Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 481 – 498
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doi:10.3906/mat-2106-84

Solvability of Gripenberg's equations of fractional order with perturbation term in weighted L_p-spaces on $\mathbb{R}^+$

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Received: 20.06.2021 • **Accepted/Published Online:** 08.09.2021 • **Final Version:** 21.01.2022

Abstract: This article deals with the solvability of Gripenberg's equations of fractional order with a perturbation term in weighted Lebesgue spaces on $\mathbb{R}^+ = [0, \infty)$ via the fixed point hypothesis and the measure of noncompactness. The uniqueness of the solutions for the studied problem is discussed. An example is included to validate our results. The results presented in the article extend and generalize some former results in the available literature.

Key words: Measure of noncompactness, Gripenberg's equations, fractional calculus, weighted Lebesgue spaces, Schauder fixed point theorem

1. Introduction

The integral equations of Gripenberg's type of the form

$$x(\tau) = k\left(g_1(\tau) + \int_0^\tau a_1(\tau - s)x(s)ds\right)\left(g_2(\tau) + \int_0^\tau a_2(\tau - s)x(s)ds\right), \ \tau \in \mathbb{R}^+$$

have numerous applications in mathematical modeling, in particular in the models of the spread of diseases, that do not induce permanent immunity. Some qualitative properties of the solutions of models such as existence and uniqueness of continuous solutions were studied in [1, 12, 20, 29].

Additionally, numerous types of infectious diseases models consist of discontinuous data functions, so it is recommended to examine the integrable solutions of models.

The purpose of the article is to examine the solutions of the equation

$$x(\tau) = h(\tau, x(\varphi_3(\tau))) + \Big(g_1(\tau) + g_3(\tau) \cdot (Gx)(\varphi_1(\tau))\Big)\left(g_2(\tau) + \frac{1}{\Gamma(\alpha)}\int_0^\tau \frac{u(s, x(\varphi_2(s)))}{(\tau - s)^{1-\alpha}}ds\right), \ \tau \in \mathbb{R}^+ \quad (1.1)$$

in weighted Lebesgue spaces $L_1^N(\mathbb{R}^+)$, where $N > 0, \ 0 < \alpha < 1$.

In [22] the authors have discussed the existence of solutions in weighted L_p-spaces for Hammerstein integral equations

$$x(\tau) = h(\tau) + \int_0^\infty a(\tau, s)f(s, x(s))\,ds$$

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2010 *AMS Mathematics Subject Classification:* 45G10, 47H30, 47N20

and integral equations of Urysohn-type

$$x(\tau) = h(\tau) + \int_0^\infty u(\tau, s, x(s))\, ds$$

using Schauder fixed point hypothesis. The authors in [24] have discussed the existence results of Volterra integral equation by utilizing the Banach contraction mapping principle in the considered spaces.

Recall that, the idea of the measure of noncompactness were utilized to demonstrate the presence of continuous or L_p-solutions of the integral equations on compact or noncompact domains in [3, 5, 13, 15–17, 21, 25, 33] and for qualitative properties of the outcomes (cf. [9, 10, 26, 32]).

Further, in [8] the authors examined the solutions of the equation

$$x(\tau) = h(\tau, x(\tau)) + \left(p_1(\tau) + \int_0^\tau k_1(\tau, s) g_1(s, x(s)) ds\right)\left(p_2(\tau) + \int_0^\tau k_2(\tau, s) g_2(s, x(s)) ds\right)$$

in the space $L_1([0,1])$ and these results were extended to an unbounded interval in [11].

In [27] the author used the measure of weak noncompactness with Krasnoselskii fixed point theory to find $L_1(\mathbb{R}^+)$-solutions of the equation

$$x(\tau) = h(\tau, x(\varphi_3(\tau))) + g\bigg(\tau,\ f(\tau, x(\varphi_2(\tau))) \int_0^\tau u(\tau, s, x(\varphi_1(s)))\bigg),\ \tau \in \mathbb{R}^+.$$

Nevertheless, to investigate our results, we use triple of weighted spaces $(L_1^N,\ L_p^N,\ L_q^N)$. One more challenge is to examine the integrable solutions for fractional integral equation on unbounded domains $\mathbb{R}^+$.

Hence, we dedicate our investigations to expand the previously mentioned results as particular cases of Equation (1.1) in the weighted Lebesgue spaces on $L_1^N(\mathbb{R}^+)$ using a different set of assumptions. Moreover, the uniqueness of the solutions of Equation (1.1) are also discussed. At the end, we give an example to validate the applicably of our results. Our methodology utilizes Schauder's fixed point hypothesis and appropriate measure of noncompactness.

2. Notation and auxiliary facts

Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^+ = [0, \infty)$ and Φ be the empty set. Denote by $L_p^N = L_p^N(\mathbb{R}^+)$, $1 \le p < \infty$ the weighted Lebesgue spaces, which is the spaces of measurable functions x with

$$\|x\|_{L_p^N} = \left(\int_0^\infty e^{-N\tau} |x(\tau)|^p\, d\tau\right)^{\frac{1}{p}} < \infty, \quad N > 0.$$

When $N = 0$ we obtain classical L_p-spaces for $1 \le p < \infty$ with standard norm.

Let $C(D)$ refer to the space of real and continuous functions on a closed and bounded set $\Phi \ne D \subset \mathbb{R}$. Let $\Phi \ne X \subset C(D)$ be bounded set and $T > 0$.

For $\varepsilon > 0$ and $x \in X$, we denote by $\omega^T(x, \varepsilon)$ the modulus of continuity of the function x, on the interval $[0, T]$, where

$$\omega^T(x, \varepsilon) = \sup\{|x(\tau_2) - x(\tau_1)| :\ \tau_1, \tau_2 \in [0, T],\ |\tau_1 - \tau_2| \le \varepsilon\}.$$

Suppose that $(E, \|\cdot\|)$ is a Banach space and has zero element θ and $B_r = \{x \in L_1^N : \|x\|_{L_1^N} \leq r\},\ r > 0$. Let $\Phi \neq \mathcal{M}_E$ and $\mathcal{N}_E^W$ be the family of all bounded subsets and the subfamily containing all relatively weakly compact sets of E, respectively. The symbols Conv$Y$ and $\bar{Y}^W$ stand for the convex closed hull and the weak closure of a set Y, respectively.

Definition 2.1 [6] *The operator $\mu : \mathcal{M}_E \to [0,\ \infty)$ refers to a regular measure of weak noncompactness in E if it fulfills:*

(i) $\mu(Y) = 0 \iff Y \in \mathcal{N}_E^W$.

(ii) $Y \subset X \Rightarrow \mu(Y) \leq \mu(X)$.

(iii) $\mu(\bar{Y}) = \mu(ConvY) = \mu(Y)$.

(iv) $\mu(\lambda Y) = |\lambda|\ \mu(Y),\ for\ \lambda \in \mathbb{R}$.

(v) $\mu(Y + X) \leq \mu(Y) + \mu(X)$.

(vi) $\mu(Y \bigcup X) = \max\{\mu(Y),\ \mu(Y)\}$.

(vii) If $\Phi \neq Y_n \subset E$ is a sequence of closed and bounded sets, $Y_n = \bar{Y}_n^W$ s.t. $Y_{n+1} \subset Y_n,\ n = 1, 2, 3, \cdots$, and $\lim_{n\to\infty} \mu(Y_n) = 0$, then $Y_\infty = \bigcap_{n=1}^{\infty} Y_n \neq \Phi$.

De Blasi presented the next measure of weak noncompactness β (cf. [7]):

$$\beta(X) = \inf\{r > 0 : \text{ there exists a weakly compact subset } Y \text{ of } E \text{ s.t. } X \subset Y + B_r\}.$$

Now, for a bounded set $\Phi \neq X \subset L_1$, $\varepsilon > 0$ and $T > 0$, consider the following:

$$c^T(X) = \lim_{\varepsilon\to 0}\left\{\sup_{x\in X}\left\{\sup\left(\int_D |x(\tau)|\,d\tau : D \subset [0,T], meas(D) \leq \varepsilon\right)\right\}\right\},$$

$$c(X) = \lim_{T\to\infty} c^T(X), \tag{2.1}$$

where meas(D) refers to the Lebesgue measure of the set D and let

$$d(X) = \lim_{T\to\infty} \sup\left\{\int_T^\infty |x(\tau)|\,d\tau : x \in X\right\}. \tag{2.2}$$

Put

$$\gamma(X) = c(X) + d(X). \tag{2.3}$$

From [2, 7] we know the function $\gamma(X)$ represents a measure of weak noncompactness in L_1 and $\beta(X) \leq \gamma(X) \leq 2\beta(X)$.

