Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article

Turk J Math
(2022) 46: 3192 – 3200
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doi:10.55730/1300-0098.3327

Novel exact solutions to Navier–Stokes momentum equations describing an incompressible fluid

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Received: 01.09.2021 • **Accepted/Published Online:** 16.09.2022 • **Final Version:** 09.11.2022

Abstract: An analytical solution to the incompressible Navier–Stokes momentum equations for a divergence-free flow $\boldsymbol{\nabla} \cdot \vec{u}\,(\vec{x}, t) = 0$ with time-dependent dynamic viscosity $\mu\,(t)$ is presented. The demonstrated methodology holds for the physically relevent three dimensions. The constructed flow velocities $\vec{u}\,(\vec{x}, t)$ are eigenvectors of the vector operator curl. Moreover, vortex $\vec{\omega}\,(\vec{x}, t)$, helicity $H\,(\vec{x}, t)$, enstrophy $\mathcal{E}\,(t)$ and enstrophy evolution $\frac{\mathrm{d}\mathcal{E}(t)}{\mathrm{d}t}$ are explicitly determined.

Key words: Flow behavior, fluid dynamics, partial differential equations

1. Introduction

Flows are of major interest for scientific and engineering applications. Important exemplary utilizations include blood flows, astrophysics, plasma magneto-hydrodynamics, pollution analyses, ocean currents, and weather forecasting as well as automobiles, air conditioners, fans, water flows in pipes and power station designs [2, 14, 21, 22, 26, 28]. Especially in the aerospace industry flows play an essential role since the behavior around the wing is decisive for the flight attitude of aircraft. However, obtaining quantitative results is extremely complex since the mathematical structure of the underlying equations is quite involved. As a consequence, research on fluid mechanics is one of the main operation domains for high-performance computers [29, 35].

Flows of linear-viscous Newtonian fluids are characterized by the Navier–Stokes momentum equations obtained from the conservation laws for energy, momentum and mass whereas the Euler equations are describing inviscid flows [11, 12], which can equally be derived from self-gravitating dust matter hydrodynamic equations if pressure is included [1]. Basically, the Navier–Stokes momentum equations declare Newton's second law for fluid motion in combination with the observation that the internal stress within a fluid correlates to pressure and the diffusive viscous term. In addition, magnetohydrodynamics can be studied by Navier–Stokes momentum equations if they are coupled with Maxwell's equations. Note that turbulence theory is also an important area of fluid mechanics studied by Navier–Stokes momentum equations [18].

The modern approach for studying Navier–Stokes momentum equations by numerical methods is the adoption of advanced computational fluid dynamics, which would be unreasonably time-consuming in the absence of scientific computing. Nonetheless, in some functional utilizations, despite this advanced numerical ansatz obtaining results might get too complex such that investigations depend on statistical approaches for

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2010 *AMS Mathematics Subject Classification:* 35Qxx, 35Q30, 76D05

constructing solutions to the equations. Moreover, quantum computing and direct simulation Monte Carlo approaches are of major relevance [10, 16]. A separate method is the analytical investigation of the associated equations for deriving elementary characteristics such as the existence and smoothness of Navier–Stokes solutions on $\mathbb{R}^3$ or $\mathbb{R}^3/\mathbb{Z}^3$. Accordingly, constructing exact solutions is quite relevant and was studied for special cases [15, 17, 20, 36, 37]. Furthermore, practical applications like designing an aircraft by analytical results is of great relevance [30].

The purpose of this study is the investigation and construction of a novel exact solution class to the incompressible Navier–Stokes momentum equations with time-dependent dynamic viscosity and divergence-free flow $\vec{u}(\vec{x},t)$. In conjunction with the flow $\vec{u}(\vec{x},t)$, analytical results for the vortex $\vec{\omega}(\vec{x},t)$, helicity $H(\vec{x},t)$, enstrophy $\mathcal{E}(t)$ and enstrophy evolution $\frac{\mathrm{d}\mathcal{E}(t)}{\mathrm{d}t}$ were derived. The class of solutions is related to well-known [4–6, 34] Beltrami vector fields. Correspondingly, various Beltrami vector fields are derived for the purpose of constructing Navier–Stokes solutions.

