Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 499 – 515
© TÜBİTAK
doi:10.3906/mat-2106-90

Boundary value problems for a second-order (p, q)-difference equation with integral conditions

İlker GENÇTÜRK*
Department of Mathematics, Faculty of Arts and Sciences, Kırıkkale University, Kırıkkale, Turkey

Received: 21.06.2021 • **Accepted/Published Online:** 10.08.2021 • **Final Version:** 21.01.2022

Abstract: Our purpose in this paper is to obtain some new existence results of solutions for a boundary value problem for a (p, q)-difference equations with integral conditions, by using fixed point theorems. Examples illustrating the main results are also presented.

Key words: (p, q)-difference equations, boundary value problems, integral conditions, existence, fixed point theorems.

1. Introduction

FH Jackson firstly introduced the subject of q-derivative and q-difference operator for a function f on $[0, \infty)$ by

$$D_q f(t) = \frac{f(qt) - f(t)}{(q-1)t}, \quad t \neq 0, \tag{1.1}$$

$$D_q f(0) = \lim_{t \to 0} D_q f(t), \quad t = 0, \tag{1.2}$$

and gave its properties in a systematic way [20]. Carmichael presented the general theory of linear quantum difference equations [7]. This subject has gained an important place in the literature with the results obtained from various studies, see [1–4, 9, 12, 14, 19, 25, 29].

One of the reasons why quantum calculus is so important is that it has various applications in physics, e.g., [10, 11, 13, 24]. Studies on quantum calculus have revealed postquantum calculus. While quantum calculus deals with a q-number with one base, the postquantum calculus deals with p and q-numbers with two independent variables p and q. Many mathematicians and physicists have studied the (p, q)-calculus in many different research fields, we refer the readers to [6, 8, 18, 21, 27, 28] for details. One of these fields is related with (p, q)-difference equations.

Boundary value problems are studied for (p, q)-difference equations as in q-difference equations, e.g., [15, 22, 26]. In [26], the authors studied existence and uniqueness of solutions for a boundary value problem (p, q)-differential equation with nonlocal integral boundary condition. They presented some results by using Banach fixed point theorem, Boyd and Wong fixed point theorems for nonlinear contractions, and Leray–Schauder nonlinear alternative.

*Correspondence: ilkergencturk@gmail.com
2010 *AMS Mathematics Subject Classification:* 05A30, 39A13, 34A12.

In this paper, we study the existence of solution for a (p,q)-difference equation with integral boundary conditions of the form

$$\begin{cases} D_{p,q}^{2}u(t)+f(t,u(t))=0,\ t\in(0,1)\,, \\ u(0)=\int\limits_{0}^{1}u(t)d_{p,q}t, \quad u(1)=\int\limits_{0}^{1}tu(t)d_{p,q}t, \end{cases} \tag{1.3}$$

where $0<q<p\leq 1, D_{p,q}$ is (p,q)-difference operator and $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a given function.

A main motivation of our study is the following boundary value problem:

$$\begin{cases} u''(t)+f(t,u(t))=0,\ t\in(0,1)\,, \\ u(0)=\int\limits_{0}^{1}u(t)dt, \quad u(1)=\int\limits_{0}^{1}tu(t)dt \end{cases} \tag{1.4}$$

which Guezane-Lakoud et al. obtained by using Banach contraction principle and Leray–Schauder nonlinear alternative (see [17]). In addition, in [30], the authors studied the existence results for the following second-order boundary value problem:

$$\begin{cases} D_{q}^{2}u(t)+f(t,u(t))=0,\ t\in(0,1)\,, \\ u(0)=\int\limits_{0}^{1}u(t)d_{q}t, \quad u(1)=\int\limits_{0}^{1}tu(t)d_{q}t, \end{cases} \tag{1.5}$$

by using Banach's contraction principle, Leray–Schauder nonlinear alternative, Boyed and Wong fixed point theorem, and Krasnoselskii's fixed point theorem.

At the end of this section, we state the organization of the paper as follows. Section 2 contains some basic definitions and lemmas of the (p,q)-calculus that will be needed in the next sections. The main existence results are given in Section 3. Finally, in Section 4, some illustrative examples are given to justify the obtained results.

