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Infinitely many positive solutions for an iterative system of conformable fractional order dynamic boundary value problems on time scales

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Abstract: In this paper, we establish infinitely many positive solutions for the iterative system of conformable fractional order dynamic equations on time scales

$$\mathcal{T}_{\alpha}^{\Delta}\left[\mathcal{T}_{\beta}^{\Delta}\left(\vartheta_{n}(t)\right)\right] = \varphi(t)\mathtt{f}_{n}\left(\vartheta_{n+1}(t)\right), \ t \in (0,1)_{\mathbb{T}}, \ 1 < \alpha, \beta \leq 2,$$
$$\vartheta_1(t) = \vartheta_{\ell+1}(t), \ t \in (0,1)_{\mathbb{T}}, \ n = 1, 2, \cdots, \ell,$$

satisfying two-point Riemann–Stieltjes integral boundary conditions

$$\vartheta_{n}(0) = 0, \ \vartheta_{n}(1) = \int_0^1 \vartheta_{n}(\tau)\Box\mathtt{g}(\tau), \ n = 1, 2, \cdots, \ell,$$
$$(\mathcal{T}_{\beta}^{\Delta}\vartheta_{n})(0) = 0, \ (\mathcal{T}_{\beta}^{\Delta}\vartheta_{n})(1) = \int_0^1 (\mathcal{T}_{\beta}^{\Delta}\vartheta_{n})(\tau)\Box\mathtt{g}(\tau), \ n = 1, 2, \cdots, \ell,$$

where $\mathcal{T}_{\star}^{\Delta}$ denotes the conformable fractional derivative of order $\star \in \{\alpha, \beta\}$ on time scale $\mathbb{T}$, by an application of Krasnoselskii's fixed point theorem on a Banach space.

Key words: Conformable fractional derivative, time scale, positive solution, fixed point theorem, cone

1. Introduction

Fractional calculus is a generalization of classical integer order calculus and has gained momentum recently. Unlike integer order derivatives, the fractional derivative is a nonlocal operator, which gives the future states depend on the current state as well as the history of all previous states. From this point of view, fractional differential equations provide a powerful tool for mathematical modeling of complex phenomena in science and engineering practices, see [5, 6, 8, 17, 18, 23, 29, 30, 32, 37] and references therein. In the theory of classical and fractional order differential equations, various theorems have been extensively deployed by researchers in establishing the existence, uniqueness and multiple solutions of boundary value problems, see [26–28, 38] and the references therein.

The differential equations, difference equations and dynamic equations on time scales are three theories which play important role for modeling in the environment. Among them, the theory of dynamic equations

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on time scales is recent and was introduced by Hilger [22] in his PhD thesis in 1988 with three main features: unification, extension and discretization. Time scale is any closed and nonempty subset of the real numbers, so we can extend known results from continuous and discrete analysis to a more general settings. As a matter of fact, this theory allows us to consider time scales which possess hybrid behaviours both continuous and discrete. These types of time scales play an important role for applications, since most of the phenomena in the environment are neither only discrete nor only continuous, but they possess both behaviors [7, 9, 35]. Moreover, basic results on this issue have been well documented in the articles [2, 3] and monographs of Bohner and Peterson[15, 16].

Many researchers have started to deal with discrete versions of fractional calculus using the theory of time scales, see [1, 34, 36] and references therein. Recently, Benkhettou et al. [11] introduced a conformable fractional(CF) calculus on an arbitrary time scale, which provides a natural extension of the conformable fractional calculus. For more results in this line, see [19, 25, 33]. In [31], Sheng et al. studied existence and multiplicity of positive solutions to the CF dynamic boundary value problem,

$$\mathcal{T}_{\alpha}^{\Delta}\left[\varphi_{\mathtt{p}}\big(\mathcal{T}_{\alpha}^{\Delta}\big(\vartheta(t)\big)\big)\right]=\mathtt{f}\left(t,\vartheta(t)\right),\ t\in[0,1]_{\mathbb{T}},\ 1<\alpha\leq 2,$$
$$\vartheta(0)=\vartheta(\sigma(1))=\mathcal{T}_{\alpha}^{\Delta}\vartheta(0)=\mathcal{T}_{\alpha}^{\Delta}\vartheta(\sigma(1))=0$$

by an application of fixed point theorems on cone. In [20], Gulsen et al. derived sufficient conditions for the existence of a solution to the CF Sturm–Liouville boundary value problem

$$\mathcal{T}_{\alpha}^{\Delta}\left[\mathcal{T}_{\alpha}^{\Delta}\big(\vartheta(t)\big)\right]+\lambda\varphi(t)\mathtt{f}\left(t,\vartheta(t)\right)=0,\ t\in[\rho(a),b]_{\mathbb{T}},\ 0<\alpha\leq 1,$$
$$\mathcal{T}_{\alpha}^{\Delta}(\rho(\vartheta(a)))=\delta\vartheta(b)+\beta\mathcal{T}_{\alpha}^{\Delta}\vartheta(b)=0,$$

by an application of Schauder's fixed point theorem. In [10], Bendouma and Hammoudi established the existence of solutions for the CF dynamic boundary value problem

$$\mathcal{T}_{\alpha}^{\Delta}\vartheta(t)=\mathtt{f}(t,\sigma(\vartheta(t))),\ t\in[a,b]_{\mathbb{T}},\ 0<\alpha\leq 1,$$
$$\mathtt{B}(\vartheta(a),\vartheta)=0\ \text{ or }\ \mathtt{H}(\vartheta,\vartheta(\sigma(b)))=0,$$

by means of the upper and lower solutions method together with Schauder's fixed point theorem. In [13], Bohner and Hatipoglu established the general solutions and stability criteria for the following CF dynamic cobweb model on time scales,

$$\begin{aligned}&\mathtt{D}(\sigma(t))=a(t)+b(t)[p(t)+\mathcal{T}_{\alpha}^{\Delta}p(t)],\ t\in\mathbb{T}^{k},\ 0<\alpha\leq 1,\\&\mathtt{S}(\sigma(t))=a_1(t)+b_1(t)p(t),\\&\mathtt{D}(t)=\mathtt{S}(t),\end{aligned}$$

with initial condition $p(t_0)=p_0$. Motivated by aforementioned studies, in this paper we consider the CF dynamic boundary value problem on time scales,

$$\left.\begin{aligned}\mathcal{T}_{\alpha}^{\Delta}\left[\mathcal{T}_{\beta}^{\Delta}\big(\vartheta_{\mathtt{n}}(t)\big)\right]=\varphi(t)\mathtt{f}_{\mathtt{n}}\left(\vartheta_{\mathtt{n}+1}(t)\right),\ t\in(0,1)_{\mathbb{T}},\ 1<\alpha,\beta\leq 2,\\ \vartheta_1(t)=\vartheta_{\ell+1}(t),\ t\in(0,1)_{\mathbb{T}},\ \mathtt{n}=1,2,\cdots,\ell,\end{aligned}\right\}\tag{1.1}$$

satisfying two-point Riemann–Stieltjes integral boundary conditions

$$\left.\begin{aligned} \vartheta_{\mathtt{n}}(0)=0,\ \vartheta_{\mathtt{n}}(1)=\int_0^1 \vartheta_{\mathtt{n}}(\tau)\Box\mathtt{g}(\tau),\ \mathtt{n}=1,2,\cdots,\ell,\\ (\mathcal{T}_{\beta}^{\Delta}\vartheta_{\mathtt{n}})(0)=0,\ (\mathcal{T}_{\beta}^{\Delta}\vartheta_{\mathtt{n}})(1)=\int_0^1 (\mathcal{T}_{\beta}^{\Delta}\vartheta_{\mathtt{n}})(\tau)\Box\mathtt{g}(\tau),\ \mathtt{n}=1,2,\cdots,\ell,\end{aligned}\right\} \tag{1.2}$$

where $\mathcal{T}_{\star}^{\Delta}$ denotes the conformable fractional derivative of order $\star\in\{\alpha,\beta\}$ on time scale $\mathbb{T}$, $\varphi(t)=\prod_{j=1}^{m}\varphi_j(t)$ and $\varphi_j(t)\in L_{\Delta}^{\mathtt{p}_j}[0,1]_{\mathbb{T}}(\mathtt{p}_j\geq 1)$ has a singularity in $(0,1/2)_{\mathbb{T}}$, $\int_0^1\vartheta_{\mathtt{n}}(\tau)\Box\mathtt{g}(\tau)$ denotes Riemann–Stieltjes integral of $\vartheta_{\mathtt{n}}(\tau)$ with respect to $\mathtt{g}$, $\mathtt{g}:[0,1]_{\mathbb{T}}\to\mathbb{R}$ is a function of bounded variation and $\Box\mathtt{g}$ is a signed measure and established infinitely many positive solutions by an application of Krasnoselskii's fixed point theorem on a Banach space. We assume the following conditions hold throughout the paper:

$(\mathcal{H}_1)$ $\mathtt{f}_n:[0,+\infty)\to[0,+\infty)$ is continuous for $n=1,2,\cdots,\ell$,

$(\mathcal{H}_2)$ there exists a sequence $\{t_k\}_{k=1}^{\infty}$ such that $0<t_{k+1}<t_k<\frac{1}{2}$,

$$\lim_{k\to\infty}t_k=t^*<\frac{1}{2},\ \lim_{t\to t_k}\varphi_j(t)=+\infty,\ k\in\mathbb{N},\ j=1,2,3,\cdots,m$$

and each $\varphi_j(t)$ does not vanish identically on any subinterval of $[0,1]_{\mathbb{T}}$. Moreover, there exists $\varphi_j^*>0$ such that

$$\varphi_j^*<\varphi_j(t)<\infty\ \text{ a.e. on }\ [0,1]_{\mathbb{T}}.$$

$(\mathcal{H}_3)$ $\mathtt{g}$ be nondecreasing and of bounded variation function such that $0<\mathtt{g}^*<1$, where

$$\mathtt{g}^*=\int_0^1\tau\,\Box\mathtt{g}(\tau).$$

2. Preliminaries and Green's function

A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. It follows that the jump operators $\sigma,\rho:\mathbb{T}\to\mathbb{T}$, and the graininess $\mu:\mathbb{T}\to[0,+\infty)$ are defined by $\sigma(t)=\inf\{\tau\in\mathbb{T}:\tau>t\}$, $\rho(t)=\sup\{\tau\in\mathbb{T}:\tau<t\}$, and $\mu(t)=\rho(t)-t$, respectively.

- The point $t\in\mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t)=t$, $\rho(t)<t$, $\sigma(t)=t$, $\sigma(t)>t$, respectively.
- If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k=\mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_k=\mathbb{T}$.
- If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k=\mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^k=\mathbb{T}$.
- A function $\mathtt{f}:\mathbb{T}\to\mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of all rd-continuous functions $\mathtt{f}:\mathbb{T}\to\mathbb{R}$ is denoted by $C_{rd}=C_{rd}(\mathbb{T})=C_{rd}(\mathbb{T},\mathbb{R})$.