Theorem 2.2 *[18] Let $Q \neq \Phi$ be a convex, bounded, and closed subset of the Banach space E. Let $H : Q \to Q$ be a completely continuous mapping. Hence, H has at least one fixed point in Q.*

Definition 2.3 *[4] Suppose that a function $u(\tau, x):\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ fulfills the Carathéodory conditions, i.e. it is continuous in x for almost all $\tau\in\mathbb{R}^+$ and measurable in τ for any $x\in\mathbb{R}$. Then to every measurable function x, we can assign the superposition (Nemytskii) operator F_u as follows.*

$$F_u(x)(\tau) = u(\tau, x(\tau)), \quad \tau\in\mathbb{R}^+.$$

Moreover, for every functions $u\in L_1$ and $\varphi:\mathbb{R}^+\to\mathbb{R}^+$, we denote the operator $F_{\varphi,u}$, as

$$F_{\varphi,u}(x)(\tau) = u(\tau, x(\varphi(\tau))), \quad \tau\in\mathbb{R}^+.$$

Lemma 2.4 *Assume that the function u fulfills the Carathéodory conditions and $\exists$ a function $a\in L_q^N$ and $b\geq 0$ s.t.*

$$|u(\tau,x)|\leq a(\tau) + b\cdot|x|^{\frac{p}{q}} \tag{2.4}$$

for all $\tau\in\mathbb{R}^+$ and $x\in\mathbb{R}$. Then the operator $F_u: L_p^N\to L_q^N$ $(p,q\geq 1)$ is continuous.

Proof By using (2.4), we have

$$\begin{aligned}
\|F_u\|_{L_q^N} &= \left(\int_0^\infty e^{-N\tau}|u(\tau,x(\tau))|^q\,d\tau\right)^{\frac{1}{q}}\\
&\leq \left(\int_0^\infty e^{-N\tau}\left(a(\tau)+b\cdot|x(\tau)|^{\frac{p}{q}}\right)^q\,d\tau\right)^{\frac{1}{q}}\\
&\leq \left(\int_0^\infty e^{-N\tau}|a(\tau)|^q\,d\tau\right)^{\frac{1}{q}} + b\left(\int_0^\infty e^{-N\tau}|x(\tau)|^p\,d\tau\right)^{\frac{p}{pq}}\\
&\leq \|a\|_{L_q^N} + b\|x\|_{L_p^N}^{\frac{p}{q}}.
\end{aligned}$$

Thus $F_u: L_p^N\to L_q^N$ and is continuous by using the well known results in [4]. □

Furthermore, let D^c stands for the complement of D and let J be an interval in the next theorems.

Theorem 2.5 *Let $m: J\to\mathbb{R}$ be a measurable function. $\exists$ a closed set $D_\varepsilon\subset J$, $\varepsilon>0$ s.t. $m|_{D_\varepsilon}$ is continuous with $meas(D_\varepsilon^c)\leq\varepsilon$.*

Theorem 2.6 *[31] Let $u: J\times\mathbb{R}\to\mathbb{R}$ be a function fulfilling Carathéodory conditions. $\exists$ a closed set $D_\varepsilon\subset J$, $\varepsilon>0$ s.t. $f|_{D_\varepsilon\times\mathbb{R}}$ is continuous with $meas(D_\varepsilon^c)\leq\varepsilon$.*

These results were proved in [28] for general operator $G: Z\times X\to Y$, where Z is a topological space with finite Borel regular measure, X is a metrizable separable locally compact space, and Y is topological spaces.

Next, we give the definition of fractional operator.

Definition 2.7 *[23, 26, 30] Let $y\in L_1$, and $\alpha\in\mathbb{R}^+$. The fractional integral of Riemman-Liouville type (R-L) of the function y of order α is given by*

$$I^\alpha y(\tau) = \frac{1}{\Gamma(\alpha)}\int_0^\tau \frac{y(s)}{(\tau-s)^{1-\alpha}}\,ds, \quad \alpha>0, \quad \tau>0,$$

where $\Gamma(\alpha)=\int_0^\infty e^{-s}s^{\alpha-1}\,ds$.

3. Main results

Let us write (1.1) as follows:

$$x = (Hx) = F_{\varphi_3,h}(x) + (Bx)\cdot(Ax),$$

$$(Bx)(\tau) = g_1(\tau) + \ g_3(\tau)\cdot(Gx)(\varphi_1(\tau)),\ (Ax)(\tau) = g_2(\tau) + I^{\alpha}F_{\varphi_2,u}(x)(\tau),$$

where $F_{\varphi_3,h}, F_{\varphi_2,u}$ are the superposition operators, as in Definition 2.3, I^{α} as in Definition 2.7 and (Gx) is an operator maps L_1^N into L_q^N continuously.

Let $\frac{1}{p} + \frac{1}{q} = 1$ and write the assumptions:

(i) $g_i : \mathbb{R}^+ \to \mathbb{R},\ i = 1,2,3$, where $g_1 \in L_q^N$, $g_2 \in L_p^N$ and g_3 is bounded function s.t. $\sup_{\tau\in\mathbb{R}^+}|g_3(\tau)| \le M$.

(ii) Suppose that $u, h, G : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ fulfill Carathéodory conditions. Further, assume that $G : L_1^N \to L_q^N$ is continuous.

(iii) $\exists\ b_i \ge 0\ (i = 1,2,3)$ and positive functions $a_1 \in L_q^N,\ a_2 \in L_p^N$, and $a_3 \in L_1^N$ s.t.

$$|Gx| \le a_1(\tau) + b_1|x|^{\frac{1}{q}}, \quad |u(\tau,x)| \le a_2(\tau) + b_2|x|^{\frac{1}{p}}$$

and

$$|h(\tau,x)| \le a_3(\tau) + b_3|x|$$

for all $\tau \in \mathbb{R}^+$ and $x \in \mathbb{R}$.

(iv) $\varphi_i : \mathbb{R}^+ \to \mathbb{R}^+$ are increasing and absolutely continuous functions. Further, $\exists\ M_i > 0$ s.t. $\varphi_i' \ge M_i$ a.e. on $(0,\infty)$ and $\varphi_i^{-1}(\tau) \ge \tau,\ i = 1,2,3$.

(v) Assume that $\exists\ N > 0$, s.t.

$$W = e^{N^2}\left(\frac{b_3}{M_3} + \frac{Mb_1b_2p^{\alpha}}{N^{\alpha}M_1^{\frac{1}{q}}M_2^{\frac{1}{p}}}\right) < 1.$$

(vi) Assume that r is a positive number satisfying

$$\begin{aligned}\|a_3\|_{L_1^N} \quad &+ \quad \left(\|g_1\|_{L_q^N} + M\|a_1\|_{L_q^N}\right)\left(\|g_2\|_{L_p^N} + \left(\frac{p}{N}\right)^{\alpha}\|a_2\|_{L_p^N}\right) + \left(\|g_1\|_{L_q^N} + M\|a_1\|_{L_q^N}\right)\frac{b_2p^{\alpha}}{N^{\alpha}M_2^{\frac{1}{p}}}\cdot r^{\frac{1}{p}} \\ &+ \quad \left(\|g_2\|_{L_p^N} + \left(\frac{p}{N}\right)^{\alpha}\|a_2\|_{L_p^N}\right)\frac{Mb_1}{M_1^{\frac{1}{q}}}\cdot r^{\frac{1}{q}} + \left(\frac{b_3}{M_3} + \frac{Mb_1b_2p^{\alpha}}{N^{\alpha}M_1^{\frac{1}{q}}M_2^{\frac{1}{p}}}\right)\cdot r \le r.\end{aligned}$$

Proposition 3.1 *We can easily check the followings:*

(a) *Assume that the functions* $g \in L_q^N$ *and* $f \in L_p^N$ *s.t.* $\frac{1}{p} + \frac{1}{q} = 1$, *then we have the following inequality*

$$\begin{aligned}
\|g \cdot f\|_{L_1^N} &= \int_0^\infty e^{-N\tau}|g(\tau)f(\tau)|d\tau \\
&= \int_0^\infty (e^{-N\tau} * e^{\frac{N\tau}{p}})e^{-\frac{N\tau}{p}}|g(\tau)f(\tau)|d\tau \\
&\leq \left(\int_0^\infty \left(e^{-N\tau(1-\frac{1}{p})}|g(\tau)|\right)^q d\tau\right)^{\frac{1}{q}} \left(\int_0^\infty \left(e^{-\frac{N\tau}{p}}|f(\tau)|\right)^p d\tau\right)^{\frac{1}{p}} \\
&\leq \|g\|_{L_q^N} \cdot \|f\|_{L_p^N}.
\end{aligned}$$

(b) *The operator* $I^\alpha,\ \alpha > 0$ *is bounded in* $L_p^N,\ 1 \leq p \leq \infty$ *(cf. [30, Theorem 5.7] see also [23]) s.t.*

(1) $\|I^\alpha f\|_{L_p^N} \leq (\frac{p}{N})^\alpha \|f\|_{L_p^N}$ *for* $1 \leq p < \infty$.