Exact solutions are of particular interest since they can be used for stability analyses and control purposes of numerical solutions while also revealing relationships between various physical parameters. A number of exact solutions are well-known [31, 32], whereby typically three approaches (similarity solutions, consideration of basically unidirectional flows or Beltrami flows) were used to derive these. Exact solutions based on Beltrami flows were generally used in the literature for cases where (i) two of the three components of the vortex are equal to zero corresponding to planar or axisymmetric two-dimensional flows and zero cross-flows [13], (ii) pseudo-plane flows are considered and only the third component vanishes [24, 25], (iii) the third component of the vortex is determined by the streamfunction while an addition of an auxiliary function is performed [33] and generalizations are considered [3, 7], (iv) Beltrami-Trkal flows are of relevance [19, 27]. Typically, a constant dynamic viscosity is considered in these cases. In contrast, this study thematises time-dependent dynamic viscosities in combination with a gravitational potential. Moreover, the presented vortex has in general nonvanishing three components. In particular, the solutions are generalizations of Trkalian fluid flows. Furthermore, to the best of our knowledge, four of the five presented expressions for Beltrami vector fields are novel.

2. Construction of various Beltrami vector fields

A three-dimensional Beltrami vector field $\vec{F}(\vec{x})$, which is parallel to its own curl [8] and thus satisfies

$$\boldsymbol{\nabla} \times \vec{F}(\vec{x}) = \lambda \vec{F}(\vec{x}), \tag{2.1}$$

where λ is a constant, can be expressed by

$$\vec{F}(\vec{x}) = \boldsymbol{\nabla} \times \vec{\psi}(\vec{x}) + \frac{1}{\lambda}\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \vec{\psi}(\vec{x}), \qquad \triangle\vec{\psi}(\vec{x}) = -\lambda^2\vec{\psi}(\vec{x}), \tag{2.2}$$

while $\vec{\psi}(\vec{x})$ satisfies the Helmholtz equation [5]. Solving this equation in spherical coordinates (r,θ,φ) is well-known and yields

$$\vec{\psi}(r,\theta,\varphi) = \sum_{k=1}^{3}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(a_{klm}j_l(\lambda r) + b_{klm}y_l(\lambda r)\right)Y_l^m(\theta,\varphi)\,\vec{\mathrm{e}}_k, \tag{2.3}$$

where a_{klm} as well as b_{klm} are arbitrary constants, $j_l(x)$ as well as $y_l(x)$ are spherical Bessel functions, $Y_l^m(\theta,\varphi)$ are spherical harmonics and the set of vectors $\{\vec{\mathrm{e}}_1,\vec{\mathrm{e}}_2,\vec{\mathrm{e}}_3\}$ is the standard basis of the three-dimensional space $\mathbb{R}^3$.

Reinserting this expression in Eq. (2.2) while assuming spherical symmetry, i.e. independence from azimuthal angle θ as well as polar angle φ, and setting $\frac{a_{k00}}{2\sqrt{\pi}\lambda} = c_1 a_k$, $\frac{b_{k00}}{2\sqrt{\pi}\lambda} = c_2 a_k$ as well as $\psi(r) = c_1 \frac{\sin(\lambda r)}{r} + c_2 \frac{\cos(\lambda r)}{r}$ results in

$$\vec{F}(r,\theta,\varphi) = \psi'(r)\,\vec{\mathrm{e}}_r \times \vec{a} + \left(\lambda\psi(r) + \frac{\psi'(r)}{\lambda r}\right)\vec{a} + \frac{1}{\lambda}\left(\psi''(r) - \frac{\psi'(r)}{r}\right)(\vec{\mathrm{e}}_r\cdot\vec{a})\,\vec{\mathrm{e}}_r, \tag{2.4}$$

where the set of vectors $\{\vec{\mathrm{e}}_r, \vec{\mathrm{e}}_\theta, \vec{\mathrm{e}}_\varphi\}$ is the basis given by spherical unit vectors. Note that $\vec{F}(r,\theta,\varphi)$ is rotationally invariant. Moreover, it is linear with regard to $\vec{a} = \sum_{k=1}^{3} a_k \vec{\mathrm{e}}_k$, i.e. a linear combination of these Beltrami vector fields with varying vectors $\vec{a}_j$ leads again to a Beltrami vector field containing the same symmetry. Due to the relevance of $\vec{a}$ these class of Beltrami vector fields are denoted as $\vec{F}_1(\vec{x},\vec{a})$. These vector fields are well-known for the case $c_2 = 0$ and are applied in plasma physics for studying optimal force-free spherical plasma configurations [23].