- A function $\mathtt{f} : \mathbb{T} \to \mathbb{R}$ is called ld-continuous provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. The set of all ld-continuous functions $\mathtt{f} : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{ld} = C_{ld}(\mathbb{T}) = C_{ld}(\mathbb{T}, \mathbb{R})$.

- By an interval time scale, we mean the intersection of a real interval with a given time scale. i.e. $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$ other intervals can be defined similarly. Moreover, if $I = [a, b]_{\mathbb{T}}$ then we define $I_{\Delta} = [a, \rho(b)]_{\mathbb{T}}$ and $I_{\nabla} = [\sigma(a), b]_{\mathbb{T}}$. By $I_{\Box}$ we denote one of them, where $\Box$ means either Δ or ∇. Similarly, we use $\Box$ as a common notation for the two kinds of derivatives on time scales: one can read $\mathtt{f}^{\Box}$ either as $\mathtt{f}^{\Delta}$ or as $\mathtt{f}^{\nabla}$.

Let $\mathtt{f}$ be a real valued and bounded function on the interval I. Let us take a partition $\mathtt{P} = \{t_0, t_1, \cdots, t_n\}$ of I. Denote $I_{\Box j} = [t_{j-1}, t_j]_{\Box}$, $j = 1, 2, \cdots, n$, and

$$m_{\Box j} = \inf_{t \in I_{\Box j}} \mathtt{f}(t), \quad M_{\Box j} = \sup_{t \in I_{\Box j}} \mathtt{f}(t).$$

The upper Darboux–Stieltjes $\Box$-sum of $\mathtt{f}$ with respect to the partition $\mathtt{P}$, denoted by $U_{\Box}(\mathtt{P}, \mathtt{f}, \mathtt{g})$, is defined by

$$U_{\Box}(\mathtt{P}, \mathtt{f}, \mathtt{g}) = \sum_{j=1}^{n} M_{\Box j} \Delta \mathtt{g}_j,$$

and the lower Darboux–Stieltjes $\Box$-sum of $\mathtt{f}$ with respect to the partition $\mathtt{P}$, denoted by $L_{\Box}(\mathtt{P}, \mathtt{f}, \mathtt{g})$, is defined by

$$L_{\Box}(\mathtt{P}, \mathtt{f}, \mathtt{g}) = \sum_{j=1}^{n} m_{\Box j} \Delta \mathtt{g}_j,$$

where $\mathtt{g}$ is a continuous function on I and $\Delta \mathtt{g}_j = \mathtt{g}(t_j) - \mathtt{g}(t_{j-1})$. For more details, see [24].

Definition 2.1 *[24] Let $I = [a, b]_{\mathbb{T}}$, where $a, b \in \mathbb{T}$ and $\mathcal{P}(I)$ the set of all partitions of I. The upper Darboux–Stieltjes $\Box$-integral from a to b with respect to function $\mathtt{g}$ is defined by*

$$\overline{\int_a^b} \mathtt{f} \Box \mathtt{g} = \inf_{\mathtt{P} \in \mathcal{P}(I)} U_{\Box}(\mathtt{P}, \mathtt{f}, \mathtt{g}).$$

Similarly we can define the upper Darboux–Stieltjes $\Box$-integral. If $\overline{\int_a^b} \mathtt{f}(t) \Box \mathtt{g}(t) = \underline{\int_a^b} \mathtt{f}(t) \Box \mathtt{g}(t)$ then we say that $\mathtt{f}$ is $\Box$-integrable with respect to $\mathtt{g}$ on I, and the common value of the integrals, denoted by $\int_a^b \mathtt{f}(t) \Box \mathtt{g}(t) = \int_a^b \mathtt{f} \Box \mathtt{g}$ is called the Riemann–Stieltjes (or simply Stieltjes) $\Box$-integral of f with respect to $\mathtt{g}$ on I.

Definition 2.2 *[14] Let μ_{Δ} and μ_{∇} be the Lebesgue $\Delta-$ measure and the Lebesgue $\nabla-$ measure on $\mathbb{T}$, respectively. If $A \subset \mathbb{T}$ satisfies $\mu_{\Delta}(A) = \mu_{\nabla}(A)$, then we call A is measurable on $\mathbb{T}$, denoted $\mu(A)$ and this value is called the Lebesgue measure of A. Let P denote a proposition with respect to $t \in \mathbb{T}$.*

(i) If there exists $E_1 \subset A$ with $\mu_{\Delta}(E_1) = 0$ such that P holds on $A \backslash E_1$, then P is said to hold Δ–a.e. on A.

(ii) If there exists $E_2 \subset A$ with $\mu_\nabla(E_2) = 0$ such that P holds on $A \backslash E_2$, then P is said to hold ∇–a.e. on A.

Definition 2.3 *[4] Let $E \subset \mathbb{T}$ be a Δ–measurable set and $p \in \bar{\mathbb{R}} \equiv \mathbb{R} \cup \{-\infty, +\infty\}$ be such that $p \geq 1$ and let $f : E \to \bar{\mathbb{R}}$ be Δ-measurable function. We say that f belongs to $L^p_\nabla(E)$ provided that either*

$$\int_E |\mathtt{f}|^p(\tau)\Delta\tau < \infty \quad \text{if} \quad \mathtt{p} \in \mathbb{R},$$

or there exists a constant $M \in \mathbb{R}$ such that

$$|\mathtt{f}| \leq M, \quad \Delta - a.e. \text{ on } E \quad \text{if} \ \mathtt{p} = +\infty.$$

Lemma 2.4 *[12] Let (X, M, μ_Δ) and (Y, L, ν_Δ) be two finite-dimensional time scales measure spaces. If $\mathtt{f} : X \times Y \to \mathbb{R}$ is a Δ-integrable function and if we define the functions*

$$\varphi(y) = \int_X \mathtt{f}(x, y) d\mu_\Delta(x) \quad \text{for} \ \ y \in Y,$$

and

$$\mathtt{Q}(x) = \int_Y \mathtt{f}(x, y) d\nu_\Delta(y) \quad \text{for} \ \ x \in X,$$

then φ is Δ-integrable on Y and is Δ-integrable on X and

$$\int_X d\mu_\Delta(x) \int_Y \mathtt{f}(x, y) d\nu_\Delta(y) = \int_Y d\nu_\Delta(y) \int_X \mathtt{f}(x, y) d\mu_\Delta(x).$$

Definition 2.5 *[11] Let $\alpha \in (1, 2]$ and $\mathtt{f} : \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}^k$. For $t > 0$, we define $\mathcal{T}^\Delta_\alpha \mathtt{f}(t)$ to be the number (provided it exists) with the property that, given any $\varepsilon > 0$, there is a δ-neighborhood $V_t \subset \mathbb{T}$ of $t, \delta > 0$, such that*

$$\left|[\mathtt{f}(\sigma(t)) - f(\tau)]t^{2-\alpha} - \mathcal{T}^\Delta_\alpha \mathtt{f}(t)[\sigma(t) - \tau]\right| \leq \varepsilon |\sigma(t) - \tau|.$$

We call $\mathcal{T}^\Delta_\alpha \mathtt{f}(t)$ the conformable fractional derivative of $\mathtt{f}$ of order α at t, and we define the conformable fractional derivative at 0 as

$$\mathcal{T}^\Delta_\alpha \mathtt{f}(0) = \lim_{t \to 0^+} \mathcal{T}^\Delta_\alpha \mathtt{f}(t).$$

Definition 2.6 *[11] Let $\alpha \in (1, 2]$ and $\mathtt{f}$ be two times delta differentiable at $t \in \mathbb{T}^k$. Then*

$$\mathcal{T}^\Delta_\alpha \mathtt{f}(t) = t^{2-\alpha} \mathtt{f}(t).$$

Definition 2.7 *[31] Let $\mathtt{f} : \mathbb{T} \to \mathbb{R}$ be a regulated function and $\alpha \in (1, 2]$. Then the α-fractional integral of $\mathtt{f}$ is defined by*

$$\mathcal{I}^\alpha \mathtt{f}(t) = \mathcal{I}^2 \left(t^{\alpha-2} \mathtt{f}(t)\right) = \int_0^1 (t - \tau)\tau^{\alpha-2} \mathtt{f}(\tau)\Delta\tau.$$

Lemma 2.8 *[31] Let $t > 0, \alpha \in (1,2]$ and the function $\mathtt{f} : [0,\infty)_{\mathbb{T}} \to \mathbb{R}$ be rd-continuous, then*

$$\mathcal{T}_{\alpha}^{\Delta}\mathcal{I}^{\alpha}\mathtt{f}(t) = \mathtt{f}(t).$$

Lemma 2.9 *[31] Let $\alpha \in (1,2]$, $\mathtt{f}$ be a α-differentiable function at $t > 0$, then $\mathcal{T}_{\alpha}^{\Delta}\mathtt{f}(t) = 0$ for $t \in [0,1]_{\mathbb{T}}$ if and only if $\mathtt{f}(t) = a_0 + a_1 t$, where a_0 and a_1 are real constants.*

Lemma 2.10 *[31] Assume that $\vartheta \in C(0,+\infty)_{\mathbb{T}}$ with a fractional derivative of order $\alpha \in (1,2]$. Then*

$$\mathcal{I}^{\alpha}\mathcal{T}_{\alpha}^{\Delta}\vartheta(t) = \vartheta(t) + c_0 + c_1 t,$$

where c_0, c_1 are real constants.

Lemma 2.11 *Let $y \in \mathcal{C}((0,1)_{\mathbb{T}})$. Then the boundary value problem*

$$\mathcal{T}_{\beta}^{\Delta}\vartheta_1(t) + y(t) = 0,\ t \in (0,1)_{\mathbb{T}}, \tag{2.1}$$

$$\vartheta_1(0) = 0,\ \vartheta_1(1) = \int_0^1 \vartheta_1(\tau)\Box\mathsf{g}(\tau), \tag{2.2}$$

has a unique solution

$$\vartheta_1(t) = \int_0^1 \aleph_{\beta}(t,\tau)y(\tau)d\tau, \tag{2.3}$$

where $\aleph_{\beta}(t,\tau)$ is the Green's function and is given by

$$\aleph_{\beta}(t,\tau) = \mathcal{K}_{\beta}(t,\tau) + \frac{t}{1-\mathsf{g}^*}\mathcal{G}_{\beta}(\tau),$$

$$\mathcal{K}_{\beta}(t,\tau) = \begin{cases} \tau^{\beta-1}(1-t), & \tau \le t, \\ t(1-\tau)\tau^{\beta-2}, & t \le \tau, \end{cases}$$

and

$$\mathcal{G}_{\beta}(\tau) = \int_0^1 \mathcal{K}_{\beta}(\tau_1,\tau)\Box\mathsf{g}(\tau_1).$$

Proof From Lemma 2.10, the equivalent integral equation to (2.1) is given by

$$\begin{aligned}\vartheta_1(t) &= -\mathcal{I}^{\beta}y(t) + A + Bt \\ &= -\int_0^t (t-\tau)\tau^{\beta-2}y(\tau)\Delta\tau + A + Bt,\end{aligned}$$

for some $A, B \in \mathbb{R}$. Using boundary conditions (2.2), we obtain $A = 0$ and

$$B = \int_0^1 \vartheta_1(\tau)\Box\mathsf{g}(\tau) + \int_0^1 (1-\tau)\tau^{\beta-2}y(\tau)\Delta\tau.$$