2. Preliminaries

In this section, we review some of the standard facts on (p,q)-calculus (see [27]). Without loss of generality, we can assume that $p+q\neq 1$ for $0<q<p\leq 1$ throughout the paper.

Assume that $f:[0,T]\to\mathbb{R},\ T>0$. Then the (p,q)-derivative of function f is defined by the formula

$$D_{p,q}f(t)=\frac{f(pt)-f(qt)}{(p-q)t}, \quad t\neq 0, \tag{2.1}$$

$$D_{p,q}f(0)=\lim_{t\to 0}D_{p,q}f(t), \quad t=0. \tag{2.2}$$

Remark 2.1 *We note that the function $D_{p,q}f(t)$ is defined on $[0,T/p]$ which is extended from $[0,T]$ of a function $f(t)$. Moreover, f is called (p,q)-differentiable on $[0,T/p]$ if $D_{p,q}f(t)$ exists $\forall t\in[0,T/p]$.*

Assume that $f:[0,T]\to\mathbb{R},\ T>0$. We define the (p,q)-integral of f by the formula

$$\int_0^t f(s)d_{p,q}s=(p-q)t\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}t\right) \tag{2.3}$$

whenever the right hand side converges.

Remark 2.2 *We note that the domain of function* $\int_0^t f(s)d_{p,q}s$ *is* $[0,pT]$ *which constricts from* $[0,T]$ *of a function* $f(t)$ *since* $0<p\leq 1$.

In the following theorem, we have compiled some basic facts properties of (p,q)-differentiation and (p,q)-integration, respectively.

Theorem 2.3 *Assume that* $f,g:[0,T]\to\mathbb{R},\ T>0,$ *are* (p,q)*-differentiable functions on* $[0,T/p]$. *Then*

(i.) $f+g:[0,T]\to\mathbb{R}$ *is* (p,q)*-differentiable on* $[0,T/p]$ *and*

$$D_{p,q}\left(f(t)+g(t)\right)=D_{p,q}f(t)+D_{p,q}g(t). \tag{2.4}$$

(ii.) For any constant c, $cf:[0,T]\to\mathbb{R}$ *is* (p,q)*-differentiable on* $[0,T/p]$ *and*

$$D_{p,q}\left(cf(t)\right)=cD_{p,q}f(t). \tag{2.5}$$

(iii.) $fg:[0,T]\to\mathbb{R}$ *is* (p,q)*-differentiable on* $[0,T/p]$ *and*

$$D_{p,q}\left(fg\right)(t)=f(pt)D_{p,q}g(t)+g(qt)D_{p,q}f(t), \tag{2.6}$$

$$D_{p,q}\left(fg\right)(t)=g(pt)D_{p,q}f(t)+f(qt)D_{p,q}g(t). \tag{2.7}$$

(iv.) If $g(t)\neq 0$, *then* f/g *is* (p,q)*-differentiable on* $[0,T/p]$ *with*

$$D_{p,q}\left(\frac{f}{g}\right)(t)=\frac{g(qt)D_{p,q}f(t)-f(qt)D_{p,q}g(t)}{g(pt)g(qt)}, \tag{2.8}$$

$$D_{p,q}\left(\frac{f}{g}\right)(t)=\frac{g(pt)D_{p,q}f(t)-f(pt)D_{p,q}g(t)}{g(pt)g(qt)}. \tag{2.9}$$

Theorem 2.4 *Assume that* $f,g:[0,T]\to\mathbb{R},\ T>0$ *are continuous functions and constants* $0<q<p\leq 1, a,b\in[0,T]$. *The following formulas hold:*

(i.) The (p,q)*-integration by parts is given by*

$$\int_a^b f(pt)D_{p,q}g(t)d_{p,q}t=f(t)g(t)\Big|_a^b-\int_a^b g(qt)D_{p,q}f(t)d_{p,q}t. \tag{2.10}$$

(ii.)

$$D_{p,q}\int_0^t f(s)d_{p,q}s=f(t). \tag{2.11}$$

(iii.)