(2) $\|I^\alpha f\|_{L_p^N} \leq (\frac{1}{N})^\alpha \|f\|_{L_p^N}$ *for* $p = \infty$.

(3) *The operator* $I^\alpha : L_p^N \to L_p^N$ *and is continuous.*

(c) *Let,* $D \subset \mathbb{R}^+$ *with* $meas(D) \leq \varepsilon,\ \varepsilon > 0$ *be arbitrary. For* $0 \leq \tau \leq N,\ N > 0$ *and by using assumption (iv), we have*

(1) $e^{-N^2} \int_D |x(\tau)|d\tau \leq \int_D e^{-N\tau}|x(\tau)|d\tau \leq \int_D |x(\tau)|d\tau.$

(2) $\|(x(\varphi_i))^{\frac{1}{p}}\|_{L_p^N} \leq \frac{1}{M_i^{\frac{1}{p}}} \left(\int_0^\infty e^{-Ns}|x(\varphi_i(s))|\varphi_i'(s)\ ds \right)^{\frac{1}{p}} \leq \frac{1}{M_i^{\frac{1}{p}}} \|x\|_{L_1^N}^{\frac{1}{p}}.$

(3) $\|(x(\varphi_i))^{\frac{1}{p}}\|_{L_p^N(D)} \leq \frac{1}{M_i^{\frac{1}{p}}} \left(\int_{\varphi_i(D)} e^{-Nu}|x(u)|du \right)^{\frac{1}{p}} \leq \frac{1}{M_i^{\frac{1}{p}}} \left(\int_{\varphi_i(D)} |x(u)|du \right)^{\frac{1}{p}}.$

(d) *Assumption (vi) takes the form* $C + Dr^{\frac{1}{p}} + Kr^{\frac{1}{q}} + Lr \leq r$. *If* $r = 1$, *we would have* $C + D + K + L \leq 1$.

Theorem 3.2 *Let the assumptions (i)–(vi) be fulfilled, then Equation (1.1) has at least one integrable solution* $x \in L_1^N$ *on* $\mathbb{R}^+$.

Proof Step 1. By assumptions (ii), (iii) and Lemma 2.4 we get $F_{\varphi_2,u}$ is continuous mappings from L_1^N into L_p^N and $F_{\varphi_3,h}$ is continuous mappings from L_1^N into itself. Since the operator $I^\alpha : L_p^N \to L_p^N$ is continuous, assumption (i) gives that $A :\ L_1^N \to L_p^N$ is continuous and $B :\ L_1^N \to L_q^N$ is continuous. At last, the Hölder inequality gives us that $H : L_1^N \to L_1^N$ is continuous.

Step 2. In view of our assumptions and Proposition 3.1, we get

$$\begin{aligned}
\|Hx\|_{L_1^N} &= \|F_{\varphi_3,h} + Bx\cdot Ax\|_{L_1^N}\\
&\le \|F_{\varphi_3,h}\|_{L_1^N} + \|Bx\cdot Ax\|_{L_1^N}\\
&\le \|h(\tau,x(\varphi_3))\|_{L_1^N} + \|Bx\|_{L_q^N}\cdot\|Ax\|_{L_p^N}\\
&\le \Big\|a_3 + b_3|x(\varphi_3)|\Big\|_{L_1^N} + \Big\|g_1 + g_3\cdot Gx(\varphi_1)\Big\|_{L_q^N}\cdot\Big\|g_2 + I^\alpha F_{\varphi_2,u}(x)\Big\|_{L_p^N}\\
&\le \|a_3\|_{L_1^N} + b_3\|x(\varphi_3)\|_{L_1^N}\\
&\quad +\Big(\|g_1\|_{L_q^N} + M\|Gx(\varphi_1)\|_{L_q^N}\Big)\cdot\Big(\|g_2\|_{L_p^N} + \|I^\alpha F_{\varphi_2,u}(x)\|_{L_p^N}\Big)\\
&\le \|a_3\|_{L_1^N} + \frac{b_3}{M_3}\|x\|_{L_1^N} + \Big(\|g_1\|_{L_q^N} + M\Big(\|a_1\|_{L_q^N} + b_1\|(x(\varphi_1))^{\frac{1}{q}}\|_{L_q^N}\Big)\Big)\\
&\quad \times\Big(\|g_2\|_{L_p^N} + \Big(\frac{p}{N}\Big)^\alpha\Big\|u(\tau,x(\varphi_2))\Big\|_{L_p^N}\Big)\\
&\le \|a_3\|_{L_1^N} + \frac{b_3}{M_3}\|x\|_{L_1^N} + \Big(\|g_1\|_{L_q^N} + M\|a_1\|_{L_q^N} + \frac{b_1 M}{M_1^{\frac{1}{q}}}\|x\|^{\frac{1}{q}}_{L_1^N}\Big)\\
&\quad \times\Big(\|g_2\|_{L_p^N} + \Big(\frac{p}{N}\Big)^\alpha\Big\|a_2 + b_2(x(\varphi_2))^{\frac{1}{p}}\Big\|_{L_p^N}\Big)\\
&\le \|a_3\|_{L_1^N} + \frac{b_3}{M_3}\|x\|_{L_1^N} + \Big(\|g_1\|_{L_q^N} + M\|a_1\|_{L_q^N} + \frac{b_1 M}{M_1^{\frac{1}{q}}}\|x\|^{\frac{1}{q}}_{L_1^N}\Big)\\
&\quad \times\Big(\|g_2\|_{L_p^N} + \Big(\frac{p}{N}\Big)^\alpha\Big(\|a_2\|_{L_p^N} + \frac{b_2}{M_2^{\frac{1}{p}}}\|x\|^{\frac{1}{p}}_{L_1^N}\Big)\Big).
\end{aligned}$$

Thus $H: L_1^N \to L_1^N$. For $x\in B_r$, where r is as in assumption (vi) and $B_r = \{m\in L_1^N:\ \|m\|_{L_1^N}\le r\}$. Then

$$\begin{aligned}
\|Hx\|_{L_1^N} &\le \|a_3\|_{L_1^N} + \Big(\|g_1\|_{L_q^N} + M\|a_1\|_{L_q^N}\Big)\Big(\|g_2\|_{L_p^N} + \Big(\frac{p}{N}\Big)^\alpha\|a_2\|_{L_p^N}\Big)\\
&+ \Big(\|g_1\|_{L_q^N} + M\|a_1\|_{L_q^N}\Big)\frac{b_2 p^\alpha}{N^\alpha M_2^{\frac{1}{p}}}\cdot r^{\frac{1}{p}} + \Big(\|g_2\|_{L_p^N} + \Big(\frac{p}{N}\Big)^\alpha\|a_2\|_{L_p^N}\Big)\frac{Mb_1}{M_1^{\frac{1}{q}}}\cdot r^{\frac{1}{q}}\\
&+ \Big(\frac{b_3}{M_3} + \frac{Mb_1b_2p^\alpha}{N^\alpha M_1^{\frac{1}{q}}M_2^{\frac{1}{p}}}\Big)\cdot r\le r.
\end{aligned}$$

Thus $H: B_r\to B_r$ is continuous.