So,

$$\begin{aligned}\vartheta_1(t) &= -\int_0^t (t-\tau)\tau^{\beta-2}y(\tau)\Delta\tau + t\int_0^1 (1-\tau)\tau^{\beta-2}y(\tau)\Delta\tau + t\int_0^1 \vartheta_1(\tau)\Box g(\tau)\\ &= \int_0^1 \mathcal{K}_\beta(t,\tau)y(\tau)\Delta\tau + t\int_0^1 \vartheta_1(\tau)\Box g(\tau). \end{aligned} \tag{2.4}$$

By simple algebraic calculations, we get

$$\int_0^1 \vartheta_1(\tau)\Box g(\tau) = \frac{1}{1-g^*}\int_0^1\int_0^1 \mathcal{K}_\beta(\tau_1,\tau)\Box g(\tau_1)\Delta\tau. \tag{2.5}$$

Substituing (2.5) into (2.4), we finally have (2.3). □

Lemma 2.12 *The function $\mathcal{K}_\beta(t,\tau)$ has the following properties:*

(i) $\mathcal{K}_\beta(t,\tau)$ *is nonnegative and continuous on* $[0,1]_{\mathbb{T}}\times[0,1]_{\mathbb{T}}$,

(ii) $\mathcal{K}_\beta(t,\tau)\le\mathcal{K}_\beta(\tau,\tau)$ *for* $t,\tau\in[0,1]_{\mathbb{T}}$,

(iii) there exists $\lambda\in(0,1/2)_{\mathbb{T}}$ *such that* $\lambda\mathcal{K}_\beta(\tau,\tau)\le\mathcal{K}_\beta(t,\tau)$ *for* $t\in[\lambda,1-\lambda]_{\mathbb{T}},\tau\in[0,1]_{\mathbb{T}}$.

Proof It is easy to establish the result (i). It can be seen that $\mathcal{K}_\beta(t,\tau)$ is decreasing with respect to t for $\tau\le t$ and increasing with respect to t for $t\le s$. So, by this monotonicity of $\mathcal{K}_\beta(t,\tau)$, we have

$$\mathcal{K}_\beta(t,\tau)\le\mathcal{K}_\beta(\tau,\tau)=\tau^{\beta-1}(1-\tau),\quad \tau\in[0,1]_{\mathbb{T}}.$$

This proves (ii). Now we prove (iii). For $\tau\le t$, we have

$$\frac{\mathcal{K}_\beta(t,\tau)}{\mathcal{K}_\beta(\tau,\tau)}=\frac{\tau^{\beta-1}(1-t)}{\tau^{\beta-1}(1-\tau)}=\frac{1-t}{1-\tau}\ge 1-t\ge\lambda,$$

and for $t\le\tau$,

$$\frac{\mathcal{K}_\beta(t,\tau)}{\mathcal{K}_\beta(\tau,\tau)}=\frac{t(1-\tau)\tau^{\beta-2}}{\tau^{\beta-1}(1-\tau)}=\frac{t}{\tau}\ge\frac{\lambda}{\tau}\ge\lambda.$$

This completes proof. □

Lemma 2.13 *Let* $\mathcal{G}^*_\beta(\tau)=\mathcal{K}_\beta(\tau,\tau)+\frac{1}{1-g^*}\mathcal{G}_\beta(\tau)$. *Then the Green's function* $\aleph_\beta(t,\tau)$ *has the following properties:*

(i) $\aleph_\beta(t,\tau)$ *is nonnegative and continuous on* $[0,1]_{\mathbb{T}}\times[0,1]_{\mathbb{T}}$,

(ii) $\aleph_\beta(t,\tau)\le\mathcal{G}^*_\beta(\tau)$ *for* $t,\tau\in[0,1]_{\mathbb{T}}$,

(iii) there exists $\lambda\in(0,1/2)_{\mathbb{T}}$ *such that* $\lambda\mathcal{G}^*_\beta(\tau)\le\aleph_\beta(t,\tau)$ *for* $t\in[\lambda,1-\lambda]_{\mathbb{T}},\tau\in[0,1]_{\mathbb{T}}$.

Proof The results (i) and (ii) are obvious. To prove (iii), let $\lambda \in (0,1/2)_{\mathbb{T}}$ and $\tau \in [0,1]_{\mathbb{T}}$. Then from Lemma 2.12, we have

$$\aleph_\beta(t,\tau) = \mathcal{K}_\beta(t,\tau) + \frac{t}{1-\mathrm{g}^*}\mathcal{G}_\beta(\tau) \geq \lambda\mathcal{K}_\beta(\tau,\tau) + \frac{\lambda}{1-\mathrm{g}^*}\mathcal{G}_\beta(\tau) = \lambda\mathcal{G}^*_\beta(\tau).$$

This completes proof. □

Lemma 2.14 *Let $z \in \mathcal{C}((0,1)_{\mathbb{T}})$. Then the boundary value problem*

$$\mathcal{T}^\Delta_\alpha\left[\mathcal{T}^\Delta_\beta\left(\vartheta_\ell(t)\right)\right] = z(t),\ t \in (0,1)_{\mathbb{T}}, \tag{2.6}$$

$$\left.\begin{aligned} \vartheta_1(0) = 0,\ \vartheta_1(1) = \int_0^1 \vartheta_1(\tau)\Box\mathrm{g}(\tau), \\ (\mathcal{T}^\Delta_\beta\vartheta_1)(0) = 0,\ (\mathcal{T}^\Delta_\beta\vartheta_1)(1) = \int_0^1 (\mathcal{T}^\Delta_\beta\vartheta_1)(\tau)\Box\mathrm{g}(\tau), \end{aligned}\right\} \tag{2.7}$$

has a unique solution

$$\vartheta_1(t) = \int_0^1 \aleph_\beta(t,\tau)\left[\int_0^1 \aleph_\alpha(\tau,s)z(s)\Delta s\right]\Delta\tau, \tag{2.8}$$

where $\aleph_\beta(t,\tau)$ is defined in Lemma 2.11 and

$$\aleph_\alpha(t,\tau) = \mathcal{K}_\alpha(t,\tau) + \frac{t}{1-\mathrm{g}^*}\mathcal{G}_\alpha(\tau),$$

$$\mathcal{K}_\alpha(t,\tau) = \begin{cases} \tau^{\alpha-1}(1-t), & \tau \leq t, \\ t(1-\tau)\tau^{\alpha-2}, & t \leq \tau, \end{cases}$$

and

$$\mathcal{G}_\alpha(\tau) = \int_0^1 \mathcal{K}_\alpha(\tau_1,\tau)\Box\mathrm{g}(\tau_1).$$

Proof Let $y_1(t) = \mathcal{T}^\Delta_\beta\vartheta_1(t)$ for $0 < t < 1$. Then the boundary value problem

$$\mathcal{T}^\Delta_\alpha\left[\mathcal{T}^\Delta_\beta\left(\vartheta_1(t)\right)\right] - z(t) = 0,\ t \in [0,1]_{\mathbb{T}},$$

$$\mathcal{T}^\Delta_\beta\vartheta_1(0) = 0,\ \mathcal{T}^\Delta_\beta\vartheta_1(1) = \int_0^1 (\mathcal{T}^\Delta_\beta\vartheta_1)(\tau)\Box\mathrm{g}(\tau)$$

is equivalent to the problem

$$\left.\begin{aligned} \mathcal{T}^\Delta_\alpha y_1(t) - z(t) = 0,\ t \in [0,1]_{\mathbb{T}}, \\ y_1(0) = 0,\ y_1(1) = \int_0^1 y_1(\tau)\Box\mathrm{g}(\tau). \end{aligned}\right\} \tag{2.9}$$

By Lemma 2.11, the boundary value problem (2.9) has unique solution $y_1(t) = -\int_0^1 \aleph_\alpha(t,\tau)z(\tau)\Delta\tau$. That is

$$\mathcal{T}^\Delta_\beta\vartheta_1(t) + \int_0^1 \aleph_\alpha(t,\tau)z(\tau)\Delta\tau = 0. \tag{2.10}$$

Again by Lemma 2.11, the differential equation (2.10) with boundary conditions

$$\vartheta_1(0) = 0 \quad \text{and} \quad \vartheta_1(1) = \int_0^1 \vartheta_1(\tau)\Box \mathtt{g}(\tau)$$

has a unique solution

$$\vartheta_1(t) = \int_0^1 \aleph_\beta(t,\tau)\left[\int_0^1 \aleph_\alpha(\tau,s)z(s)\Delta s\right]\Delta\tau.$$

This completes the proof. □

Lemma 2.15 *The function $\mathcal{K}_\alpha(t,\tau)$ has the following properties:*

(i) $\mathcal{K}_\alpha(t,\tau)$ is nonnegative and continuous on $[0,1]_\mathbb{T}\times[0,1]_\mathbb{T}$,

(ii) $\mathcal{K}_\alpha(t,\tau)\le\mathcal{K}_\alpha(\tau,\tau)$ for $t,\tau\in[0,1]_\mathbb{T}$,

(iii) there exists $\lambda\in(0,1/2)_\mathbb{T}$ such that $\lambda\mathcal{K}_\alpha(\tau,\tau)\le\mathcal{K}_\alpha(t,\tau)$ for $t\in[\lambda,1-\lambda]_\mathbb{T},\tau\in[0,1]_\mathbb{T}$.

Lemma 2.16 *Let $\mathcal{G}^*_\alpha(\tau) = \mathcal{K}_\alpha(\tau,\tau)+\frac{1}{1-\mathtt{g}^*}\mathcal{G}_\alpha(\tau)$. Then the Green's function $\aleph_\alpha(t,\tau)$ has the following properties:*

(i) $\aleph_\alpha(t,\tau)$ is nonnegative and continuous on $[0,1]_\mathbb{T}\times[0,1]_\mathbb{T}$,

(ii) $\aleph_\alpha(t,\tau)\le\mathcal{G}^*_\alpha(\tau)$ for $t,\tau\in[0,1]_\mathbb{T}$,

(iii) there exists $\lambda\in(0,1/2)_\mathbb{T}$ such that $\lambda\mathcal{G}^*_\alpha(\tau)\le\aleph_\alpha(t,\tau)$ for $t\in[\lambda,1-\lambda]_\mathbb{T},\tau\in[0,1]_\mathbb{T}$.