In addition, $\vec{F}_2(\vec{x},\vec{a}) = \boldsymbol{\nabla}_{\vec{a}}\vec{F}_1(\vec{x},\vec{a})$ is also a Beltrami vector field with rotational symmetry around the axis $\vec{a}$. It simplifies to

$$\begin{aligned}\vec{F}_2(\vec{x},\vec{a}) = &\left(\psi''(r) - \frac{\psi'(r)}{r}\right)(\vec{\mathrm{e}}_r\cdot\vec{a})\,\vec{\mathrm{e}}_r\times\vec{a} + \left(\lambda\psi'(r) + \frac{2}{\lambda r}\left(\psi''(r) - \frac{\psi'(r)}{r}\right)\right)(\vec{\mathrm{e}}_r\cdot\vec{a})\,\vec{a}\\ &+ \frac{1}{\lambda}\left(\left(\frac{3}{r^2}\psi'(r) - \frac{3}{r}\psi''(r) + \psi'''(r)\right)(\vec{\mathrm{e}}_r\cdot\vec{a})^2 + \left(\psi''(r) - \frac{\psi'(r)}{r}\right)\frac{\vec{a}^2}{r}\right)\vec{\mathrm{e}}_r\end{aligned} \tag{2.5}$$

while

$$\vec{\mathrm{e}}_r\cdot\vec{F}_2(\vec{x},\vec{a})\Big|_{\lambda=1} = \frac{1}{r^3}\left(\frac{3-r^2}{r}\sin r - 3\cos r\right)\left(\vec{a}^2 - 3(\vec{\mathrm{e}}_r\cdot\vec{a})^2\right) \tag{2.6}$$

holds for $\lambda = 1$. Thus, all spheres with radii r_j satisfying $\frac{3-r_j^2}{r_j}\sin r_j - 3\cos r_j = 0$ are invariant submanifolds for any Beltrami vector field $\vec{F}_2(\vec{x},\vec{a})\Big|_{\lambda=1}$. Note that a linear combination of $\vec{F}_1(\vec{x},\vec{a})$ with arbitrary vectors $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_3$ results in an axisymmetric Beltrami vector field while this property does not hold for $\vec{F}_2(\vec{x},\vec{a})$.

Furthermore, a third class of Beltrami vector fields is given by

$$\vec{F}_3\left(\vec{x},\vec{a},\vec{b}\right) = \vec{F}_2\left(\vec{x},\vec{a}+\vec{b}\right)\Big|_{\lambda=1} - \vec{F}_2(\vec{x},\vec{a})\Big|_{\lambda=1} - \vec{F}_2\left(\vec{x},\vec{b}\right)\Big|_{\lambda=1} \tag{2.7}$$

and simplifies to

$$\begin{aligned}\vec{F}_3\left(\vec{x},\vec{a},\vec{b}\right) =&\frac{1}{r^2}\left(\frac{3-r^2}{r}\sin r - 3\cos r\right)\left((\vec{\mathrm{e}}_r\cdot\vec{a})\,\vec{\mathrm{e}}_r\times\vec{b} + \left(\vec{\mathrm{e}}_r\cdot\vec{b}\right)\vec{\mathrm{e}}_r\times\vec{a}\right)\\ &+\frac{2}{r^3}\left(\left(\frac{6r^2-15}{r}\sin r - \left(r^2-15\right)\cos r\right)(\vec{\mathrm{e}}_r\cdot\vec{a})\left(\vec{\mathrm{e}}_r\cdot\vec{b}\right)\right.\\ &\left.+\left(\frac{3-r^2}{r}\sin r - 3\cos r\right)\left(\vec{a}\cdot\vec{b}\right)\right)\vec{\mathrm{e}}_r\\ &+\frac{1}{r^3}\left(\frac{6-3r^2}{r}\sin r + \left(r^2-6\right)\cos r\right)\left((\vec{\mathrm{e}}_r\cdot\vec{a})\,\vec{b} + \left(\vec{\mathrm{e}}_r\cdot\vec{b}\right)\vec{a}\right),\end{aligned} \tag{2.8}$$