Step 3. In what follows, let $\Phi\neq X\subset B_r$. For arbitrary fixed $\varepsilon>0$, $T>0$ s.t. for any $D\subset[0,T]$ with

$meas(D) \leq \varepsilon$. By using Proposition 3.1 for $x \in X$, we derive

$$\begin{aligned}
e^{-N^2}\int_D |(Hx)(\tau)|d\tau &\leq \int_D e^{-N\tau}|(Hx)(\tau)|d\tau \\
&\leq \|F_{\varphi_3,h}x\|_{L_1^N(D)} + \|Bx\|_{L_q^N(D)} \cdot \|Ax\|_{L_p^N(D)} \\
&\leq \|a_3\|_{L_1^N(D)} + b_3\|x(\varphi_3)\|_{L_1^N(D)} \\
&+ \left[\|g_1\|_{L_q^N(D)} + M\|a_1\|_{L_q^N(D)} + Mb_1\|(x(\varphi_1))^{\frac{1}{q}}\|_{L_q^N(D)}\right]\left[\|g_2\|_{L_p^N(D)}\right. \\
&+ \left.\left(\frac{p}{N}\right)^{\alpha}\left(\|a_2\|_{L_p^N(D)} + b_2\|(x(\varphi_2))^{\frac{1}{p}}\|_{L_p^N(D)}\right)\right] \\
&\leq \|a_3\|_{L_1^N(D)} + \frac{b_3}{M_3}\int_{\varphi_3(D)} |x(u)|du \\
&+ \left[\|g_1\|_{L_q^N(D)} + M\|a_1\|_{L_q^N(D)} + \frac{Mb_1}{M_1^{\frac{1}{q}}}\left(\int_{\varphi_1(D)} |x(u)|du\right)^{\frac{1}{q}}\right]\left[\ \|g_2\|_{L_p^N(D)}\right. \\
&+ \left.\left(\frac{p}{N}\right)^{\alpha}\|a_2\|_{L_p^N(D)} + \frac{b_2 p^{\alpha}}{N^{\alpha}M_2^{\frac{1}{p}}}\left(\int_{\varphi_2(D)} |x(u)|du\right)^{\frac{1}{p}}\right].
\end{aligned}$$

Since $g_1, a_1 \in L_q^N$, $g_2, a_2 \in L_p^N$ and $a_3 \in L_1^N$, we have

$$\lim_{\varepsilon\to 0}\left\{\sup_{x\in X}\left\{\sup\left(\ \|a_3\|_{L_1^N(D)} :\ D \subset \mathbb{R}^+,\ meas(D) \leq\ \varepsilon\ \right)\right\}\right\} = 0,$$

$$\lim_{\varepsilon\to 0}\left\{\sup_{x\in X}\left\{\sup\left(\ \|g_1\|_{L_q^N(D)} + M\|a_1\|_{L_q^N(D)} :\ D \subset \mathbb{R}^+,\ meas(D) \leq\ \varepsilon\ \right)\right\}\right\} = 0$$

and

$$\lim_{\varepsilon\to 0}\left\{\sup_{x\in X}\left\{\sup\left(\ \|g_2\|_{L_p^N(D)} + \left(\frac{p}{N}\right)^{\alpha}\|a_2\|_{L_p^N(D)} :\ D \subset \mathbb{R}^+,\ meas(D) \leq \varepsilon\right)\right\}\right\} = 0.$$

From Equation (2.1) and assumption (v), we have

$$c(HX) \leq W \cdot c(X). \tag{3.1}$$

For $T > 0$ and $x \in X$, we get

$$\begin{aligned}
e^{-N^2}\int_T^{\infty} |(Hx)(\tau)|d\tau &\leq \|a_3\|_{L_1^N(T)} + \frac{b_3}{M_3}\int_{\varphi_3(T)}^{\infty} |x(u)|du \\
&+ \left[\|g_1\|_{L_q^N(T)} + M\|a_1\|_{L_q^N(T)} + \frac{Mb_1}{M_1^{\frac{1}{q}}}\left(\int_{\varphi_1(T)}^{\infty} |x(u)|du\right)^{\frac{1}{q}}\right]\left[\ \|g_2\|_{L_p^N(T)}\right. \\
&+ \left.\left(\frac{p}{N}\right)^{\alpha}\|a_2\|_{L_p^N(T)} + \frac{b_2 p^{\alpha}}{N^{\alpha}M_2^{\frac{1}{p}}}\left(\int_{\varphi_2(T)}^{\infty} |x(u)|du\right)^{\frac{1}{p}}\right].
\end{aligned}$$

Then as $T \to \infty$, we have $\varphi_i(T) \to \infty,\ i = 1, 2, 3$ and from Equation (2.2), we obtain

$$d(HX) \leq W \cdot d(X). \tag{3.2}$$

By adding Equations (3.1) and (3.2) and from Equation (2.3), we have

$$\gamma(HX) \leq W \cdot \gamma(X). \tag{3.3}$$

Step 4. Recall that B_r is defined in Step 2 and define $B_r^1 = Conv(H(B_r))$, $B_r^2 = Conv(H(B_r^1))$ and so on. We have the sequence $\{B_r^n\}$, which is decreasing i.e. $B_r^{n+1} \subset B_r^n$ for $n = 1, 2, \cdots$. Clearly, all sets from that sequence are convex and closed, so weakly closed. From Step 3, that $\gamma(HX) \leq W\gamma(X)$ for all bounded set $X \subset B_r$, we have

$$\gamma(B_r^n) \leq W^n \gamma(B_r),$$

which implies $\lim_{n\to\infty} \gamma(B_r^n) = 0$. Then, axiom (vii) of Definition 2.1 gives us the set $Q = \cap_{n=1}^{\infty} B_r^n \neq \Phi$ is convex, closed, and weakly compact (in virtue of $\gamma(Q) = 0$). Moreover, $H(Q) \subset Q$.

Step 5. Since $\gamma(Q) = 0$, and for arbitrary sequence $\{x_n\} \subset Q$, $\exists\ T, \forall n$, s.t. the next inequality holds

$$\int_T^{\infty} e^{-N\tau} |x_n(\tau)| d\tau \leq \frac{\varepsilon}{4}. \tag{3.4}$$

Considering the functions $u(\tau, x)$, $h(\tau, x)$ and $Gx(\tau)$ on $[0, T] \times \mathbb{R}$, then by Theorems 2.5 and 2.6 we can find a closed set $D_\varepsilon \subset [0, T]$, s.t. $meas(D_\varepsilon^c) \leq \varepsilon$, $g_i\big|_{D_\varepsilon}$, $i = 1, 2, 3$ are continuous and $u\big|_{D_\varepsilon \times \mathbb{R}}$, $h\big|_{D_\varepsilon \times \mathbb{R}}$, $G\big|_{D_\varepsilon \times \mathbb{R}}$ are continuous.

Denoting by $U_n(\tau) = (Bx_n).(Ax_n)(\tau)$, $n \in \mathbb{N}$. First, we will prove that the operator $(U_n)(\tau)$ is equicontinuous and equibounded in the space $C(D_\varepsilon)$. Thus

$$\begin{aligned}
|U_n(\tau)| &\leq \Big(|g_1(\tau)| + |g_3(\tau)||(Gx_n)(\varphi_1(\tau))|\Big)\Big(|g_2(\tau)| + \frac{1}{\Gamma(\alpha)} \int_0^{\tau} \frac{|u(s, x_n(\varphi_2(s)))|}{(\tau - s)^{1-\alpha}} ds\Big) \\
&\leq \Big(d_5 + M\Big(|a_1(\tau)| + b_1 |x_n(\varphi_1(\tau))|^{\frac{1}{q}}\Big)\Big) \cdot \Big(d_6 + \frac{1}{\Gamma(\alpha)} \int_0^{\tau} \frac{|a_2(s)| + b_2 |x_n(\varphi_2(s))|^{\frac{1}{p}}}{(\tau - s)^{1-\alpha}} ds\Big) \\
&\leq \Big(d_5 + M\Big(d_1 + b_1 d_3^{\frac{1}{q}}\Big)\Big) \cdot \Big(d_6 + \frac{1}{\Gamma(\alpha)} \int_0^{\tau} \frac{d_2 + b_2 d_4^{\frac{1}{p}}}{(\tau - s)^{1-\alpha}} ds\Big) \\
&\leq \Big(d_5 + M\Big(d_1 + b_1 d_3^{\frac{1}{q}}\Big)\Big) \cdot \Big(d_6 + \frac{d_2 + b_2 d_4^{\frac{1}{p}}}{\Gamma(\alpha + 1)} T^{\alpha}\Big),
\end{aligned}$$

where $|a_1(\tau)| \leq d_1, |a_2(\tau)| \leq d_2, |x_n(\varphi_1(\tau))| \leq d_3, |x_n(\varphi_2(\tau))| \leq d_4, |g_1(\tau)| \leq d_5$, and $|g_2(\tau)| \leq d_6$ for $\tau \in D_\varepsilon$. The previous estimate yields that $\{U_n\}$ is equibounded in $C(D_\varepsilon)$.