Note that an ℓ–tuple $(\vartheta_1,\vartheta_2,\cdots,\vartheta_\ell)$ is a solution of (1.1)–(1.2) if and only if

$$\begin{aligned}\vartheta_1(t) = \int_0^1 \aleph_\beta(t,\tau_1)\Big[\int_0^1 \aleph_\alpha(\tau_1,\tau_2)\varphi(\tau_2)\mathtt{f}_1\Big[\int_0^1 \aleph_\beta(\tau_2,\tau_3)\Big[\int_0^1 \aleph_\alpha(\tau_3,\tau_4)\varphi(\tau_4)\mathtt{f}_2\cdots\\ \mathtt{f}_{\ell-1}\Big[\int_0^1 \aleph_\beta(\tau_{2\ell-2},\tau_{2\ell-1})\Big[\int_0^1 \aleph_\alpha(\tau_{2\ell-1},\tau_{2\ell})\varphi(\tau_{2\ell})\mathtt{f}_\ell(\vartheta_1(\tau_{2\ell}))\Delta\tau_{2\ell}\Big]\Delta\tau_{2\ell-1}\cdots\Delta\tau_2\Big]\Delta\tau_1\end{aligned}$$

and

$$\begin{aligned}&\vartheta_\mathtt{n}(t) = \int_0^1 \aleph_\beta(t,\tau)\left[\int_0^1 \aleph_\alpha(\tau,s)\varphi(s)\mathtt{f}_\mathtt{n}(\vartheta_{\mathtt{n}+1}(\mathtt{s}))\Delta\mathtt{s}\right]\Delta\tau,\ \mathtt{n}=2,3,\cdots,\ell,\\ &\vartheta_{\ell+1}(t)=\vartheta_1(t),\ t\in(0,1)_\mathbb{T}.\end{aligned}$$

Denote the Banach space $\mathcal{C}((0,1)_\mathbb{T},\mathbb{R})$ by $\mathcal{B}$ with the norm $\|\vartheta\| = \max_{t\in[0,1]}|\vartheta(t)|$. For $\lambda\in(0,1/2)_\mathbb{T}$, the cone $\mathscr{N}_\lambda\subset\mathcal{B}$ is defined by

$$\mathscr{N}_\lambda = \Big\{\vartheta\in\mathcal{B} : x(t)\ge 0,\ \min_{t\in[\lambda,\,1-\lambda]}\vartheta(t)\ge\lambda\|\vartheta\|\Big\},$$

For any $\vartheta_1 \in \mathscr{N}_\lambda$, define an operator $\Omega : \mathscr{N}_\lambda \to \mathcal{B}$ by

$$(\Omega\vartheta_1)(t) = \int_0^1 \aleph_\beta(t,\tau_1)\Big[\int_0^1 \aleph_\alpha(\tau_1,\tau_2)\varphi(\tau_2)\mathtt{f}_1\Big[\int_0^1 \aleph_\beta(\tau_2,\tau_3)\Big[\int_0^1 \aleph_\alpha(\tau_3,\tau_4)\varphi(\tau_4)\mathtt{f}_2\cdots$$
$$\mathtt{f}_{\ell-1}\Big[\int_0^1 \aleph_\beta(\tau_{2\ell-2},\tau_{2\ell-1})\Big[\int_0^1 \aleph_\alpha(\tau_{2\ell-1},\tau_{2\ell})\varphi(\tau_{2\ell})\mathtt{f}_\ell(\vartheta_1(\tau_{2\ell}))\Delta\tau_{2\ell}\Big]\Delta\tau_{2\ell-1}\cdots\Delta\tau_2\Big]\Delta\tau_1.$$

Lemma 2.17 *For each $\lambda \in (0,1/2)_{\mathbb{T}}$, $\Omega(\mathscr{N}_\lambda) \subset \mathscr{N}_\lambda$ and $\Omega : \mathscr{N}_\lambda \to \mathscr{N}_\lambda$ is completely continuous.*

Proof Let $\lambda \in (0,1/2)_{\mathbb{T}}$. Since $\mathtt{f}_\ell(\vartheta_{\ell+1}(\tau))$ is nonnegative for $\tau \in [0,1]_{\mathbb{T}}$, $\vartheta_1 \in \mathscr{N}_\lambda$. Since $\aleph_\beta(t,\tau)$, $\aleph_\alpha(t,\tau)$ are nonnegative for all $t,\tau \in [0,1]_{\mathbb{T}}$, it follows that $\Omega(\vartheta_1(t)) \ge 0$ for all $t \in [0,1]_{\mathbb{T}}$, $\vartheta_1 \in \mathscr{N}_\lambda$ Now, by Lemma 2.13 and 2.16, we have

$$\begin{aligned}
&\min_{t\in[\lambda,1-\lambda]_{\mathbb{T}}}(\Omega\vartheta_1)(t)\\
&= \min_{t\in[\lambda,1-\lambda]_{\mathbb{T}}}\Bigg\{\int_0^1 \aleph_\beta(t,\tau_1)\Big[\int_0^1 \aleph_\alpha(\tau_1,\tau_2)\varphi(\tau_2)\mathtt{f}_1\Big[\int_0^1 \aleph_\beta(\tau_2,\tau_3)\Big[\int_0^1 \aleph_\alpha(\tau_3,\tau_4)\varphi(\tau_4)\mathtt{f}_2\cdots\\
&\qquad \mathtt{f}_{\ell-1}\Big[\int_0^1 \aleph_\beta(\tau_{2\ell-2},\tau_{2\ell-1})\Big[\int_0^1 \aleph_\alpha(\tau_{2\ell-1},\tau_{2\ell})\varphi(\tau_{2\ell})\mathtt{f}_\ell(\vartheta_1(\tau_{2\ell}))\Delta\tau_{2\ell}\Big]\Delta\tau_{2\ell-1}\cdots\Delta\tau_2\Big]\Delta\tau_1\Bigg\}\\
&\ge \lambda\int_0^1 \mathcal{G}^*_\beta(\tau_1)\Big[\int_0^1 \aleph_\alpha(\tau_1,\tau_2)\varphi(\tau_2)\mathtt{f}_1\Big[\int_0^1 \aleph_\beta(\tau_2,\tau_3)\Big[\int_0^1 \aleph_\alpha(\tau_3,\tau_4)\varphi(\tau_4)\mathtt{f}_2\cdots\\
&\qquad \mathtt{f}_{\ell-1}\Big[\int_0^1 \aleph_\beta(\tau_{2\ell-2},\tau_{2\ell-1})\Big[\int_0^1 \aleph_\alpha(\tau_{2\ell-1},\tau_{2\ell})\varphi(\tau_{2\ell})\mathtt{f}_\ell(\vartheta_1(\tau_{2\ell}))\Delta\tau_{2\ell}\Big]\Delta\tau_{2\ell-1}\cdots\Delta\tau_2\Big]\Delta\tau_1\\
&\ge \lambda\Bigg\{\int_0^1 \aleph_\beta(t,\tau_1)\Big[\int_0^1 \aleph_\alpha(\tau_1,\tau_2)\varphi(\tau_2)\mathtt{f}_1\Big[\int_0^1 \aleph_\beta(\tau_2,\tau_3)\Big[\int_0^1 \aleph_\alpha(\tau_3,\tau_4)\varphi(\tau_4)\mathtt{f}_2\cdots\\
&\qquad \mathtt{f}_{\ell-1}\Big[\int_0^1 \aleph_\beta(\tau_{2\ell-2},\tau_{2\ell-1})\Big[\int_0^1 \aleph_\alpha(\tau_{2\ell-1},\tau_{2\ell})\varphi(\tau_{2\ell})\mathtt{f}_\ell(\vartheta_1(\tau_{2\ell}))\Delta\tau_{2\ell}\Big]\Delta\tau_{2\ell-1}\cdots\Delta\tau_2\Big]\Delta\tau_1\Bigg\}\\
&\ge \lambda\max_{t\in[0,1]_{\mathbb{T}}}|\Omega\vartheta_1(t)|.
\end{aligned}$$

Thus $\Omega(\mathscr{N}_\lambda) \subset \mathscr{N}_\lambda$. Therefore, the operator Ω is completely continuous by standard methods and by the Arzela–Ascoli theorem. □

3. infinitely many positive solutions

In establishing the existence of infinitely many positive solutions for the boundary value problem (1.1)–(1.2), we utilize following theorems.

Theorem 3.1 *[21] Let $\mathcal{B}$ be a cone in a Banach space $\mathcal{X}$ and $\mathtt{Q}_1$, $\mathtt{Q}_2$ are open sets with $0 \in \mathtt{Q}_1, \overline{\mathtt{Q}}_1 \subset \mathtt{Q}_2$. Let $\Omega : \mathcal{B} \cap (\overline{\mathtt{Q}}_2 \backslash \mathtt{Q}_1) \to \mathcal{B}$ be a completely continuous operator such that*

(a) $\|\Omega\vartheta\| \le \|\vartheta\|$, $\vartheta \in \mathcal{B} \cap \partial\mathtt{Q}_1$, and $\|\Omega\vartheta\| \ge \|\vartheta\|$, $\vartheta \in \mathcal{B} \cap \partial\mathtt{Q}_2$, or

(b) $\|\Omega\vartheta\| \geq \|\vartheta\|$, $\vartheta \in \mathcal{B} \cap \partial\mathtt{Q}_1$, *and* $\|\Omega\vartheta\| \leq \|\vartheta\|$, $\vartheta \in \mathcal{B} \cap \partial\mathtt{Q}_2$.

Then Ω *has a fixed point in* $\mathcal{B} \cap (\overline{\mathtt{Q}}_2 \backslash \mathtt{Q}_1)$.

Theorem 3.2 (Hölder's inequality [4]) *Let* $f \in L^{p_i}_{\Delta}(J)$ *with* $p_i > 1$, *for* $j = 1, 2, \cdots, n$ *and* $\sum_{j=1}^{n} \frac{1}{p_j} = 1$. *Then* $\prod_{j=1}^{n} \mathtt{f}_j \in L^1_{\Delta}(J)$ *and* $\left\|\prod_{j=1}^{n} \mathtt{f}_j\right\|_1 \leq \prod_{j=1}^{n} \|\mathtt{f}_j\|_{p_j}$. *Further, if* $f \in L^1_{\Delta}(J)$ *and* $g \in L^{\infty}_{\Delta}(J)$. *Then* $fg \in L^1_{\Delta}(J)$ *and* $\|fg\|_1 \leq \|f\|_1 \|g\|_{\infty}$, *where*

$$\|f\|_p := \begin{cases} \Big[\int_J |f|^p(s)\Delta s\Big]^{\frac{1}{p}}, & p \in \mathbb{R}, \\ inf\Big\{M \in \mathbb{R} \,/\, |f| \leq M \ \ \Delta - a.e., \text{ on } J\Big\}, & p = \infty, \end{cases}$$

and $J = [a, b)_{\mathbb{T}}$.

For $\varphi \in L^{p_i}_{\Delta}([0,1]_{\mathbb{T}})$, we have three possible cases:

$$\sum_{j=1}^{n} \frac{1}{\mathtt{p}_j} < 1, \ \sum_{j=1}^{n} \frac{1}{\mathtt{p}_j} = 1, \ \sum_{j=1}^{n} \frac{1}{\mathtt{p}_j} > 1.$$

Firstly, we establish infinitely many positive solutions for the case $\sum_{j=1}^{n} \frac{1}{\mathtt{p}_j} < 1$.