where similar to $\vec{F}_2(\vec{x},\vec{a})$

$$\vec{\mathrm{e}}_r\cdot\vec{F}_3\left(\vec{x},\vec{a},\vec{b}\right)=\frac{2}{r^3}\left(\frac{3-r^2}{r}\sin r-3\cos r\right)\left(\vec{a}\cdot\vec{b}-3\left(\vec{\mathrm{e}}_r\cdot\vec{a}\right)\left(\vec{\mathrm{e}}_r\cdot\vec{b}\right)\right)\tag{2.9}$$

holds. Hence, all spheres with radii r_j satisfying $\frac{3-r_j^2}{r_j}\sin r_j-3\cos r_j=0$ are also invariant submanifolds for any Beltrami vector field $\vec{F}_3\left(\vec{x},\vec{a},\vec{b}\right)$.

Moreover, another Beltrami vector field is given by

$$\vec{F}_4(\vec{x},\xi,\zeta)=\vec{F}_2(\vec{x},\vec{\mathrm{e}}_3)\Big|_{\lambda=1}+\xi\,\vec{F}_2(\vec{x},\vec{\mathrm{e}}_1)\Big|_{\lambda=1}+\zeta\vec{F}_3(\vec{x},\vec{\mathrm{e}}_2,\vec{\mathrm{e}}_3)\tag{2.10}$$

and simplifies to

$$\begin{aligned}\vec{F}_4(\vec{x},\xi,\zeta)=&\frac{1}{r^4}\Bigg(\left(\frac{3-r^2}{r}\sin r-3\cos r\right)\left((1-2\xi)x+yz+\zeta\left(y^2-z^2\right)\right)\\&+\frac{x}{r^2}\left(\frac{6r^2-15}{r}\sin r-\left(r^2-15\right)\cos r\right)\\&\cdot\left(z^2-\xi\left(y^2+z^2\right)+2\zeta yz\right)\Bigg)\vec{\mathrm{e}}_1\\&+\frac{1}{r^4}\Bigg(\left(\frac{3-r^2}{r}\sin r-3\cos r\right)\left((1+\xi)y-(1-\xi)xz-\zeta(3z+xy)\right)\\&+\frac{1}{r^2}\left(\frac{6r^2-15}{r}\sin r-\left(r^2-15\right)\cos r\right)\\&\cdot\left(\left(\xi x^2+z^2\right)y+\zeta z\left(y^2-x^2-z^2\right)\right)\Bigg)\vec{\mathrm{e}}_2\\&+\frac{1}{r^4}\Bigg(\left(\frac{3-r^2}{r}\sin r-3\cos r\right)\left((\xi-2)z-\xi xy-\zeta(3y-xz)\right)\\&+\frac{1}{r^2}\left(\frac{6r^2-15}{r}\sin r-\left(r^2-15\right)\cos r\right)\\&\cdot\left(\left((\xi-1)x^2-y^2\right)z+\zeta y\left(z^2-x^2-y^2\right)\right)\Bigg)\vec{\mathrm{e}}_3,\end{aligned}\tag{2.11}$$

where ξ as well as ζ are arbitrary constants and $r=\sqrt{x^2+y^2+z^2}$. Note that expression (2.11) is written down in Cartesian coordinates in contrast to Eqs. (2.5) and (2.8) formulated in spherical coordinates because the formulation in (2.11) becomes even more bulky otherwise. Moreover, the expression exhibits no symmetry. Furthermore, in analogy to the previous vector fields all spheres with radii r_j satisfying $\frac{3-r_j^2}{r_j}\sin r_j-3\cos r_j=0$ are invariant submanifolds for any Beltrami vector field $\vec{F}_4(\vec{x},\xi,\zeta)$ since

$$\begin{aligned}\vec{\mathrm{e}}_r\cdot\vec{F}_4(\vec{x},\xi,\zeta)=&\frac{1}{r^3}\left(\frac{3-r^2}{r}\sin r-3\cos r\right)\\&\cdot\left((\xi-2)\cos^2\theta-\frac{\xi-2+3\xi\cos(2\varphi)}{2}\sin^2\theta-3\zeta\sin(2\theta)\sin\varphi\right)\end{aligned}\tag{2.12}$$

holds. Therefore, spheres with radii $0 \le r \le r_j$ and spherical shells characterized by $r_j \le r \le r_{1+j}$ are compact invariant regions if r_j are sorted in an increasing manner for increasing j.