$$\int_0^t D_{p,q}f(s)d_{p,q}s = f(t) - f(0). \tag{2.12}$$

(iv.)

$$\int_a^t D_{p,q}f(s)d_{p,q}s = f(t) - f(a) \quad \textit{for } a \in (0,t). \tag{2.13}$$

Theorem 2.5 *Assume that a function $f : [0,T] \to \mathbb{R},\ T > 0$. Then for $t \in [0, p^2T]$, we have*

$$\int_0^t \int_0^s f(r)d_{p,q}r d_{p,q}s = \frac{1}{p}\int_0^t (t-qs)f\left(\frac{s}{p}\right)d_{p,q}s. \tag{2.14}$$

Proof Using integration by parts formula (2.10), we have

$$\begin{aligned}
\int_0^t \int_0^s f(r)d_{p,q}r d_{p,q}s &= \int_0^t D_{p,q}(s)\int_0^s f(r)d_{p,q}r d_{p,q}s \\
&= \left(s\int_0^{s/p} f(r)d_{p,q}r\right)\Bigg|_{s=0}^{s=t} - \frac{1}{p}\int_0^t qsf\left(\frac{s}{p}\right)d_{p,q}s \\
&= t\int_0^{t/p} f(s)d_{p,q}s - \frac{1}{p}\int_0^t qsf\left(\frac{s}{p}\right)d_{p,q}s.
\end{aligned}$$

By changing variables in the first integrand of the right side of the last equality, we obtain (2.14). □

Remark 2.6 *Another proof of Theorem 2.5 is given in [15, 26].*

Remark 2.7 *We note that in a particular case of $p = 1$, all of the above theorems, then all of the presented results are reduced to corresponding results of the usual q-calculus.*

Lemma 2.8 *For any $h \in C([0,1], \mathbb{R})$, the boundary value problem*

$$\begin{cases} D_{p,q}^2 u(t) + h(t) = 0, \\ u(0) = \int_0^1 u(t)d_{p,q}t, \quad u(1) = \int_0^1 tu(t)d_{p,q}t \end{cases} \tag{2.15}$$

is equivalent to the integral equation

$$u(t) = -\frac{1}{p}\int_0^t (t-qs)h\left(\frac{s}{p}\right)d_{p,q}s + \frac{1}{p}\frac{p+q}{p+q-1}\int_0^1 (1-qs)h\left(\frac{s}{p}\right)d_{p,q}s$$
$$-\frac{(p^2-q^2)}{p^3(p+q-1)(p^2+pq+q^2)}\int_0^1 (s-qs^2)\Bigg(\Big[\left(p^2+pq+q^2\right)((1-t)(p+q-1)+s)\Big]$$
$$+\Big[p^2+(p+q)(q-1)\Big]\Bigg)h\left(\frac{s}{p^2}\right)d_{p,q}s. \tag{2.16}$$

Proof Applying double (p,q)-integration to the first equation of (2.15), then changing the order of (p,q)-integration, we have

$$u(t) = A + Bt - \frac{1}{p}\int_0^t (t-qs)h(s/p)d_{p,q}s. \tag{2.17}$$

Using the first integral condition, we get

$$u(0) = A = \int_0^1 u(s)d_{p,q}s. \tag{2.18}$$

By replacing A in (2.17), we get

$$u(t) = \int_0^1 u(s)d_{p,q}s + Bt - \frac{1}{p}\int_0^t (t-qs)h(s/p)d_{p,q}s. \tag{2.19}$$

Using the second integral condition, we obtain

$$B = \frac{1}{p}\int_0^1 (1-qs)h(s/p)d_{p,q}s - \int_0^1 u(s)d_{p,q}s + \int_0^1 su(s)d_{p,q}s. \tag{2.20}$$

By replacing B in (2.19), we have

$$u(t) = -\frac{1}{p}\int_0^t (t-qs)h(s/p)d_{p,q}s + \frac{t}{p}\int_0^1 (1-qs)h(s/p)d_{p,q}s$$
$$+(1-t)\int_0^1 u(s)d_{p,q}s + t\int_0^1 su(s)d_{p,q}s. \tag{2.21}$$