Next, for arbitrary $\tau_1, \tau_2 \in D_\varepsilon$ with $\tau_1 < \tau_2$, we get

$$\begin{aligned}
& |U_n(\tau_2) - U_n(\tau_1)| \\
= & \left|\Big(g_1(\tau_2) + \ g_3(\tau_2)\cdot(Gx_n)(\varphi_1(\tau_2))\Big)\left(g_2(\tau_2) + \frac{1}{\Gamma(\alpha)}\int_0^{\tau_2}\frac{u(s, x_n(\varphi_2(s)))}{(\tau_2 - s)^{1-\alpha}}ds\right)\right. \\
- & \left.\Big(g_1(\tau_1) + \ g_3(\tau_1)\cdot(Gx_n)(\varphi_1(\tau_1))\Big)\left(g_2(\tau_1) + \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}\frac{u(s, x_n(\varphi_2(s)))}{(\tau_1 - s)^{1-\alpha}}ds\right)\right| \\
\leq & \Big|g_1(\tau_2)g_2(\tau_2) - g_1(\tau_1)g_2(\tau_1)\Big| + \left|\frac{g_3(\tau_2)(Gx_n)(\varphi_1(\tau_2))}{\Gamma(\alpha)}\int_0^{\tau_2}\frac{u(s, x_n(\varphi_2(s)))}{(\tau_2 - s)^{1-\alpha}}ds\right. \\
& \left.-\frac{g_3(\tau_1)(Gx_n)(\varphi_1(\tau_1))}{\Gamma(\alpha)}\int_0^{\tau_1}\frac{u(s, x_n(\varphi_2(s)))}{(\tau_1 - s)^{1-\alpha}}ds\right| \\
& +\left|\frac{g_1(\tau_2)}{\Gamma(\alpha)}\int_0^{\tau_2}\frac{u(s, x_n(\varphi_2(s)))}{(\tau_2 - s)^{1-\alpha}}ds - \frac{g_1(\tau_1)}{\Gamma(\alpha)}\int_0^{\tau_1}\frac{u(s, x_n(\varphi_2(s)))}{(\tau_1 - s)^{1-\alpha}}ds\right| \\
& +\Big|g_2(\tau_2)g_3(\tau_2)\cdot(Gx_n)(\varphi_1(\tau_2)) - g_2(\tau_1)g_3(\tau_1)\cdot(Gx_n)(\varphi_1(\tau_1))\Big| \\
= & I_1 + I_2 + I_3 + I_4. \qquad (3.5)
\end{aligned}$$

First, we estimate I_2:

$$\begin{aligned}
I_2 = & \left|\frac{g_3(\tau_2)(Gx_n)(\varphi_1(\tau_2))}{\Gamma(\alpha)}\int_0^{\tau_2}\frac{u(s, x_n(\varphi_2(s)))}{(\tau_2 - s)^{1-\alpha}}ds - \frac{g_3(\tau_1)(Gx_n)(\varphi_1(\tau_1))}{\Gamma(\alpha)}\int_0^{\tau_1}\frac{u(s, x_n(\varphi_2(s)))}{(\tau_1 - s)^{1-\alpha}}ds\right| \\
\leq & \left|\frac{g_3(\tau_2)(Gx_n)(\varphi_1(\tau_2))}{\Gamma(\alpha)}\int_0^{\tau_2}\frac{u(s, x_n(\varphi_2(s)))}{(\tau_2 - s)^{1-\alpha}}ds - \frac{g_3(\tau_2)(Gx_n)(\varphi_1(\tau_1))}{\Gamma(\alpha)}\int_0^{\tau_1}\frac{u(s, x_n(\varphi_2(s)))}{(\tau_1 - s)^{1-\alpha}}ds\right| \\
+ & \left|\frac{g_3(\tau_2)(Gx_n)(\varphi_1(\tau_1))}{\Gamma(\alpha)}\int_0^{\tau_1}\frac{u(s, x_n(\varphi_2(s)))}{(\tau_1 - s)^{1-\alpha}}ds - \frac{g_3(\tau_1)(Gx_n)(\varphi_1(\tau_1))}{\Gamma(\alpha)}\int_0^{\tau_1}\frac{u(s, x_n(\varphi_2(s)))}{(\tau_1 - s)^{1-\alpha}}ds\right| \\
\leq & |g_3(\tau_2)|\left|\frac{(Gx_n)(\varphi_1(\tau_2))}{\Gamma(\alpha)}\int_0^{\tau_2}\frac{u(s, x_n(\varphi_2(s)))}{(\tau_2 - s)^{1-\alpha}}ds - \frac{(Gx_n)(\varphi_1(\tau_2))}{\Gamma(\alpha)}\int_0^{\tau_1}\frac{u(s, x_n(\varphi_2(s)))}{(\tau_1 - s)^{1-\alpha}}ds\right. \\
& \left.+\frac{(Gx_n)(\varphi_1(\tau_2))}{\Gamma(\alpha)}\int_0^{\tau_1}\frac{u(s, x_n(\varphi_2(s)))}{(\tau_1 - s)^{1-\alpha}}ds - \frac{(Gx_n)(\varphi_1(\tau_1))}{\Gamma(\alpha)}\int_0^{\tau_1}\frac{u(s, x_n(\varphi_2(s)))}{(\tau_1 - s)^{1-\alpha}}ds\right| \\
& +|g_3(\tau_2) - g_3(\tau_1)|\left|\frac{(Gx_n)(\varphi_1(\tau_1))}{\Gamma(\alpha)}\int_0^{\tau_1}\frac{u(s, x_n(\varphi_2(s)))}{(\tau_1 - s)^{1-\alpha}}ds\right| \\
\leq & M\left|\frac{(Gx_n)(\varphi_1(\tau_2))}{\Gamma(\alpha)}\right|\left|\int_0^{\tau_2}\frac{u(s, x_n(\varphi_2(s)))}{(\tau_2 - s)^{1-\alpha}}ds - \int_0^{\tau_1}\frac{u(s, x_n(\varphi_2(s)))}{(\tau_1 - s)^{1-\alpha}}ds\right| \\
& +\frac{M}{\Gamma(\alpha)}\Big|(Gx_n)(\varphi_1(\tau_2)) - (Gx_n)(\varphi_1(\tau_1))\Big|\left|\int_0^{\tau_1}\frac{u(s, x_n(\varphi_2(s)))}{(\tau_1 - s)^{1-\alpha}}ds\right| \\
& +\omega^T(g_3, |\tau_2 - \tau_1|)(d_1 + b_1 d_3^{\frac{1}{q}})\frac{(d_2 + b_2 d_4^{\frac{1}{p}})T^\alpha}{\Gamma(\alpha + 1)}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{M(d_1+b_1d_3^{\frac{1}{q}})}{\Gamma(\alpha)}\Big|\int_0^{\tau_2}\frac{u(s,x_n(\varphi_2(s)))}{(\tau_2-s)^{1-\alpha}}ds-\int_0^{\tau_2}\frac{u(s,x_n(\varphi_2(s)))}{(\tau_1-s)^{1-\alpha}}ds\Big| \\
&\quad+\frac{M(d_1+b_1d_3^{\frac{1}{q}})}{\Gamma(\alpha)}\Big|\int_0^{\tau_2}\frac{u(s,x_n(\varphi_2(s)))}{(\tau_1-s)^{1-\alpha}}ds-\int_0^{\tau_1}\frac{u(s,x_n(\varphi_2(s)))}{(\tau_1-s)^{1-\alpha}}ds\Big| \\
&\quad+M\omega^T(G,|\tau_2-\tau_1|)\frac{(d_2+b_2d_4^{\frac{1}{p}})T^\alpha}{\Gamma(\alpha+1)}+\omega^T(g_3,|\tau_2-\tau_1|)(d_1+b_1d_3^{\frac{1}{q}})\frac{(d_2+b_2d_4^{\frac{1}{p}})T^\alpha}{\Gamma(\alpha+1)} \\
&\leq \frac{M(d_1+b_1d_3^{\frac{1}{q}})(d_2+b_2d_4^{\frac{1}{p}})}{\Gamma(\alpha)}\Bigg(\int_0^{\tau_2}|(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}|ds+\int_{\tau_1}^{\tau_2}(\tau_1-s)^{\alpha-1}ds\Bigg) \\
&\quad+\frac{(d_2+b_2d_4^{\frac{1}{p}})T^\alpha}{\Gamma(\alpha+1)}\Big[M\omega^T(G,|\tau_2-\tau_1|)+(d_1+b_1d_3^{\frac{1}{q}})\omega^T(g_3,|\tau_2-\tau_1|)\Big] \\
&\leq \frac{M(d_1+b_1d_3^{\frac{1}{q}})(d_2+b_2d_4^{\frac{1}{p}})}{\alpha\Gamma(\alpha)}\Bigg(-(\tau_2-s)^\alpha|_0^{\tau_2}+(\tau_1-s)^\alpha|_0^{\tau_2}-(\tau_1-s)^\alpha|_{\tau_1}^{\tau_2}\Bigg) \\
&\quad+\frac{(d_2+b_2d_4^{\frac{1}{p}})T^\alpha}{\Gamma(\alpha+1)}\Big[M\omega^T(G,|\tau_2-\tau_1|)+(d_1+b_1d_3^{\frac{1}{q}})\omega^T(g_3,|\tau_2-\tau_1|)\Big] \\
&\leq \frac{(d_2+b_2d_4^{\frac{1}{p}})}{\Gamma(\alpha+1)}\Big[M(d_1+b_1d_3^{\frac{1}{q}})(\tau_2^\alpha-\tau_1^\alpha)+MT^\alpha\omega^T(G,|\tau_2-\tau_1|) \\
&\quad+T^\alpha(d_1+b_1d_3^{\frac{1}{q}})\omega^T(g_3,|\tau_2-\tau_1|)\Big].
\end{aligned}
$$