Theorem 3.3 *Assume that* $(\mathcal{H}_1) - (\mathcal{H}_3)$ *hold, let* $\{\lambda_k\}_{k=1}^{\infty}$ *be such that* $t_{k+1} < \lambda_k < t_k$, $k = 1, 2, 3, \cdots$. *Let* $\{\mathtt{R}_k\}_{k=1}^{\infty}$ *and* $\{\mathtt{r}_k\}_{k=1}^{\infty}$ *be such that* $\mathtt{R}_{k+1} < \lambda_k \mathtt{r}_k < \mathtt{L}\mathtt{r}_k < \mathtt{R}_k$, $k \in \mathbb{N}$, *where*

$$\mathtt{L} = \max\left\{\left[\lambda_1^2 \prod_{j=1}^{n} \varphi_j^* \int_{\lambda_1}^{1-\lambda_1} \mathcal{G}^*_{\beta}(\tau)\Delta\tau\Big[\int_{\lambda_1}^{1-\lambda_1} \mathcal{G}^*_{\alpha}(\tau)\Delta\tau\Big]\right]^{-1}, \ 1\right\}.$$

Further, assume that $\mathtt{f}_i$ *satisfies*

$(\mathcal{J}_1)$ $\mathtt{f}_\ell(\vartheta(t)) \leq \mathtt{A}_1 \mathtt{R}_k$ *for all* $0 \leq \vartheta(t) \leq \mathtt{R}_k$, $t \in [0,1]_{\mathbb{T}}$, *where* $\mathtt{A}_1 < \left[\|\mathcal{G}^*_{\alpha}\|_{\mathtt{q}} \prod_{j=1}^{n} \|\varphi_j\|_{\mathtt{p}_j} \int_0^1 \mathcal{G}^*_{\beta}(\tau)\Delta\tau\right]^{-1}$.

$(\mathcal{J}_2)$ $\mathtt{f}_\ell(\vartheta(t)) \geq \mathtt{L}\mathtt{r}_k$ *for all* $\lambda_k \mathtt{r}_k \leq \vartheta(t) \leq \mathtt{r}_k$, $t \in [\lambda_k, 1-\lambda_k]_{\mathbb{T}}$.

The iterative system (1.1)–(1.2) *has infinitely many solutions* $\{(\vartheta_1^{[k]}, \vartheta_2^{[k]}, \cdots, \vartheta_n^{[k]})\}_{k=1}^{\infty}$ *such that* $\vartheta_\ell^{[k]}(t) \geq 0$ *on* $(0,1)_{\mathbb{T}}$, $\ell = 1, 2, \cdots, n$ *and* $k \in \mathbb{N}$.

Proof Consider the sequences $\{\mathtt{Q}_{1,k}\}_{k=1}^{\infty}$ and $\{\mathtt{Q}_{2,k}\}_{k=1}^{\infty}$ of open subsets of $\mathcal{B}$ defined by

$$\mathtt{Q}_{1,k} = \{\vartheta \in \mathcal{B} : \|\vartheta\| < \mathtt{R}_k\}, \ \mathtt{Q}_{2,k} = \{\vartheta \in \mathcal{B} : \|\vartheta\| < \mathtt{r}_k\}.$$

Let $\{\lambda_k\}_{k=1}^{\infty}$ be as in the hypothesis and note that $t^* < t_{k+1} < \lambda_k < t_k < \frac{1}{2}$, for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, define the cone $\mathscr{N}_{\lambda_k}$ by

$$\mathscr{N}_{\lambda_k} = \left\{ \vartheta \in \mathcal{B} : \vartheta(t) \geq 0 \;\text{ and }\; \min_{t \in [\lambda_k,\, 1-\lambda_k]} \vartheta(t) \geq \lambda_k \|\vartheta\| \right\}.$$

Let $\vartheta_1 \in \mathscr{N}_{\lambda_k} \cap \partial \mathtt{Q}_{1,k}$. Then, $\vartheta_1(\tau) \leq \mathtt{R}_k = \|\vartheta_1\|$ for all $\tau \in [0,1]_{\mathbb{T}}$. By $(\mathcal{J}_1)$ and $0 < \tau_{2\ell-2} < 1$, we have

$$\begin{aligned}
&\int_0^1 \aleph_\beta(\tau_{2\ell-2},\tau_{2\ell-1}) \Big[\int_0^1 \aleph_\alpha(\tau_{2\ell-1},\tau_{2\ell}) \varphi(\tau_{2\ell}) \mathtt{f}_\ell(\vartheta_1(\tau_{2\ell})) \Delta\tau_{2\ell} \Big] \Delta\tau_{2\ell-1} \\
&\qquad \leq \int_0^1 \mathcal{G}^*_\beta(\tau_{2\ell-1}) \Big[\int_0^1 \mathcal{G}^*_\alpha(\tau_{2\ell}) \varphi(\tau_{2\ell}) \mathtt{f}_\ell(\vartheta_1(\tau_{2\ell})) \Delta\tau_{2\ell} \Big] \Delta\tau_{2\ell-1} \\
&\qquad \leq \mathtt{A}_1 \mathtt{R}_k \int_0^1 \mathcal{G}^*_\beta(\tau_{2\ell-1}) \Big[\int_0^1 \mathcal{G}^*_\alpha(\tau_{2\ell}) \varphi(\tau_{2\ell}) \Delta\tau_{2\ell} \Big] \Delta\tau_{2\ell-1}.
\end{aligned}$$

There exists a $\mathtt{q} > 1$ such that $\dfrac{1}{\mathtt{q}} + \displaystyle\sum_{j=1}^{n} \dfrac{1}{\mathtt{p}_j} = 1$. By the first part of Theorem 3.2, we have

$$\begin{aligned}
&\int_0^1 \aleph_\beta(\tau_{2\ell-2},\tau_{2\ell-1}) \Big[\int_0^1 \aleph_\alpha(\tau_{2\ell-1},\tau_{2\ell}) \varphi(\tau_{2\ell}) \mathtt{f}_\ell(\vartheta_1(\tau_{2\ell})) \Delta\tau_{2\ell} \Big] \Delta\tau_{2\ell-1} \\
&\qquad \leq \mathtt{A}_1 \mathtt{R}_k \int_0^1 \mathcal{G}^*_\beta(\tau_{2\ell-1}) \Delta\tau_{2\ell-1} \Big[\int_0^1 \mathcal{G}^*_\alpha(\tau_{2\ell}) \varphi(\tau_{2\ell}) \Delta\tau_{2\ell} \Big] \\
&\qquad \leq \mathtt{A}_1 \mathtt{R}_k \int_0^1 \mathcal{G}^*_\beta(\tau_{2\ell-1}) \Delta\tau_{2\ell-1} \|\mathcal{G}^*_\alpha\|_{\mathtt{q}} \, \|\varphi\|_{\mathtt{p}_j} \leq \mathtt{A}_1 \mathtt{R}_k \int_0^1 \mathcal{G}^*_\beta(\tau_{2\ell-1}) \Delta\tau_{2\ell-1} \|\mathcal{G}^*_\alpha\|_{\mathtt{q}} \prod_{j=1}^{n} \|\varphi_j\|_{\mathtt{p}_j} \\
&\qquad \leq \mathtt{R}_k.
\end{aligned}$$

It follows in similar manner (for $0 < \tau_{2\ell-4} < 1$) that

$$\begin{aligned}
&\int_0^1 \aleph_\beta(\tau_{2\ell-4},\tau_{2\ell-3}) \Big[\int_0^1 \aleph_\alpha(\tau_{2\ell-3},\tau_{2\ell-2}) \varphi(\tau_{2\ell-2}) \\
&\times \mathtt{f}_{\ell-1} \Big[\int_0^1 \aleph_\beta(\tau_{2\ell-2},\tau_{2\ell-1}) \Big[\int_0^1 \aleph_\alpha(\tau_{2\ell-1},\tau_{2\ell}) \varphi(\tau_{2\ell}) \mathtt{f}_\ell(\vartheta_1(\tau_{2\ell})) \Delta\tau_{2\ell} \Big] \Delta\tau_{2\ell-1} \Big] \Delta\tau_{2\ell-2} \Big] \Delta\tau_{2\ell-3} \\
&\leq \int_0^1 \aleph_\beta(\tau_{2\ell-4},\tau_{2\ell-3}) \Big[\int_0^1 \aleph_\alpha(\tau_{2\ell-3},\tau_{2\ell-2}) \varphi(\tau_{2\ell-2}) \mathtt{f}_{\ell-1}(\mathtt{R}_k) \Delta\tau_{2\ell-2} \Big] \Delta\tau_{2\ell-3} \\
&\leq \mathtt{A}_1 \mathtt{R}_k \int_0^1 \mathcal{G}^*_\beta(\tau_{2\ell-3}) \Big[\int_0^1 \mathcal{G}^*_\alpha(\tau_{2\ell-2}) \varphi(\tau_{2\ell-2}) \Delta\tau_{2\ell-2} \Big] \Delta\tau_{2\ell-3} \\
&\leq \mathtt{A}_1 \mathtt{R}_k \int_0^1 \mathcal{G}^*_\beta(\tau_{2\ell-1}) \Delta\tau_{2\ell-1} \|\mathcal{G}^*_\alpha\|_{\mathtt{q}} \, \|\varphi\|_{\mathtt{p}_j} \leq \mathtt{A}_1 \mathtt{R}_k \int_0^1 \mathcal{G}^*_\beta(\tau_{2\ell-1}) \Delta\tau_{2\ell-1} \|\mathcal{G}^*_\alpha\|_{\mathtt{q}} \prod_{j=1}^{n} \|\varphi_j\|_{\mathtt{p}_j} \\
&\leq \mathtt{R}_k.
\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$(\Omega\vartheta_1)(t) = \int_0^1 \aleph_\beta(t,\tau_1)\Big[\int_0^1 \aleph_\alpha(\tau_1,\tau_2)\varphi(\tau_2)\mathtt{f}_1\Big[\int_0^1 \aleph_\beta(\tau_2,\tau_3)\Big[\int_0^1 \aleph_\alpha(\tau_3,\tau_4)\varphi(\tau_4)\mathtt{f}_2\cdots$$
$$\mathtt{f}_{\ell-1}\Big[\int_0^1 \aleph_\beta(\tau_{2\ell-2},\tau_{2\ell-1})\Big[\int_0^1 \aleph_\alpha(\tau_{2\ell-1},\tau_{2\ell})\varphi(\tau_{2\ell})\mathtt{f}_\ell(\vartheta_1(\tau_{2\ell}))\Delta\tau_{2\ell}\Big]\Delta\tau_{2\ell-1}\cdots\Delta\tau_2\Big]\Delta\tau_1 \le \mathtt{R}_k.$$