Integrating (2.21) over $[0,1]$, we obtain

$$\int_0^1 u(s)d_{p,q}s = -\frac{p^2-q^2}{p^3}\int_0^1 (s-qs^2)h\left(\frac{s}{p^2}\right)d_{p,q}s$$
$$+\frac{1}{p}\int_0^1 (1-qs)h\left(\frac{s}{p}\right)d_{p,q}s + \int_0^1 su(s)d_{p,q}s. \tag{2.22}$$

Substituting (2.22) in (2.21), we have

$$u(t) = -\frac{1}{p}\int_0^t (t-qs)h\left(\frac{s}{p}\right)d_{p,q}s + \frac{1}{p}\int_0^1 (1-qs)h\left(\frac{s}{p}\right)d_{p,q}s$$
$$-\frac{(1-t)\left(p^2-q^2\right)}{p^3}\int_0^1 (s-qs^2)h\left(\frac{s}{p^2}\right)d_{p,q}s + \int_0^1 su(s)d_{p,q}s. \tag{2.23}$$

By integrating the equality (2.23) multiplied by s over $[0,1]$, we obtain

$$\int_0^1 su(s)d_{p,q}s = -\frac{(p-q)(p+q)}{p^3(p+q-1)}\int_0^1 (s^2-qs^3)h\left(\frac{s}{p^2}\right)d_{p,q}s + \frac{1}{p(p+q-1)}\int_0^1 (1-qs)h\left(\frac{s}{p}\right)d_{p,q}s$$
$$-\frac{\left(p^2-q^2\right)}{p^3}\frac{\left(p^2+(p+q)(q-1)\right)}{(p+q-1)\left(pq+p^2+q^2\right)}\int_0^1 (s-qs^2)h\left(\frac{s}{p^2}\right)d_{p,q}s. \tag{2.24}$$

On substituting (2.23) into (2.24), we get

$$u(t) = -\frac{1}{p}\int_0^t (t-qs)h\left(\frac{s}{p}\right)d_{p,q}s + \frac{1}{p}\frac{p+q}{p+q-1}\int_0^1 (1-qs)h\left(\frac{s}{p}\right)d_{p,q}s$$
$$-\frac{\left(p^2-q^2\right)}{p^3\left(p+q-1\right)\left(p^2+pq+q^2\right)}\int_0^1 (s-qs^2)\Bigg(\Big[\left(p^2+pq+q^2\right)\left((1-t)(p+q-1)+s\right)\Big]$$
$$+\Big[p^2+(p+q)(q-1)\Big]\Bigg)h\left(\frac{s}{p^2}\right)d_{p,q}s.$$

The proof is now completed. □

We will denote by $\mathcal{C} = C([0,1],\mathbb{R})$ the Banach space of all continuous functions from $[0,1]$ to $\mathbb{R}$, endowed with the norm defined by $||u|| = \sup\{|u(t)|,\ t \in [0,1]\}$. We regard $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ as being defined an operator as

$$
\begin{aligned}
(\mathcal{F}u)(t) = &-\frac{1}{p}\int_0^t (t-qs) f\left(s, u(ps)\right) d_{p,q}s + \frac{1}{p}\frac{p+q}{p+q-1}\int_0^1 (1-qs) f\left(s, u(ps)\right) d_{p,q}s \\
&- \frac{\left(p^2-q^2\right)}{p^3\left(p+q-1\right)\left(p^2+pq+q^2\right)}\int_0^1 (s-qs^2)\Bigg(\Big[\left(p^2+pq+q^2\right)\left((1-t)(p+q-1)+s\right)\Big] \\
&+\Big[p^2+(p+q)(q-1)\Big]\Bigg) f\left(s, u(p^2 s)\right) d_{p,q}s.
\end{aligned}
\tag{2.25}
$$

It should be noted that problem (1.3) has solutions if and only if the operator $\mathcal{F}$ has fixed points. For simplicity, we set a constant $\mathcal{A}$ as

$$
\begin{aligned}
\mathcal{A} = &\frac{1}{(p+q)} + \frac{1}{p+q-1} + \frac{(p-q)}{p\left(p^2+pq+q^2\right)} \\
&+ \frac{1}{\left(p+q-1\right)\left(p^2+pq+q^2\right)\left(p^2+q^2\right)} + \frac{(p-q)\left(p^2+(p+q)(q-1)\right)}{p\left(p+q-1\right)\left(p^2+pq+q^2\right)^2}.
\end{aligned}
\tag{2.26}
$$

3. Existence results

In this section, we will deal with our main results. Our first result is based on the concept of Banach's fixed point theorem.