By mean value theorem: $\exists\ z,\ \tau_1<z<\tau_2$ s.t. $\tau_2^\alpha-\tau_1^\alpha=\alpha z^{\alpha-1}(\tau_2-\tau_1)\leq(\tau_2-\tau_1)$, then we have

$$
I_2\leq\frac{(d_2+b_2d_4^{\frac{1}{p}})}{\Gamma(\alpha+1)}\Big[M(d_1+b_1d_3^{\frac{1}{q}})(\tau_2-\tau_1)+MT^\alpha\omega^T(G,|\tau_2-\tau_1|)+T^\alpha(d_1+b_1d_3^{\frac{1}{q}})\omega^T(g_3,|\tau_2-\tau_1|)\Big]. \tag{3.6}
$$

To evaluate I_1, we have

$$
I_1=\Big|g_1(\tau_2)g_2(\tau_2)-g_1(\tau_1)g_2(\tau_1)\Big|\leq d_5\omega^T(g_2,|\tau_2-\tau_1|)+d_6\omega^T(g_1,|\tau_2-\tau_1|). \tag{3.7}
$$

To evaluate I_3, we have

$$
\begin{aligned}
I_3 &= \Big|\frac{g_1(\tau_2)}{\Gamma(\alpha)}\int_0^{\tau_2}\frac{u(s,x_n(\varphi_2(s)))}{(\tau_2-s)^{1-\alpha}}ds-\frac{g_1(\tau_1)}{\Gamma(\alpha)}\int_0^{\tau_1}\frac{u(s,x_n(\varphi_2(s)))}{(\tau_1-s)^{1-\alpha}}ds\Big| \\
&\leq \frac{d_5\Big(d_2+b_2d_4^{\frac{1}{p}}\Big)}{\Gamma(\alpha+1)}|\tau_2-\tau_1|+\frac{\Big(d_2+b_2d_4^{\frac{1}{p}}\Big)T^\alpha}{\Gamma(\alpha+1)}\omega^T(g_1,|\tau_2-\tau_1|)
\end{aligned} \tag{3.8}
$$

and to estimate I_4, we have

$$\begin{aligned} I_4 &= \Big| g_2(\tau_2)g_3(\tau_2)\cdot(Gx_n)(\varphi_1(\tau_2)) - g_2(\tau_1)g_3(\tau_1)\cdot(Gx_n)(\varphi_1(\tau_1))\Big| \\ &\leq d_6 M\omega^T(G,|\tau_2-\tau_1|) + d_6(d_1+b_2 d_3^{\frac{1}{q}})\omega^T(g_3,|\tau_2-\tau_1|) \\ &\quad + M(d_1+b_2 d_3^{\frac{1}{q}})\omega^T(g_2,|\tau_2-\tau_1|), \end{aligned} \tag{3.9}$$

where $\omega^T(g_i,\cdot)$, $\omega^T(G,\cdot)$ denotes the modulus continuity of the functions $g_i (i=1,2,3)$ and G on the sets D_ε and $D_\varepsilon \times \mathbb{R}$, respectively. By substituting from (3.6), (3.7), (3.8), and (3.9) in (3.5), we have an inequality which has been gotten since $Q \subset B_r$.

From the fact $\gamma(\{x_n\}) \leq \gamma(Q) = 0$, we deduce that, (3.5) tends to zero as $(\tau_2 - \tau_1) \to 0$ independently of x_n. We have $\{U_n\}$ is equicontinuous in $C(D_\varepsilon)$. Set

$$Y = \sup\{\ |U_n| :\ \tau \in D_\varepsilon,\ n \in \mathbb{N}\}.$$

Obviously, Y is finite and according to the choice of the set D_ε we conclude that the functions $u\big|_{D_\varepsilon\times[-Y,Y]}$, $h\big|_{D_\varepsilon\times[-Y,Y]}$, $G\big|_{D_\varepsilon\times[-Y,Y]}$ are uniformly continuous. So $\{Hx_n\} = \{F_{\varphi_3,h}(x_n) + U_n\}$ is equicontinuous and equibounded in $C(D_\varepsilon)$. Then, by Ascoli–Arzéla theorem [18], we deduce that $\{Hx_n\}$ is a relatively compact set in $C(D_\varepsilon)$.

We note that the above thinking is independent of the choice of ε. Then, we shall reconstruct a sequence of closed sets $D_l \subset [0,T]$ s.t. $meas(D_l^c) \to 0$ as $l \to \infty$ and s.t. $\{Hx_n\}$ is relatively compact in each space $C(D_l)$.

By utilizing the weak compact set $H(Q)$, we select a number $\delta > 0$ s.t. for every closed set $D_\delta \subset [0,T]$ with $meas(D_\delta^c) \leq \delta$, we have

$$\int_{D_\delta^c} e^{-N\tau}|(Hx)(\tau)|d\tau \leq \frac{\varepsilon}{4}, \tag{3.10}$$

for any $x \in Q$. Since, the sequence $\{Hx_n\}$ is a Cauchy sequence in the spaces $C(D_l)$, $l = 1,2,\cdots$, we can select a number l_0 s.t. $meas(D_{l_0}^c) \leq \delta$ and $meas(D_{l_0}) > 0$, and for any numbers $n,m \geq l_0$, we have

$$e^{-N\tau}|(Hx_n)(\tau) - (Hx_m)(\tau)| \leq \frac{\varepsilon}{4meas(D_{l_0})} \tag{3.11}$$

for any $\tau \in D_{l_0}$. Then, by using (3.4), (3.10), and (3.11) we obtain

$$\begin{aligned} &\int_0^\infty e^{-N\tau}|(Hx_n)(\tau) - (Hx_m)(\tau)|d\tau = \int_T^\infty e^{-N\tau}|(Hx_n)(\tau) - (Hx_m)(\tau)|d\tau \\ +\ &\int_{D_{l_0}} e^{-N\tau}|(Hx_n)(\tau) - (Hx_m)(\tau)|d\tau + \int_{D_{l_0}^c} e^{-N\tau}|(Hx_n)(\tau) - (Hx_m)(\tau)|d\tau \leq \varepsilon, \end{aligned}$$

which gives that $\{Hx_n\}$ is a Cauchy sequence in L_1^N. Hence, we obtain that the set (HQ) is relatively strongly compact in L_1^N.

Step 6. Let $Q_0 = Conv(H(Q))$ and by Mazur theorem, we have that Q_0 is compact in L_1^N. Further, the operator $H : Q_0 \to Q_0$ is continuous. Then, by applying Theorem 2.2, this fulfills the proof. □

Remark 3.3 *The results mentioned in Theorem 3.2 can be applied in the spaces* $L_p^N(\mathbb{R}^+)$, $(p > 1)$ *or in Orlicz spaces* L_φ *under appropriate set of assumptions guarantee continuity and acting conditions of the studied problems (cf. [14]).*

Remark 3.4 *One can easily check that the operators* $Gx = 1$, $Gx = x$ *and* $Gx = f(t, x)$ *are examples of the operator* Gx*, which satisfy assumptions (ii) and (iii) of Theorem 3.2. Moreover,* Gx *shall take the form of linear, nonlinear, Hammerstein, Urysohn, or fraction integral operators.*

Now, we will tackle the uniqueness of the solutions.