Note that any kind of Beltrami vector field can be expressed by

$$\vec{F}_5(\vec{x}) = \int_{S^2} \mathrm{d}^2 k \left(\cos\left(\lambda \vec{k} \cdot \vec{x}\right) \vec{k} \times \vec{T}\left(\vec{k}\right) + \sin\left(\lambda \vec{k} \cdot \vec{x}\right) \vec{T}\left(\vec{k}\right)\right), \tag{2.13}$$

where the integral is taken over the unit sphere S^2 and $\vec{k}^2 = 1$. Moreover, $\vec{T}\left(\vec{k}\right)$ is a smooth arbitrarily chosen vector field tangent, i.e. $\vec{k} \cdot \vec{T}\left(\vec{k}\right) = 0$. Remarkably, this vector field does not possess any symmetries. In addition, its norm has an upper boundary given by

$$\left|\vec{F}_5(\vec{x})\right| \le \int_{S^2} \mathrm{d}^2 k \left|\vec{T}\left(\vec{k}\right)\right| \tag{2.14}$$

due to the norm $\vec{k}^2 = 1$ and orthogonality $\vec{k} \cdot \vec{T}\left(\vec{k}\right) = 0$.

The presented vector fields $\vec{F}_1(\vec{x}, \vec{a})$, $\vec{F}_2(\vec{x}, \vec{a})$, $\vec{F}_3\left(\vec{x}, \vec{a}, \vec{b}\right)$, $\vec{F}_4(\vec{x}, \xi, \zeta)$ and $\vec{F}_5(\vec{x})$ satisfy

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \vec{F}(x) = \lambda^2 \vec{F}(x) \tag{2.15}$$

and are therefore indeed Beltrami vector fields. Moreover, Eq. (2.13) can be derived by use of the Fourier method. In addition, all presented Beltrami vector fields satisfy (2.1).

3. Constructing solutions to the incompressible Navier–Stokes momentum equations for divergence-free flows

The incompressible Navier–Stokes momentum equation for divergence-free flows $\boldsymbol{\nabla} \cdot \vec{u}(\vec{x}, t) = 0$, where $\vec{x}$ is the position vector and t the time, is given by

$$\boldsymbol{\partial}_t \vec{u}(\vec{x}, t) + (\vec{u}(\vec{x}, t) \cdot \boldsymbol{\nabla})\, \vec{u}(\vec{x}, t) = -\frac{1}{\rho} \boldsymbol{\nabla} p(\vec{x}, t) + \frac{\mu(t)}{\rho} \triangle \vec{u}(\vec{x}, t) + \boldsymbol{\nabla} \varphi(\vec{x}) \tag{3.1}$$

with constant density ρ, pressure $p(\vec{x}, t)$, time-dependent dynamic viscosity $\mu(t)$ and gravitational potential $\varphi(\vec{x})$.

Assuming separation of variables for the flow velocity $\vec{u}(\vec{x}, t)$, i.e.

$$\vec{u}(\vec{x}, t) = f(t)\, \vec{U}(\vec{x}) \tag{3.2}$$

with $\vec{U}(\vec{x})$ being a Beltrami vector field, and

$$p(\vec{x}, t) = c_3 + \rho \varphi(\vec{x}) - \frac{\rho}{2} \vec{u}^2(\vec{x}, t) \tag{3.3}$$

as pressure simplifies the Navier–Stokes momentum equation (3.1) to

$$\triangle \vec{u}(\vec{x}, t) = -\lambda^2 f(t)\, \vec{U}(\vec{x}), \qquad f'(t) = -\frac{\lambda^2}{\rho} \mu(t)\, f(t). \tag{3.4}$$

Thus, $f(t)$ is given by

$$f(t) = c_4\, \mathrm{e}^{-\frac{\lambda^2}{\rho}\int_0^t \mathrm{d}\tau\, \mu(\tau)}, \tag{3.5}$$

where c_3 and c_4 are arbitrary constants. Hence, the flow velocity (3.2) in combination with Eq. (3.5) and the pressure (3.3) represent a novel class of solutions to the incompressible Navier–Stokes momentum equation for divergence-free flows $\vec{u}(\vec{x},t)$. Note that any kind of Beltrami vector field can be chosen in this class of solutions while the exponential part in Eq. (3.5) leads to damping depending on λ, density ρ and dynamic viscosity $\mu(t)$.