Theorem 3.1 *Suppose that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the conditions*

(H_1) $|f(t,u) - f(t,v)| \leq \mathrm{L}|u-v|$, $\forall t \in [0,1]$ *and* $u, v \in \mathbb{R}$,

(H_2) $\mathrm{L}\mathcal{A} < 1$,

where L *is a Lipschitz constant, and* $\mathcal{A}$ *is given by* (2.26)*. Then problem* (1.3) *has a unique solution.*

Proof We convert problem (1.3) to a fixed point problem $u = \mathcal{F}u$, where $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ is given by (2.25). Suppose that $\sup_{t\in[0,1]} |f(t,0)| = \mathrm{M}$ and set a constant R satisfying

$$
\mathrm{R} \geqslant \frac{\mathrm{M}\mathcal{A}}{1-\mathrm{L}\mathcal{A}}.
$$

We will first show that $\mathcal{F}B_{\mathrm{R}} \subset B_{\mathrm{R}}$, where $B_{\mathrm{R}} = \{u \in \mathcal{C} : ||u|| \leq \mathrm{R}\}$.

For any $u \in B_{\mathrm{R}}$, we have

$$
\begin{aligned}
||\mathcal{F}u|| \leq & \sup_{t\in[0,1]} \Bigg| -\frac{1}{p}\int_0^t (t-qs)\big(\mathrm{L}||u||+\mathrm{M}\big)d_{p,q}s + \frac{1}{p}\frac{p+q}{p+q-1}\int_0^1 (1-qs)\big(\mathrm{L}||u||+\mathrm{M}\big)d_{p,q}s \\
& - \frac{\left(p^2-q^2\right)}{p^3\left(p+q-1\right)\left(p^2+pq+q^2\right)}\int_0^1 (s-qs^2)\bigg(\Big[\big(p^2+pq+q^2\big)\big((1-t)(p+q-1)+s\big)\Big] \\
& + \Big[p^2+(p+q)(q-1)\Big]\bigg)\big(\mathrm{L}||u||+M\big)d_{p,q}s\Bigg| \\
\leq & \left(\mathrm{LR}+\mathrm{M}\right)\sup_{t\in[0,1]} \Bigg| -\frac{1}{p}\int_0^t (t-qs)d_{p,q}s + \frac{1}{p}\frac{p+q}{p+q-1}\int_0^1 (1-qs)d_{p,q}s \\
& - \frac{\left(p^2-q^2\right)}{p^3\left(p+q-1\right)\left(p^2+pq+q^2\right)}\int_0^1 (s-qs^2)\bigg(\Big[\big(p^2+pq+q^2\big)\big((1-t)(p+q-1)+s\big)\Big] \\
& + \Big[p^2+(p+q)(q-1)\Big]\bigg)d_{p,q}s\Bigg| \\
\leq & \left(\mathrm{LR}+\mathrm{M}\right)\Bigg(\frac{1}{p}\sup_{t\in[0,1]}\left|\int_0^t (t-qs)d_{p,q}s\right| + \frac{1}{p}\frac{p+q}{p+q-1}\sup_{t\in[0,1]}\left|\int_0^1 (1-qs)d_{p,q}s\right| \\
& + \frac{\left(p^2-q^2\right)}{p^3\left(p+q-1\right)\left(pq+p^2+q^2\right)}\sup_{t\in[0,1]}\Bigg|\int_0^1 (s-qs^2)\bigg(\Big[\big(p^2+pq+q^2\big)\big((1-t)(p+q-1)+s\big)\Big] \\
& + \Big[p^2+(p+q)(q-1)\Big]\bigg)d_{p,q}s\Bigg| \\
\leq & \left(\mathrm{LR}+\mathrm{M}\right)\Bigg(\frac{1}{(p+q)}+\frac{1}{p+q-1}+\frac{(p-q)}{p\left(p^2+pq+q^2\right)} \\
& + \frac{1}{\left(p+q-1\right)\left(p^2+pq+q^2\right)\left(p^2+q^2\right)}+\frac{(p-q)\left(p^2+(p+q)(q-1)\right)}{p\left(p+q-1\right)\left(p^2+pq+q^2\right)^2}\Bigg) \\
\leq & \left(\mathrm{LR}+\mathrm{M}\right)\mathcal{A}\leq \mathrm{R}.
\end{aligned}
$$