Theorem 3.5 *Let assumptions of Theorem 3.2 are fulfilled, but replace assumption (iii) by the conditions:*

(vii) $\exists\, b_i \geq 0,\ (i = 1, 3, 3)$ *and the functions* $a_1 \in L_q^N$, $a_2 \in L_p^N$ *and* $a_3 \in L_1^N$ *s.t.*

$$|G(0)(\tau)| \leq a_1(\tau), \quad |u(\tau, 0)| \leq a_2(\tau), \quad |h(\tau, 0)| \leq a_3(\tau)$$

$$|(Gx)(\tau) - (Gy)(\tau)| \leq b_1 |x - y|^{\frac{1}{q}}, \quad |u(\tau, x) - u(\tau, y)| \leq b_2 |x - y|^{\frac{1}{p}},$$

$$\text{and} \quad |h(\tau, x) - h(\tau, y)| \leq b_3 |x - y|$$

for $x, y \in Q$*, where* Q *is as in Theorem 3.2.*

(viii) If for any constant $\mathbb{L} \geq 0$*, we have*

$$\begin{aligned} \mathbb{L} \quad \leq \quad & \frac{b_3}{M_3}\mathbb{L} + \frac{b_2 p^\alpha \|g_1\|_{L_q^N}}{N^\alpha M_2^{\frac{1}{p}}}\mathbb{L}^{\frac{1}{p}} + \frac{M b_1 \|g_2\|_{L_p^N}}{M_1^{\frac{1}{q}}}\mathbb{L}^{\frac{1}{q}} + \frac{p^\alpha M b_2}{N^\alpha M_2^{\frac{1}{p}}}\left(\|a_1\|_{L_q^N} + \frac{b_1}{M_1^{\frac{1}{q}}} r^{\frac{1}{q}}\right)\mathbb{L}^{\frac{1}{p}} \\ & + \frac{p^\alpha M b_1}{N^\alpha M_1^{\frac{1}{q}}}\left(\|a_2\|_{L_p^N} \ + \ \frac{b_2}{M_2^{\frac{1}{p}}} r^{\frac{1}{p}}\right)\mathbb{L}^{\frac{1}{q}}, \ \text{then } \mathbb{L} = 0, \end{aligned}$$

where r *is defined in Equation (3.1), then (1.1) has a unique integrable solution* $x \in L_1^N$ *in* Q*.*

Proof From assumption (vii), we derive

$$\begin{aligned} \Big| |u(\tau, x)| - |u(\tau, 0)| \Big| \quad & \leq \quad |u(\tau, x) - u(\tau, 0)| \leq b_2 |x|^{\frac{1}{p}} \\ \Rightarrow |u(\tau, x)| \quad & \leq \quad |u(\tau, 0)| + b_2 |x|^{\frac{1}{p}} \leq a_2(\tau) + b_2 |x|^{\frac{1}{p}}. \end{aligned}$$

Similarly, we have $|h(\tau, x)| \leq a_3(\tau) + b_3 |x|$ and $|Gx(\tau)| \leq a_1(\tau) + b_1 |x|^{\frac{1}{q}}$. Then by Theorem 3.2, we have (1.1) has at least one solution $x \in L_1^N$ on $\mathbb{R}^+$.

Next, assume that let x and y be any solutions of Equation (1.1). Thus we have

$$\begin{aligned}
|x(\tau)-y(\tau)| &= \Bigg|h(\tau,x(\varphi_3(\tau)))+\Big(g_1(\tau)+g_3(\tau)(Gx(\varphi_1(\tau)))\Big)\Big(g_2(\tau)+\frac{1}{\Gamma(\alpha)}\int_0^\tau\frac{u(s,x(\varphi_2(s)))}{(\tau-s)^{1-\alpha}}ds\Big)\\
&- h(\tau,y(\varphi_3(\tau)))-\Big(g_1(\tau)+g_3(\tau)(Gy(\varphi_1(\tau)))\Big)\cdot\Big(g_2(\tau)+\frac{1}{\Gamma(\alpha)}\int_0^\tau\frac{u(s,y(\varphi_2(s)))}{(\tau-s)^{1-\alpha}}ds\Big)\Bigg|\\
&\leq |h(\tau,x(\varphi_3(\tau)))-h(\tau,y(\varphi_3(\tau)))|+\frac{|g_1(\tau)|}{\Gamma(\alpha)}\int_0^\tau\frac{|u(s,x(\varphi_2(s)))-u(s,y(\varphi_2(s)))|}{(\tau-s)^{1-\alpha}}ds\\
&+|g_2(\tau)||g_3(\tau)||Gx(\varphi_1(\tau))-Gy(\varphi_1(\tau))|\\
&+\frac{|g_3(\tau)(Gx(\varphi_1(\tau)))|}{\Gamma(\alpha)}\int_0^\tau\frac{|u(s,x(\varphi_2(s)))\ -\ u(s,y(\varphi_2(s)))|}{(\tau-s)^{1-\alpha}}ds\\
&+\frac{|g_3(\tau)|}{\Gamma(\alpha)}\Big|Gx(\varphi_1(\tau))\ -\ Gy(\varphi_1(\tau))\Big|\int_0^\tau\frac{|u(s,y(\varphi_2(s)))|}{(\tau-s)^{1-\alpha}}ds\\
&\leq b_3|x(\varphi_3(\tau))-y(\varphi_3(\tau))|\\
&+\frac{|g_1(\tau)|}{\Gamma(\alpha)}\int_0^\tau\frac{b_2|x(\varphi_2(s))-y(\varphi_2(s))|^{\frac{1}{p}}}{(\tau-s)^{1-\alpha}}ds+b_1M|g_2(\tau)||x(\varphi_1(\tau))-y(\varphi_1(\tau))|^{\frac{1}{q}}\\
&+\frac{M}{\Gamma(\alpha)}\Big(a_1(\tau)+b_1|x(\varphi_1(\tau))|^{\frac{1}{q}}\Big)\cdot\int_0^\tau\frac{b_2|x(\varphi_2(s))-y(\varphi_2(s))|^{\frac{1}{p}}}{(\tau-s)^{1-\alpha}}ds\\
&+\frac{b_1M}{\Gamma(\alpha)}\Big|x(\varphi_1(\tau))-y(\varphi_1(\tau))\Big|^{\frac{1}{q}}\int_0^\tau\frac{a_2(s)\ +\ b_2|y(\varphi_2(s))|^{\frac{1}{p}}}{(\tau-s)^{1-\alpha}}ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|x-y\|_{L_1^N} &\leq b_3\|x(\varphi_3)-y(\varphi_3)\|_{L_1^N}+\frac{b_2\|g_1\|_{L_q^N}}{\Gamma(\alpha)}\left\|\int_0^\tau\frac{|x(\varphi_2(s))-y(\varphi_2(s))|^{\frac{1}{p}}}{(\tau-s)^{1-\alpha}}ds\right\|_{L_p^N}\\
&+Mb_1\|g_2\|_{L_p^N}\left\||x(\varphi_1)-y(\varphi_1)|^{\frac{1}{q}}\right\|_{L_q^N}\\
&+\frac{Mb_2}{\Gamma(\alpha)}\left\|a_1+b_1|x(\varphi_1)|^{\frac{1}{q}}\right\|_{L_q^N}\cdot\left\|\int_0^\tau\frac{|x(\varphi_2(s))-y(\varphi_2(s))|^{\frac{1}{p}}}{(\tau-s)^{1-\alpha}}ds\right\|_{L_p^N}\\
&+\frac{Mb_1}{\Gamma(\alpha)}\left\||x(\varphi_1)-y(\varphi_1)|^{\frac{1}{q}}\right\|_{L_q^N}\cdot\left\|\int_0^\tau\frac{a_2(s)\ +\ b_2|y(\varphi_2(s))|^{\frac{1}{p}}}{(\tau-s)^{1-\alpha}}\ ds\right\|_{L_p^N}\\
&\leq \frac{b_3}{M_3}\|x-y\|_{L_1^N}+\frac{p^\alpha b_2\|g_1\|_{L_q^N}}{N^\alpha}\left\|\Big(x(\varphi_2)-y(\varphi_2)\Big)^{\frac{1}{p}}\right\|_{L_p^N}+\frac{Mb_1\|g_2\|_{L_p^N}}{M_1^{\frac{1}{q}}}\|x-y\|_{L_1^N}^{\frac{1}{q}}\\
&+\frac{p^\alpha Mb_2}{N^\alpha}\Big(\|a_1\|_{L_q^N}+\frac{b_1}{M_1^{\frac{1}{q}}}\|x\|_{L_1^N}^{\frac{1}{q}}\Big)\left\|\Big(x(\varphi_2)-y(\varphi_2)\Big)^{\frac{1}{p}}\right\|_{L_p^N}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{p^{\alpha}Mb_1}{N^{\alpha}M_1^{\frac{1}{q}}}\|x-y\|_{L_1^N}^{\frac{1}{q}}\cdot\left\|a_2\ +\ b_2|y(\varphi_2)|^{\frac{1}{p}}\right\|_{L_p^N}\\
\leq\ &\frac{b_3}{M_3}\|x-y\|_{L_1^N}+\frac{b_2p^{\alpha}\|g_1\|_{L_q^N}}{N^{\alpha}M_2^{\frac{1}{p}}}\|x-y\|_{L_1^N}^{\frac{1}{p}}+\frac{Mb_1\|g_2\|_{L_p^N}}{M_1^{\frac{1}{q}}}\|x-y\|_{L_1^N}^{\frac{1}{q}}\\
&+\frac{p^{\alpha}Mb_2}{N^{\alpha}M_2^{\frac{1}{p}}}\left(\|a_1\|_{L_q^N}+\frac{b_1}{M_1^{\frac{1}{q}}}\|x\|_{L_1^N}^{\frac{1}{q}}\right)\|x-y\|_{L_1^N}^{\frac{1}{p}}+\frac{p^{\alpha}Mb_1}{N^{\alpha}M_1^{\frac{1}{q}}}\left(\|a_2\|_{L_p^N}\ +\ \frac{b_2}{M_2^{\frac{1}{p}}}\|y\|_{L_1^N}^{\frac{1}{p}}\right)\|x-y\|_{L_1^N}^{\frac{1}{q}}\\
\leq\ &\frac{b_3}{M_3}\|x-y\|_{L_1^N}+\frac{b_2p^{\alpha}\|g_1\|_{L_q^N}}{N^{\alpha}M_2^{\frac{1}{p}}}\|x-y\|_{L_1^N}^{\frac{1}{p}}+\frac{Mb_1\|g_2\|_{L_p^N}}{M_1^{\frac{1}{q}}}\|x-y\|_{L_1^N}^{\frac{1}{q}}\\
&+\frac{p^{\alpha}Mb_2}{N^{\alpha}M_2^{\frac{1}{p}}}\left(\|a_1\|_{L_q^N}+\frac{b_1}{M_1^{\frac{1}{q}}}r^{\frac{1}{q}}\right)\|x-y\|_{L_1^N}^{\frac{1}{p}}+\frac{p^{\alpha}Mb_1}{N^{\alpha}M_1^{\frac{1}{q}}}\left(\|a_2\|_{L_p^N}\ +\ \frac{b_2}{M_2^{\frac{1}{p}}}r^{\frac{1}{p}}\right)\|x-y\|_{L_1^N}^{\frac{1}{q}}.
\end{aligned}
$$

