1-m}\cosh\theta\sinh v}{b(1-m)},\frac{u^{1-m}\cosh\theta\cosh v}{b(1-m)}\Big),$$

where $m=-\frac{\sinh^2\theta}{1+\cosh^2\theta}\in(-1,0)$.

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