Therefore, $\mathcal{F}B_{\mathrm{R}}\subset B_{\mathrm{R}}$.

Next, we will show that $\mathcal{F}$ is a contraction. For any $u,v\in\mathcal{C}$ and $\forall t\in[0,1]$, we have

$$
\begin{aligned}
||\mathcal{F}u-\mathcal{F}v|| \leq & \sup_{t\in[0,1]} \mathrm{L}||u-v|| \Bigg\{ -\frac{1}{p}\int_0^t (t-qs)d_{p,q}s + \frac{1}{p}\frac{p+q}{p+q-1}\int_0^1 (1-qs)d_{p,q}s \\
& + \frac{\left(p^2-q^2\right)}{p^3\left(p+q-1\right)\left(p^2+pq+q^2\right)}\int_0^1 (s-qs^2)\bigg(\Big[\left(p^2+pq+q^2\right)\left((1-t)(p+q-1)+s\right)\Big] \\
& + \Big[p^2+(p+q)(q-1)\Big]\bigg)d_{p,q}s\Bigg\} \\
\leq & \mathrm{L}\mathcal{A}||u-v||.
\end{aligned}
$$

As $\mathrm{L}\mathcal{A}<1$, $\mathcal{F}$ is a contraction. By the Banach's contraction mapping principle, we obtain that the conclusion of the theorem which completes the proof. □

Next, we are in a position to prove the existence of solutions of problem (1.3) by using the following Leray–Schauder nonlinear alternative:

Theorem 3.2 (Nonlinear Alternative for Single Valued Maps [16]) *Assume that $\mathcal{C}$ is a closed convex subset of a Banach space $\mathbb{E}$. Let $\mathcal{U}$ be an open subset of $\mathcal{C}$, with $0\in\mathcal{U}$. Then any continuous, compact (that is, $\mathcal{F}(\mathcal{U})$ is a relatively compact subset of $\mathcal{C}$) map $\mathcal{F}:\mathcal{U}\to\mathcal{C}$ has at least one of the following properties:*

(1) *$\mathcal{F}$ has a fixed point in $\mathcal{U}$,*

(2) *There exist a $u\in\partial\mathcal{U}$ (the boundary of $\mathcal{U}$ in $\mathcal{C}$) and $\lambda\in(0,1)$ such that $u=\lambda\mathcal{F}(u)$.*

Theorem 3.3 *Assume that:*

(H_3) *There exists a continuous nondecreasing function $\psi:[0,\infty)\to(0,\infty)$ and a function $\varphi\in L^1([0,1],\mathbb{R}^+)$ with $|f(t,u)|\leq\varphi(t)\psi(\|u\|)$, $\forall(t,u)\in[0,1]\times\mathbb{R}$;*

(H_4) *There exists a constant $\mathrm{M}>0$ such that*

$$
\frac{\mathrm{M}}{\psi(\|u\|)\|\varphi\|_{L^1}\mathcal{A}}>1,
$$

where $f(x)=\|\varphi\|_{L^1}=\int_0^1\varphi(s)d_{pq}s\neq 0.$

Then problem (1.3) has at least one solution.

Proof We define $\mathcal{F}:\mathcal{C}\to\mathcal{C}$ as in (2.25). The proof will be divided into three steps.

(1) $\mathcal{F}$ maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$.