The above inequality with assumption (viii) implies that $x=y$ (a.e.), and that fulfills the proof. □

3.1. Example

We construct an example to indicate the validity of our outcomes.

Example 3.6 *For $\tau\in[0,\infty)$, consider the equation*

$$
x(\tau)=\tau^3+\frac{x(\sin(3\tau))}{e^{N^2}}+\left(e^{-\tau}+e^{-\tau^3}\left(\tau^{\frac{5}{2}}+\frac{\sqrt{x\left(\frac{\tau}{4+\tau}\right)}}{e^{N^2}}\right)\right)\left(e^{\tau}+\int_0^{\tau}\frac{(\tau-s)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}\left(e^{5s}+\frac{\sqrt{x(\frac{2s}{2+s})}}{2+(x(\frac{2s}{2+s}))^2}\right)ds\right).
\tag{3.12}
$$

For $p=q=2$, we can see that

1. $g_1(\tau)=e^{-\tau}\in L_2^N$ *and* $\|g_1\|_{L_2^N}=\frac{1}{\sqrt{N+2}}$.
2. $g_2(\tau)=e^{\tau}\in L_2^N$ *and* $\|g_2\|_{L_2^N}=\frac{1}{\sqrt{N-2}}$.
3. $g_3(\tau)=e^{-\tau^3}$ *and* $M=\sup_{0\leq\tau<\infty}|g_3(\tau)|\leq 1$.
4. $(Gx)(\tau)=\tau^{\frac{5}{2}}+\frac{1}{e^{N^2}}\sqrt{x}$, *then* $a_1(\tau)=\tau^{\frac{5}{2}}$, $b_1=\frac{1}{e^{N^2}}$ *with* $\|a_1\|_{L_2^N}=\frac{\sqrt{120}}{N^3}$.
5. $u(\tau,x)=e^{5\tau}+\frac{\sqrt{x}}{2+x^2}$, *then* $a_2(\tau)=e^{5\tau}$, $b_2=\frac{1}{2}$ *with* $\|a_2\|_{L_2^N}=\frac{1}{\sqrt{N-10}}$.
6. $h(\tau,x)=\tau^3+\frac{1}{e^{N^2}}x$, *then* $a_3(\tau)=\tau^3$, $b_3=\frac{1}{e^{N^2}}$ *with* $\|a_3\|_{L_1^N}=\frac{6}{N^4}$.
7. $\varphi_1(\tau)=\frac{\tau}{4+\tau}$, *then* $\varphi_1'(\tau)=\frac{4}{(4+\tau)^2}\geq\frac{1}{16}=M_1$ *and* $\varphi_1^{-1}(\tau)=\frac{4\tau}{1-\tau}\geq\ \tau$.
8. $\varphi_2(\tau)=\frac{2\tau}{2+\tau}$, *then* $\varphi_2'(\tau)=\frac{4}{(2+\tau)^2}\geq\frac{1}{4}=M_2$ *and* $\varphi_2^{-1}(\tau)=\frac{2\tau}{2-\tau}\geq\ \tau$.
9. $\varphi_3(\tau)=\sin(3\tau)$, *then* $\varphi_3'(\tau)=3\cos(3\tau)\geq 2=M_3$ *and* $\varphi_3^{-1}(\tau)=\frac{\sin^{-1}\tau}{3}\geq\ \tau$.

10. For $N > 128$, we have

$$W = e^{N^2}\left(\frac{b_3}{M_3} + \frac{M b_1 b_2 p^{\alpha}}{N^{\alpha} M_1^{\frac{1}{q}} M_2^{\frac{1}{p}}}\right) \leq \left(\frac{1}{2} + 4\sqrt{\frac{2}{N}}\right) < 1.$$

11. Let $r = 1$, and note that

$$\begin{aligned}
&\|a_3\|_{L_1^N} + \left(\|g_1\|_{L_2^N} + M\|a_1\|_{L_2^N}\right)\left(\|g_2\|_{L_2^N} + \sqrt{\frac{2}{N}}\|a_2\|_{L_2^N} + b_2\sqrt{\frac{2}{NM_2}}\right) \\
&+\left(\|g_2\|_{L_2^N} + \sqrt{\frac{2}{N}}\|a_2\|_{L_2^N}\right)\frac{Mb_1}{\sqrt{M_1}} + \left(\frac{b_3}{M_3} + Mb_1b_2\sqrt{\frac{2}{NM_1M_2}}\right) \\
&\leq \frac{\sqrt{6}}{N^4} + \left(\frac{1}{\sqrt{N+2}} + \frac{\sqrt{120}}{N^3}\right)\left(\frac{1}{\sqrt{N-2}} + \sqrt{\frac{2}{N}} + \sqrt{\frac{2}{N(N-10)}}\right) \\
&+\frac{4}{e^{N^2}}\left(\frac{1}{\sqrt{N-2}} + \sqrt{\frac{2}{N(N-10)}}\right) + \frac{e^{-N^2}}{2}\left(1 + 8\sqrt{\frac{2}{N}}\right) < 1.
\end{aligned}$$

Hence, due to Theorem 3.2, Equation (3.12) has at least one integrable solution $x \in L_1^N$ on $\mathbb{R}^+$.

Moreover, we have

1. $|G(0)(\tau)| = \tau^{\frac{5}{2}}$ *and* $|G(x) - G(y)| \leq \frac{1}{e^{N^2}}|x - y|^{\frac{1}{2}}$.
2. $|u(\tau, 0)| = e^{5\tau}$ *and* $|u(\tau, x) - u(\tau, y)| \leq \frac{1}{2}|x - y|^{\frac{1}{2}}$.
3. $|h(\tau, 0)| = \tau^3$ *and* $|h(\tau, x) - h(\tau, y)| \leq \frac{1}{e^{N^2}}|x - y|$.
4. *Assumption (viii) is satisfied for $\mathbb{L} = \|x - y\|_{L_1^N} = 0$, i.e. $x = y$ (a.e.).*

Hence, due to Theorem 3.5, Equation (3.12) has a unique integrable solution $x \in L_1^N$.

4. Conclusion

This article dealt with the integral equations of the Gripenberg's type of fractional order with a perturbation term (1.1) in weighted Lebesgue spaces $L_1^N, N > 0$ on $\mathbb{R}^+ = [0, \infty)$. With the aid of Schauder's fixed point hypothesis and appropriate measure of noncompactness, we proved that considered problem has at least one integrable solution in the considered spaces under a general set of assumptions. Indeed, two new theorems, Theorems 3.2 and 3.5, are proved s.t. Equation (1.1) has at least one integrable solution and a unique integrable solution $x \in L_1^N$, respectively. At the end, an example is constructed to indicate the validity of these theorems. The results here extend and generalize some former results in the available literature.

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