For our further purposes we consider the Beltrami vector fields $\vec{F}_4(\vec{x},\xi,\zeta)$ and $\vec{F}_5(\vec{x})$, i.e. $\vec{U}(\vec{x})$ is replaced by $\vec{F}_4(\vec{x},\xi,\zeta)$ and $\vec{F}_5(\vec{x})$. On one hand, in the case of the Beltrami vector field $\vec{F}_5(\vec{x})$ the solution of the incompressible Navier-Stokes momentum equation involves two arbitrary piecewise continuous functions expressed in Eq. (2.13) and does not have geometric symmetries, which is unusual in comparison with previously known exact solutions [15, 17, 20, 31, 36, 37]. Furthermore, if $\mu(t) \geq \tilde{\mu} > 0$ holds, exact solutions (3.2) involve the constants $\tilde{\mu}$, λ and $\int_{S^2} \mathrm{d}^2k \left|\vec{T}\left(\vec{k}\right)\right|$ which define the Reynolds number

$$\mathrm{Re} = \frac{1}{\tilde{\mu}\,|\lambda|}\int\limits_{S^2} \mathrm{d}^2k \left|\vec{T}\left(\vec{k}\right)\right|. \tag{3.6}$$

Furthermore, considering the norm of the velocity $\vec{u}(\vec{x},t)$ leads to

$$|\vec{u}(\vec{x},t)| \leq \tilde{\mu}\,|\lambda|\ \mathrm{Re}\,\mathrm{e}^{-\frac{\lambda^2\tilde{\mu}t}{\rho}} \tag{3.7}$$

$$\int\limits_0^\infty \mathrm{d}t\ |\vec{u}(\vec{x}(t),t)| \leq \frac{\rho}{|\lambda|}\,\mathrm{Re} \tag{3.8}$$

with $\vec{u}(\vec{x},t) = \dot{\vec{x}}(t)$ due to Eqs. (2.14), (3.2), (3.5) and (3.6). Thus, both expressions in Eqs. (3.7) and (3.8) have a finite upper bound.

On the other hand, choosing $\vec{F}_4(\vec{x},\xi,\zeta)$ as Beltrami vector field in Eq. (3.2) in combination with $r(t) = |\vec{x}(t)|$ leads to

$$\begin{aligned}\dot{r}(t) =& \frac{c_4}{r^4(t)}\mathrm{e}^{-\frac{\lambda^2}{\rho}\int_0^t \mathrm{d}\tau\,\mu(\tau)}\left(\frac{3-r^2(t)}{r(t)}\sin r(t) - 3\cos r(t)\right)\\ &\cdot\left((\xi-2)\cos^2\theta(t) - \frac{\xi-2+3\xi\cos(2\varphi(t))}{2}\sin^2\theta(t) - 3\zeta\sin(2\theta(t))\sin\varphi(t)\right),\end{aligned} \tag{3.9}$$

which describes the dynamical system with regard to the radius $r(t)$ and depends on the arbitrary parameters c_4, ξ and ζ. However, for $\varphi(t) = 0$ the dependence on ζ vanishes. Obviously, the system has in general no symmetries or conserved quantities. Moreover, all spheres with radii r_j satisfying $\frac{3-r_j^2}{r_j}\sin r_j - 3\cos r_j = 0$ are invariant submanifolds of this system. Thus, the radii of these spheres satisfy $\tan r_j = \frac{3r_j}{3-r_j^2}$ which has infinitely many roots. Any sphere with $0 \leq r \leq r_j$ and any spherical shell with $r_j \leq r \leq r_{1+j}$ are invariant compact regions to the obtained dynamical system. Note that $\tan r_j = \frac{3r_j}{3-r_j^2}$ simplifies to $0 = -\frac{3}{\pi j} + \mathcal{O}\left(j^{-3}\right)$ for $r_j = \pi j$ which shows that the radii are asymptotically $(j \to \infty)$ given by $r_j = \pi j$.