Since $\mathtt{R}_k = \|\vartheta_1\|$ for $\vartheta_1 \in \mathscr{N}_{\lambda_k}\cap\partial\mathtt{Q}_{1,k}$, we get

$$\|\Omega\vartheta_1\| \le \|\vartheta_1\|. \tag{3.1}$$

Let $t \in [\lambda_k, 1-\lambda_k]_{\mathbb{T}}$. Then, $\mathtt{r}_k = \|\vartheta_1\| \ge \vartheta_1(t) \ge \min_{t\in[\lambda_k,1-\lambda_k]_{\mathbb{T}}}\vartheta_1(t) \ge \lambda_k\|\vartheta_1\| \ge \lambda_k\mathtt{r}_k$. By $(\mathscr{J}_2)$ and for $\tau_{2\ell-2}\in[\lambda_k,1-\lambda_k]_{\mathbb{T}}$, we have

$$\begin{aligned}
&\int_0^1 \aleph_\beta(\tau_{2\ell-2},\tau_{2\ell-1})\Big[\int_0^1 \aleph_\alpha(\tau_{2\ell-1},\tau_{2\ell})\varphi(\tau_{2\ell})\mathtt{f}_\ell(\vartheta_1(\tau_{2\ell}))\Delta\tau_{2\ell}\Big]\Delta\tau_{2\ell-1}\\
&\qquad\ge \lambda_k\int_0^1 \mathcal{G}^*_\beta(\tau_{2\ell-1})\Big[\int_0^1 \aleph_\alpha(\tau_{2\ell-1},\tau_{2\ell})\varphi(\tau_{2\ell})\mathtt{f}_\ell(\vartheta_1(\tau_{2\ell}))\Delta\tau_{2\ell}\Big]\Delta\tau_{2\ell-1}\\
&\qquad\ge \lambda_k\int_{\lambda_k}^{1-\lambda_k} \mathcal{G}^*_\beta(\tau_{2\ell-1})\Big[\lambda_k\int_{\lambda_k}^{1-\lambda_k} \mathcal{G}^*_\alpha(\tau_{2\ell})\varphi(\tau_{2\ell})\mathtt{f}_\ell(\vartheta_1(\tau_{2\ell}))\Delta\tau_{2\ell}\Big]\Delta\tau_{2\ell-1}\\
&\qquad\ge \lambda_k^2\int_{\lambda_k}^{1-\lambda_k} \mathcal{G}^*_\beta(\tau_{2\ell-1})\Big[\mathtt{L}\mathtt{r}_k\prod_{j=1}^{n}\varphi_j^*\int_{\lambda_k}^{1-\lambda_k} \mathcal{G}^*_\alpha(\tau_{2\ell})\Delta\tau_{2\ell}\Big]\Delta\tau_{2\ell-1}\\
&\qquad\ge \mathtt{L}\mathtt{r}_k\lambda_1^2\prod_{j=1}^{n}\varphi_j^*\int_{\lambda_1}^{1-\lambda_1} \mathcal{G}^*_\beta(\tau_{2\ell-1})\Delta\tau_{2\ell-1}\Big[\int_{\lambda_1}^{1-\lambda_1} \mathcal{G}^*_\alpha(\tau_{2\ell})\Delta\tau_{2\ell}\Big]\\
&\qquad\ge \mathtt{r}_k.
\end{aligned}$$

Continuing with bootstrapping argument, we get

$$(\Omega\vartheta_1)(t) = \int_0^1 \aleph_\beta(t,\tau_1)\Big[\int_0^1 \aleph_\alpha(\tau_1,\tau_2)\varphi(\tau_2)\mathtt{f}_1\Big[\int_0^1 \aleph_\beta(\tau_2,\tau_3)\Big[\int_0^1 \aleph_\alpha(\tau_3,\tau_4)\varphi(\tau_4)\mathtt{f}_2\cdots$$
$$\mathtt{f}_{\ell-1}\Big[\int_0^1 \aleph_\beta(\tau_{2\ell-2},\tau_{2\ell-1})\Big[\int_0^1 \aleph_\alpha(\tau_{2\ell-1},\tau_{2\ell})\varphi(\tau_{2\ell})\mathtt{f}_\ell(\vartheta_1(\tau_{2\ell}))\Delta\tau_{2\ell}\Big]\Delta\tau_{2\ell-1}\cdots\Delta\tau_2\Big]\Delta\tau_1 \ge \mathtt{r}_k.$$

Thus, if $\vartheta_1 \in \mathscr{N}_{\lambda_k}\cap\partial\mathtt{Q}_{2,k}$, then

$$\|\Omega\vartheta_1\| \ge \|\vartheta_1\|. \tag{3.2}$$

It is evident that $0\in\mathtt{Q}_{2,k}\subset\overline{\mathtt{Q}}_{2,k}\subset\mathtt{Q}_{1,k}$. From (3.1) and (3.2), it follows from Theorem 3.1 that the operator Ω has a fixed point $\vartheta_1^{[k]}\in\mathscr{N}_{\lambda_k}\cap\big(\overline{\mathtt{Q}}_{1,k}\backslash\mathtt{Q}_{2,k}\big)$ such that $\vartheta_1^{[k]}(t)\ge 0$ on $(0,1)_{\mathbb{T}}$, and $k\in\mathbb{N}$. Next setting $\vartheta_{n+1}=\vartheta_1$, we obtain infinitely many positive solutions $\{(\vartheta_1^{[k]},\vartheta_2^{[k]},\cdots,\vartheta_n^{[k]})\}_{k=1}^{\infty}$ of (1.1)–(1.2) given iteratively by

$$\vartheta_\ell(t) = \int_0^1 \aleph_\beta(t,\tau)\left[\int_0^1 \aleph_\alpha(\tau,s)\mathtt{f}_\ell(\vartheta_{\ell+1}(s))\Delta s\right]\Delta\tau,\ \ell = n, n-1,\cdots,2,1.$$

The proof is completed. □

For $\sum_{j=1}^{n}\mathtt{p}_j = 1$, we have the following theorem.

Theorem 3.4 *Assume that* $(\mathcal{H}_1)-(\mathcal{H}_3)$ *hold, let* $\{\lambda_k\}_{k=1}^{\infty}$ *be such that* $t_{k+1}<\lambda_k<t_k$, $k=1,2,3,\cdots$. *Let* $\{\mathtt{R}_k\}_{k=1}^{\infty}$ *and* $\{\mathtt{r}_k\}_{k=1}^{\infty}$ *be such that* $\mathtt{R}_{k+1}<\lambda_k\mathtt{r}_k<\mathtt{L}\mathtt{r}_k<\mathtt{R}_k$, $k\in\mathbb{N}$, *where*

$$\mathtt{L}=\max\left\{\left[\lambda_1^2\prod_{j=1}^{n}\varphi_j^*\int_{\lambda_1}^{1-\lambda_1}\mathcal{G}_{\beta}^*(\tau)\Delta\tau\Big[\int_{\lambda_1}^{1-\lambda_1}\mathcal{G}_{\alpha}^*(\tau)\Delta\tau\Big]\right]^{-1},\ 1\right\}.$$

Further, assume that $\mathtt{f}_i$ *satisfies*

$(\mathcal{J}_3)$ $\mathtt{f}_\ell(\vartheta(t))\leq \mathtt{A}_2\mathtt{R}_k$ *for all* $0\leq\vartheta(t)\leq\mathtt{R}_k$, $t\in[0,1]_{\mathbb{T}}$, *where*

$$\mathtt{A}_2<\left[\|\mathcal{G}_{\alpha}^*\|_{\infty}\prod_{j=1}^{n}\|\varphi_j\|_{\mathtt{p}_j}\int_0^1\mathcal{G}_{\beta}^*(\tau)\Delta\tau\right]^{-1}.$$

$(\mathcal{J}_4)$ $\mathtt{f}_\ell(\vartheta(t))\geq\mathtt{L}\mathtt{r}_k$ *for all* $\lambda_k\mathtt{r}_k\leq\vartheta(t)\leq\mathtt{r}_k$, $t\in[\lambda_k,1-\lambda_k]_{\mathbb{T}}$.

The iterative system (1.1)–(1.2) *has infinitely many solutions* $\{(\vartheta_1^{[k]},\vartheta_2^{[k]},\cdots,\vartheta_n^{[k]})\}_{k=1}^{\infty}$ *such that* $\vartheta_\ell^{[k]}(t)\geq 0$ *on* $(0,1)_{\mathbb{T}}$, $\ell=1,2,\cdots,n$ *and* $k\in\mathbb{N}$.

Proof For a fixed k, let $\mathtt{Q}_{1,k}$ be as in the proof of Theorem 3.3 and let $\vartheta_1\in\mathscr{N}_{\lambda_k}\cap\partial\mathtt{Q}_{1,k}$. Again $\vartheta_1(\tau)\leq\mathtt{R}_k=\|\vartheta_1\|$, for all $\tau\in[0,1]_{\mathbb{T}}$. By $(\mathcal{J}_3)$ and for $\tau_{2\ell-2}\in[0,1]_{\mathbb{T}}$, we have

$$\begin{aligned}
&\int_0^1\aleph_{\beta}(\tau_{2\ell-2},\tau_{2\ell-1})\Big[\int_0^1\aleph_{\alpha}(\tau_{2\ell-1},\tau_{2\ell})\varphi(\tau_{2\ell})\mathtt{f}_\ell(\vartheta_1(\tau_{2\ell}))\Delta\tau_{2\ell}\Big]\Delta\tau_{2\ell-1}\\
&\qquad\leq\int_0^1\mathcal{G}_{\beta}^*(\tau_{2\ell-1})\Big[\int_0^1\mathcal{G}_{\alpha}^*(\tau_{2\ell})\varphi(\tau_{2\ell})\mathtt{f}_\ell(\vartheta_1(\tau_{2\ell}))\Delta\tau_{2\ell}\Big]\Delta\tau_{2\ell-1}\\
&\qquad\leq\mathtt{A}_1\mathtt{R}_k\int_0^1\mathcal{G}_{\beta}^*(\tau_{2\ell-1})\Big[\int_0^1\mathcal{G}_{\alpha}^*(\tau_{2\ell})\varphi(\tau_{2\ell})\Delta\tau_{2\ell}\Big]\Delta\tau_{2\ell-1}\\
&\qquad\leq\mathtt{A}_1\mathtt{R}_k\int_0^1\mathcal{G}_{\beta}^*(\tau_{2\ell-1})\Delta\tau_{2\ell-1}\|\mathcal{G}_{\alpha}^*\|_{\infty}\,\|\varphi\|_{\mathtt{p}_j}\\
&\qquad\leq\mathtt{A}_1\mathtt{R}_k\int_0^1\mathcal{G}_{\beta}^*(\tau_{2\ell-1})\Delta\tau_{2\ell-1}\|\mathcal{G}_{\alpha}^*\|_{\infty}\prod_{j=1}^{n}\|\varphi_j\|_{\mathtt{p}_j}\\
&\qquad\leq\mathtt{R}_k.
\end{aligned}$$