In addition to the flow $\vec{u}(\vec{x},t)$ the vortex $\vec{\omega}(\vec{x},t)$ and the helicity $H(\vec{x},t)$ are explicitly determined as

$$\vec{\omega}(\vec{x},t) = \boldsymbol{\nabla} \times \vec{u}(\vec{x},t) = \lambda f(t)\,\vec{U}(\vec{x}), \qquad H(\vec{x},t) = \vec{u}(\vec{x},t)\cdot\vec{\omega}(\vec{x},t) = \lambda f^2(t)\,\vec{U}^2(\vec{x}) \tag{3.10}$$

while the enstrophy $\mathcal{E}(t)$ and its evolution $\frac{\mathrm{d}\mathcal{E}(t)}{\mathrm{d}t}$ simplify to

$$\mathcal{E}(t) = \int_{\mathbb{R}^3} \mathrm{d}^3x\,\vec{\omega}^2(\vec{x},t) = \lambda^2 f^2(t) \int_{\mathbb{R}^3} \mathrm{d}^3x\,\vec{U}^2(\vec{x}), \tag{3.11}$$

$$\frac{\mathrm{d}\mathcal{E}(t)}{\mathrm{d}t} = -\frac{2\lambda^4}{\rho}\mu(t)\,f^2(t)\int_{\mathbb{R}^3} \mathrm{d}^3x\,\vec{U}^2(\vec{x}). \tag{3.12}$$

Remarkably, solutions for enstrophy are generally constructed by studying a closure problem derived by use of the nonlinearity. Nonetheless, the analytical expression (3.2) grants permission to a direct analysis, leading to the expressions for the enstrophy (3.11) and the enstrophy evolution (3.12). Note that due to the construction, the strong Beltrami property, i.e. $\vec{\omega}(\vec{x},t)\times\vec{u}(\vec{x},t) = 0$, holds. Furthermore, the sign of the helicity $H(\vec{x},t)$ obviously depends just on the sign of λ. Thus, the chirality of the flow is completely determined by λ which is a constant embedded in the framework of Beltrami vector fields as denoted in Eq. (2.1).

Note that in agreement with the literature [9, 37] all presented solutions for the flow velocity $\vec{u}(\vec{x},t)$, vortex $\vec{\omega}(\vec{x},t)$, helicity $H(\vec{x},t)$, enstrophy $\mathcal{E}(t)$ and enstrophy evolution $\frac{\mathrm{d}\mathcal{E}(t)}{\mathrm{d}t}$ show no singularity or blow-up with regard to the time t (if the dynamic viscosity $\mu(t)$ is integrable) and are smooth since the flow velocity is divergence-free, i.e. $\boldsymbol{\nabla}\cdot\vec{u}(\vec{x},t) = 0$. Moreover, solution (3.2) is a generalization of well-known [27] Trkalian fluid flows characterized by a constant dynamic viscosity μ.

4. Conclusions

A three-dimensional exact solution to the incompressible Navier–Stokes momentum equations for divergence-free fluids with time-dependent dynamic viscosity $\mu(t)$ was derived. Analytic results for flows $\vec{u}(\vec{x},t)$ that generalize Trkalian fluid flows, i.e. flows with $f(t) = c_4\,\mathrm{e}^{-\frac{\lambda^2\mu t}{\rho}}$, in combination with expressions for the vortex $\vec{\omega}(\vec{x},t)$, helicity $H(\vec{x},t)$, enstrophy $\mathcal{E}(t)$ and enstrophy evolution $\frac{\mathrm{d}\mathcal{E}(t)}{\mathrm{d}t}$ were obtained. All results are smooth with regard to the time t as expected due to the flow $\vec{u}(\vec{x},t)$ being divergence-free.

Acknowledgment

The author acknowledges financial support by the Scientific and Technological Research Council of Turkey within program 1515 for research and laboratory developments within project no. 5189901. Moreover, he is grateful to the R&D and Technology Directorate at Turkish Aerospace, the Institute of Materials Science and Nanotechnology at Bilkent University, and the Department of Engineering Sciences at Middle East Technical University for providing the needed infrastructure.

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