It follows in similar manner (for $\tau_{2\ell-4} \in [0,1]_{\mathbb{T}},$) that

$$\begin{aligned}
&\int_0^1 \aleph_\beta(\tau_{2\ell-4},\tau_{2\ell-3})\Big[\int_0^1 \aleph_\alpha(\tau_{2\ell-3},\tau_{2\ell-2})\varphi(\tau_{2\ell-2})\\
&\qquad\times \mathtt{f}_{\ell-1}\Big[\int_0^1 \aleph_\beta(\tau_{2\ell-2},\tau_{2\ell-1})\Big[\int_0^1 \aleph_\alpha(\tau_{2\ell-1},\tau_{2\ell})\varphi(\tau_{2\ell})\mathtt{f}_\ell(\vartheta_1(\tau_{2\ell}))\Delta\tau_{2\ell}\Big]\Delta\tau_{2\ell-1}\Big]\Delta\tau_{2\ell-2}\Big]\Delta\tau_{2\ell-3}\\
&\quad\le \int_0^1 \aleph_\beta(\tau_{2\ell-4},\tau_{2\ell-3})\Big[\int_0^1 \aleph_\alpha(\tau_{2\ell-3},\tau_{2\ell-2})\varphi(\tau_{2\ell-2})\mathtt{f}_{\ell-1}(\mathtt{R}_k)\Delta\tau_{2\ell-2}\Big]\Delta\tau_{2\ell-3}\\
&\quad\le \mathtt{A}_1\mathtt{R}_k\int_0^1 \mathcal{G}^*_\beta(\tau_{2\ell-3})\Big[\int_0^1 \mathcal{G}^*_\alpha(\tau_{2\ell-2})\varphi(\tau_{2\ell-2})\Delta\tau_{2\ell-2}\Big]\Delta\tau_{2\ell-3}\\
&\quad\le \mathtt{A}_1\mathtt{R}_k\int_0^1 \mathcal{G}^*_\beta(\tau_{2\ell-1})\Delta\tau_{2\ell-1}\|\mathcal{G}^*_\alpha\|_\infty\prod_{j=1}^n\|\varphi_j\|_{\mathtt{p}_j}\\
&\quad\le \mathtt{R}_k.
\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned}
(\Omega\vartheta_1)(t) = &\int_0^1 \aleph_\beta(t,\tau_1)\Big[\int_0^1 \aleph_\alpha(\tau_1,\tau_2)\varphi(\tau_2)\mathtt{f}_1\Big[\int_0^1 \aleph_\beta(\tau_2,\tau_3)\Big[\int_0^1 \aleph_\alpha(\tau_3,\tau_4)\varphi(\tau_4)\mathtt{f}_2\cdots\\
&\mathtt{f}_{\ell-1}\Big[\int_0^1 \aleph_\beta(\tau_{2\ell-2},\tau_{2\ell-1})\Big[\int_0^1 \aleph_\alpha(\tau_{2\ell-1},\tau_{2\ell})\varphi(\tau_{2\ell})\mathtt{f}_\ell(\vartheta_1(\tau_{2\ell}))\Delta\tau_{2\ell}\Big]\Delta\tau_{2\ell-1}\cdots\Delta\tau_2\Big]\Delta\tau_1 \le \mathtt{R}_k.
\end{aligned}$$

Since $\mathtt{R}_k = \|\vartheta_1\|$ for $\vartheta_1 \in \mathscr{N}_{\lambda_k}\cap\partial\mathtt{Q}_{1,k},$ we get

$$\|\Omega\vartheta_1\| \le \|\vartheta_1\|. \tag{3.3}$$

Now define $\mathtt{Q}_{2,k} = \{\vartheta \in \mathcal{B} : \|\vartheta\| < r_k\}$. Let $\vartheta \in \mathscr{N}_{\lambda_k}\cap\partial\mathtt{Q}_{2,k}$ and let $\tau \in [\lambda_k, 1-\lambda_k]_{\mathbb{T}}$. Then, the argument leading to (3.2) can be done to the present case. This completes the proof. □

Finally, for $\sum_{j=1}^n \mathtt{p}_j > 1,$ we have the following theorem.

Theorem 3.5 *Assume that* $(\mathcal{H}_1)-(\mathcal{H}_3)$ *hold, let* $\{\lambda_k\}_{k=1}^\infty$ *be such that* $t_{k+1} < \lambda_k < t_k,\ k = 1,2,3,\cdots$. *Let* $\{\mathtt{R}_k\}_{k=1}^\infty$ *and* $\{\mathtt{r}_k\}_{k=1}^\infty$ *be such that*

$$\mathtt{R}_{k+1} < \lambda_k\mathtt{r}_k < \mathtt{L}\mathtt{r}_k < \mathtt{R}_k,\ k\in\mathbb{N},$$

where

$$\mathtt{L} = \max\left\{\left[\lambda_1^2\prod_{j=1}^n\varphi_j^*\int_{\lambda_1}^{1-\lambda_1}\mathcal{G}^*_\beta(\tau)\Delta\tau\Big[\int_{\lambda_1}^{1-\lambda_1}\mathcal{G}^*_\alpha(\tau)\Delta\tau\Big]\right]^{-1},\ 1\right\}.$$

Further, assume that $\mathtt{f}_i$ *satisfies*

$(\mathcal{J}_5)$ $\mathtt{f}_\ell(\vartheta(t)) \le \mathtt{A}_3\mathtt{R}_k$ *for all* $0 \le \vartheta(t) \le \mathtt{R}_k,\ t\in[0,1]_{\mathbb{T}},$ *where* $\mathtt{A}_3 < \left[\|\mathcal{G}^*_\alpha\|_\infty\prod_{j=1}^n\|\varphi_j\|_1\int_0^1\mathcal{G}^*_\beta(\tau)\Delta\tau\right]^{-1}.$

$(\mathcal{J}_6)$ $\mathtt{f}_\ell(\vartheta(t)) \geq \mathtt{L}\mathtt{r}_k$ *for all* $\lambda_k \mathtt{r}_k \leq \vartheta(t) \leq \mathtt{r}_k$, $t \in [\lambda_k, 1-\lambda_k]_{\mathbb{T}}$.

The iterative system (1.1)–(1.2) *has infinitely many solutions* $\{(\vartheta_1^{[k]}, \vartheta_2^{[k]}, \cdots, \vartheta_n^{[k]})\}_{k=1}^{\infty}$ *such that* $\vartheta_\ell^{[k]}(t) \geq 0$ *on* $(0,1)_{\mathbb{T}}$, $\ell = 1, 2, \cdots, n$ *and* $k \in \mathbb{N}$.

Proof The proof is similar to the proof of Theorem 3.3. So, we omit the details here. □

4. Examples

In this section, we provide two examples to check the validity of our main results.

Example 4.1 *Consider the following boundary value problem on* $\mathbb{T} = [0,1]$.

$$\left.\begin{aligned} \mathcal{T}_{3/2}\big(\mathcal{T}_{3/2}\vartheta_{\mathtt{n}}(t)\big) + \varphi(t)\mathtt{f}_{\mathtt{n}}(\vartheta_{\mathtt{n}+1}(t)) = 0,\ 0<t<1,\ \mathtt{n}=1,2,\\ \vartheta_3(t) = \vartheta_1(t),\ 0<t<1, \end{aligned}\right\} \tag{4.1}$$

$$\left.\begin{aligned} \vartheta_{\mathtt{n}}(0) = 0,\ \vartheta_{\mathtt{n}}(1) = \int_0^1 \vartheta_{\mathtt{n}}(\tau)d\mathtt{g}(\tau),\ \mathtt{n}=1,2,\\ \mathcal{T}_{3/2}\vartheta_{\mathtt{n}}(0) = 0,\ \mathcal{T}_{3/2}\vartheta_{\mathtt{n}}(1) = \int_0^1 \vartheta_{\mathtt{n}}(\tau)d\mathtt{g}(\tau),\ \mathtt{n}=1,2, \end{aligned}\right\} \tag{4.2}$$

where

$$\varphi(t) = \varphi_1(t)\varphi_2(t)$$

in which

$$\varphi_1(t) = \frac{1}{|t-\frac{1}{4}|} \qquad \text{and} \qquad \varphi_2(t) = \frac{1}{|t-\frac{1}{3}|},$$

$$\mathtt{f}_1(\vartheta) = \mathtt{f}_2(\vartheta) = \begin{cases} 188\times 10^{-4}, & \vartheta \in (10^{-4}, +\infty),\\ \frac{51\times 10^{-(4k+2)} - 188\times 10^{-4k}}{10^{-(4k+2)} - 10^{-4k}}(\vartheta - 10^{-4k}) + 188\times 10^{-4k}, & \vartheta \in \left[10^{-(4k+2)}, 10^{-4k}\right],\\ 51\times 10^{-(4k+2)}, & \vartheta \in \left(\frac{1}{5}\times 10^{-(4k+2)}, 10^{-(4k+2)}\right),\\ \frac{51\times 10^{-(4k+2)} - 188\times 10^{-(4k+4)}}{\frac{1}{5}\times 10^{-(4k+2)} - 10^{-(4k+4)}}(\vartheta - 10^{-(4k+4)}) + 188\times 10^{-(4k+4)}, & \vartheta \in \left(10^{-(4k+4)}, \frac{1}{5}\times 10^{-(4k+2)}\right],\\ 0, & \vartheta = 0, \end{cases}$$

$$\mathtt{g}(t) = \begin{cases} t, & t \in [0,1/2) \cup [2/3, 5/6),\\ \frac{1}{2}, & t \in [1/2, 2/3),\\ \frac{5}{6}, & t \in [5/6, 1]. \end{cases}$$

Let

$$t_k = \frac{31}{64} - \sum_{r=1}^{k} \frac{1}{4(r+1)^4},\ \lambda_k = \frac{1}{2}(t_k + t_{k+1}),\ k = 1, 2, 3, \cdots,$$

then

$$\lambda_1 = \frac{15}{32} - \frac{1}{648} < \frac{15}{32}$$

and

$$t_{k+1} < \lambda_k < t_k,\ \lambda_k > \frac{1}{5}.$$

It is easy to see

$$t_1 = \frac{15}{32} < \frac{1}{2},\ t_k - t_{k+1} = \frac{1}{4(k+2)^4},\ k = 1, 2, 3, \cdots.$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$, it follows that

$$t^* = \lim_{k\to\infty} t_k = \frac{31}{64} - \sum_{i=1}^{\infty} \frac{1}{4(i+1)^4} = \frac{47}{64} - \frac{\pi^4}{360} > \frac{1}{5},$$

$$\varphi_1, \varphi_2 \in L^{\mathtt{p}}_{\Delta}[0,1]_{\mathbb{T}} \quad \text{for all} \quad 0 < \mathtt{p} < 2, \quad \text{so} \quad \varphi_1^* = \varphi_2^* = \frac{1}{3}.$$

$$\lambda_1^2 \prod_{j=1}^{n} \varphi_j^* \int_{\lambda_1}^{1-\lambda_1} \mathcal{G}_{\beta}^*(\tau)d\tau \Big[\int_{\lambda_1}^{1-\lambda_1} \mathcal{G}_{\alpha}^*(\tau)d\tau\Big] \approx 0.02,$$

$$\begin{aligned}\mathtt{L} &= \max\left\{\left[\lambda_1^2 \prod_{j=1}^{n} \varphi_j^* \int_{\lambda_1}^{1-\lambda_1} \mathcal{G}_{\beta}^*(\tau)d\tau \Big[\int_{\lambda_1}^{1-\lambda_1} \mathcal{G}_{\alpha}^*(\tau)d\tau\Big]\right]^{-1},\ 1\right\}\\ &= \{50, 1\} = 50.\end{aligned}$$

and

$$\|\mathcal{G}_{\alpha}^*\|_{\mathtt{2}} \approx 12.08,\ \int_0^1 \mathcal{G}_{\beta}^*(\tau)d\tau \approx 3.22.$$

Next, let $0 < a < 1$ be fixed. Then $\varphi_1, \varphi_2 \in L^{1+a}[0,1]$. It follows that

$$\|\varphi_1\|_{1+a} = \left[-\frac{4^a(1+3^{-a})}{a}\right]^{\frac{1}{1+a}},$$

$$\|\varphi_2\|_{1+a} = \left[-\frac{3^a(1+2^{-a})}{a}\right]^{\frac{1}{1+a}}.$$

So, for $0 < a < 1$, we have

$$189.825 \leq \left[\|\mathcal{G}_{\alpha}^*\|_{\mathtt{q}} \prod_{j=1}^{n} \|\varphi_j\|_{\mathtt{p}_j} \int_0^1 \mathcal{G}_{\beta}^*(\tau)d\tau\right]^{-1}.$$

Taking $\mathtt{A}_1 = 189$. In addition if we take

$$\mathtt{R}_k = 10^{-4k},\ \mathtt{r}_k = 10^{-(4k+2)},$$

then

$$\begin{aligned}\mathtt{R}_{k+1} = 10^{-(4k+4)} &< \frac{1}{5} \times 10^{-(4k+2)} < \lambda_k \mathtt{r}_k \\ &< \mathtt{r}_k = 10^{-(4k+2)} < \mathtt{R}_k = 10^{-4k},\end{aligned}$$

$\mathtt{L}\mathtt{r}_k = 50 \times 10^{-(4k+2)} < 189 \times 10^{-4k} = \mathtt{A}_1\mathtt{R}_k$, $k = 1, 2, 3, \cdots$, and $\mathtt{g}_1$ and $\mathtt{g}_2$ satisfies the following growth conditions:

$$\mathtt{f}_1(\vartheta) = \mathtt{f}_2(\vartheta) \le \mathtt{A}_1\mathtt{R}_k = 189 \times 10^{-4k},\quad \vartheta \in \left[0, 10^{-4k}\right]$$

$$\mathtt{f}_1(\vartheta) = \mathtt{f}_2(\vartheta) \ge \mathtt{L}\mathtt{r}_k = 50 \times 10^{-(4k+2)},\quad \vartheta \in \left[\frac{1}{5} \times 10^{-(4k+2)}, 10^{-(4k+2)}\right].$$

All the conditions of Theorem 3.3 are satisfied. Therefore, by Theorem 3.3, the boundary value problem (4.1)–(4.2) has infinitely many positive solutions $\{\vartheta^{[k]}\}_{k=1}^{\infty}$ such that $10^{-(4k+2)} \le \|\vartheta^{[k]}\| \le 10^{-4k}$ for each $k = 1, 2, 3, \cdots$.

Example 4.2 *Let* $\mathbb{T} = \{0\} \cup [1/2, 1] \cup \left\{\frac{1}{2^{n+1}} : n \in \mathbb{N}\right\}$. *Consider the boundary value problem*

$$\left.\begin{aligned}\mathcal{T}_{7/4}\big(\mathcal{T}_{5/4}\vartheta_{\mathtt{n}}(t)\big) + \varphi(t)\mathtt{f}_{\mathtt{n}}(\vartheta_{\mathtt{n}+1}(t)) = 0,\ t \in (0,1)_{\mathbb{T}},\ \mathtt{n} = 1, 2,\\ \vartheta_3(t) = \vartheta_1(t),\ t \in (0,1)_{\mathbb{T}},\end{aligned}\right\} \tag{4.3}$$

$$\left.\begin{aligned}\vartheta_{\mathtt{n}}(0) = 0,\ \vartheta_{\mathtt{n}}(1) = \int_0^1 \vartheta_{\mathtt{n}}(\tau)\Delta\mathtt{g}(\tau),\ \mathtt{n} = 1, 2,\\ \mathcal{T}_{5/4}\vartheta_{\mathtt{n}}(0) = 0,\ \mathcal{T}_{5/4}\vartheta_{\mathtt{n}}(1) = \int_0^1 \vartheta_{\mathtt{n}}(\tau)\Delta\mathtt{g}(\tau),\ \mathtt{n} = 1, 2,\end{aligned}\right\} \tag{4.4}$$

where

$$\varphi(t) = \varphi_1(t)\varphi_2(t)$$

in which

$$\varphi_1(t) = \frac{1}{|t - \frac{3}{5}|} \qquad \text{and} \qquad \varphi_2(t) = \frac{1}{|t - \frac{4}{5}|},$$

$$
\mathtt{f}_1(\vartheta) = \mathtt{f}_2(\vartheta) = \begin{cases} 0.2 \times 10^{-5}, & \vartheta \in (10^{-5}, +\infty), \\ \frac{110\times 10^{-(4k+4)} - 0.2\times 10^{-(4k+1)}}{10^{-(4k+4)} - 10^{-(4k+1)}}(\vartheta - 10^{-(4k+1)}) + 0.2 \times 10^{-(4k+1)}, \\ \qquad\qquad \vartheta \in \left[10^{-(4k+4)}, 10^{-(4k+1)}\right], \\ 110 \times 10^{-(4k+4)}, & \vartheta \in \left(\frac{1}{5} \times 10^{-(4k+4)}, 10^{-(4k+4)}\right), \\ \frac{110\times 10^{-(4k+4)} - 0.2\times 10^{-(4k+5)}}{\frac{1}{5}\times 10^{-(4k+4)} - 10^{-(4k+5)}}(\vartheta - 10^{-(4k+5)}) + 0.2 \times 10^{-(4k+5)}, \\ \qquad\qquad \vartheta \in \left(10^{-(4k+5)}, \frac{1}{5} \times 10^{-(4k+4)}\right], \\ 0, & \vartheta = 0, \end{cases}
$$

$$
\mathtt{g}(t) = \begin{cases} 0, & t \in [0, 1/2), \\ t, & t \in [1/2, 1]. \end{cases}
$$

Let t_k, λ_k be the same as in example 4.1. Then $\lambda_1 = \frac{15}{32} - \frac{1}{648} < \frac{15}{32}$, $t_{k+1} < \lambda_k < t_k$, $\lambda_k > \frac{1}{5}$ and $t_1 = \frac{15}{32} < \frac{1}{2}$, $t_k - t_{k+1} = \frac{1}{4(k+2)^4}$, $k = 1, 2, 3, \cdots$. Also, $t^* = \lim_{k\to\infty} t_k = \frac{31}{64} - \sum_{i=1}^{\infty} \frac{1}{4(i+1)^4} = \frac{47}{64} - \frac{\pi^4}{360} > \frac{1}{5}$. By simple calculations, we obtain $\mathtt{g}^* = \frac{1}{4}$, $\varphi_1^* = \varphi_2^* = 5/3$, $\mathcal{G}_x^*(\tau) = \mathcal{K}_x(\tau, \tau) + \frac{4}{3}\mathcal{G}_x(\tau)$, $x \in \{\alpha, \beta\}$, where

$$
\mathcal{G}_\alpha(\tau) = \int_0^1 \mathcal{K}_\alpha(\tau_1, \tau)\Delta \mathtt{g}(\tau_1) = \int_{1/2}^1 \mathcal{K}_\alpha(\tau_1, \tau)\Delta\tau_1 = \frac{1}{8\tau^{1/4}}(4\tau^2 - 1)(1 - \tau) + \frac{1}{2}\tau^{3/4}(\tau - 1)^2,
$$

and

$$
\mathcal{G}_\beta(\tau) = \int_0^1 \mathcal{K}_\beta(\tau_1, \tau)\Delta \mathtt{g}(\tau_1) = \int_{1/2}^1 \mathcal{K}_\beta(\tau_1, \tau)\Delta\tau_1 = \frac{1}{8\tau^{3/4}}(4\tau^2 - 1)(1 - \tau) + \frac{1}{2\tau^{1/4}}(\tau - 1)^2.
$$

So, we get

$$
\lambda_1^2 \prod_{j=1}^{n} \varphi_j^* \int_{\lambda_1}^{1-\lambda_1} \mathcal{G}_\beta^*(\tau) d\tau \Big[\int_{\lambda_1}^{1-\lambda_1} \mathcal{G}_\alpha^*(\tau) d\tau \Big] \approx 0.009191748427,
$$

$$
\begin{aligned}
\mathtt{L} &= \max \left\{ \left[\lambda_1^2 \prod_{j=1}^{n} \varphi_j^* \int_{\lambda_1}^{1-\lambda_1} \mathcal{G}_\beta^*(\tau) d\tau \Big[\int_{\lambda_1}^{1-\lambda_1} \mathcal{G}_\alpha^*(\tau) d\tau \Big] \right]^{-1}, \ 1 \right\} \\
&= \{108.793, 1\} = 108.793,
\end{aligned}
$$

$\|\mathcal{G}_\alpha^*\|_2 \approx 0.03517346$, $\int_0^1 \mathcal{G}_\beta^*(\tau) d\tau \approx 3.038214786$, $\|\varphi_1\|_3 \approx 6.699800077$, $\|\varphi_2\|_6 \approx 5.839258384$.

$$
0.2410292729 \le \left[\|\mathcal{G}_\alpha^*\|_{\mathtt{q}} \prod_{j=1}^{n} \|\varphi_j\|_{\mathtt{p}_j} \int_0^1 \mathcal{G}_\beta^*(\tau) d\tau \right]^{-1}.
$$

Taking $\mathtt{A}_1 = 0.24$. In addition, if we take $\mathtt{R}_k = 10^{-(4k+1)}$, $\mathtt{r}_k = 10^{-(4k+4)}$, then

$$\mathtt{R}_{k+1} = 10^{-(4k+5)} < \frac{1}{5} \times 10^{-(4k+4)} < \lambda_k \mathtt{r}_k$$
$$< \mathtt{r}_k = 10^{-(4k+4)} < \mathtt{R}_k = 10^{-(4k+1)},$$

$\mathtt{Lr}_k = 108.793 \times 10^{-(4k+4)} < 0.24 \times 10^{-(4k+1)} = \mathtt{A}_1\mathtt{R}_k$, $k = 1, 2, 3, \cdots$, and $\mathtt{g}_1$ and $\mathtt{g}_2$ satisfies the following growth conditions:

$$\mathtt{f}_1(\vartheta) = \mathtt{f}_2(\vartheta) \leq \mathtt{A}_1\mathtt{R}_k = 0.24 \times 10^{-(4k+1)}, \quad \vartheta \in \left[0, 10^{-(4k+1)}\right]$$
$$\mathtt{f}_1(\vartheta) = \mathtt{f}_2(\vartheta) \geq \mathtt{Lr}_k = 108.793 \times 10^{-(4k+4)}, \quad \vartheta \in \left[\frac{1}{5} \times 10^{-(4k+4)}, 10^{-(4k+2)}\right].$$

All the conditions of Theorem 3.3 are satisfied. Therefore, by Theorem 3.3, the boundary value problem (4.3)–(4.4) has infinitely many positive solutions $\{\vartheta^{[k]}\}_{k=1}^{\infty}$ such that $10^{-(4k+4)} \leq \|\vartheta^{[k]}\| \leq 10^{-(4k+1)}$ for each $k = 1, 2, 3, \cdots$.

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