Let $B_{\mathrm{K}}=\{u\in C([0,1],\mathcal{R}):||u||\leq\mathrm{K}\}$ be a bounded set in $C([0,1],\mathcal{R})$ and $u\in B_{\mathrm{K}}$. Then we obtain conditions

$$
\begin{aligned}
|\mathcal{F}u(t)| \leq & \frac{1}{p}\int_0^t (t-qs)\left|f(s,u(ps))\right| d_{p,q}s + \frac{1}{p}\frac{p+q}{p+q-1}\int_0^1 (1-qs)\left|f(s,u(ps))\right| d_{p,q}s \\
& + \frac{\left(p^2-q^2\right)}{p^3\left(p+q-1\right)\left(p^2+pq+q^2\right)}\int_0^1 (s-qs^2)\Bigg(\Big[\left(p^2+pq+q^2\right)\left((1-t)(p+q-1)+s\right)\Big] \\
& \qquad + \Big[p^2+(p+q)(q-1)\Big]\Bigg)\left|f\left(s,u(p^2s)\right)\right| d_{p,q}s \\
\leq & \psi\left(||u||\right)||\varphi||_{L^1}\frac{1}{p}\int_0^t (t-qs)d_{p,q}s + \psi\left(||u||\right)||\varphi||_{L^1}\frac{1}{p}\frac{p+q}{p+q-1}\int_0^1 (1-qs)d_{p,q}s \\
& + \frac{\psi\left(||u||\right)||\varphi||_{L^1}\left(p^2-q^2\right)}{p^3\left(p+q-1\right)\left(p^2+pq+q^2\right)}\int_0^1 (s-qs^2)\Bigg(\Big[\left(p^2+pq+q^2\right)\left((1-t)(p+q-1)+s\right)\Big] \\
& \qquad + \Big[p^2+(p+q)(q-1)\Big]\Bigg)d_{p,q}s \\
= & \psi\left(||u||\right)||\varphi||_{L^1}\mathcal{A}.
\end{aligned}
$$

Thus, we obtain

$$
||\mathcal{F}u|| \leq \psi(\mathrm{K})||\varphi||_{L^1}\mathcal{A}.
$$

(2) $\mathcal{F}$ maps bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$.

Let $r_1, r_2 \in [0,1], r_1 < r_2$ and B_{K} be a bounded set of $C([0,1],\mathbb{R})$ as before, then for $u \in B_{\mathrm{K}}$, we get

$$
\begin{aligned}
& |\mathcal{F}u(r_2) - \mathcal{F}u(r_1)| \\
& \leq \frac{1}{p}\int_0^{r_1} |r_2-r_1|\varphi(s)\psi(\mathrm{K})d_{p,q}s + \int_{r_1}^{r_2} \left(r_2-qs\right)\varphi(s)\psi(\mathrm{K})d_{p,q}s \\
& \quad + \frac{\left(p^2-q^2\right)}{p^3\left(p+q-1\right)\left(pq+p^2+q^2\right)}\int_0^1 \left(s-qs^2\right)\left(p^2+pq+q^2\right)\left(p+q-1\right)\left|r_2-r_1\right|\varphi(s)\psi(\mathrm{K})d_{p,q}s.
\end{aligned}
$$

As $(r_2 - r_1) \to 0$, the right hand side of the above inequality tends to zero independence from $u \in B_{\mathrm{K}}$. Therefore, $\mathcal{F}$ is equicontinuous. As $\mathcal{F}$ satisfies the above assumptions, by aid of the Arzela-Ascoli Theorem, it follows that $\mathcal{F} : C([0,1],\mathbb{R}) \to \mathcal{C}([0,1],\mathbb{R})$ is completely continuous.

(3) Setting $u = \lambda \mathcal{F} u$, where $\lambda \in (0,1)$, we have for $t \in [0,1]$

$$
\begin{aligned}
|u(t)| &= |\lambda \mathcal{F} u(t)| \\
&\leq \frac{1}{p} \int_0^t (t - qs) \, |f(s, u(ps))| \, d_{p,q}s + \frac{1}{p} \frac{p+q}{p+q-1} \int_0^1 (1 - qs) \, |f(s, u(ps))| \, d_{p,q}s \\
&\quad + \frac{(p^2 - q^2)}{p^3 (p+q-1)(p^2+pq+q^2)} \int_0^1 (s - qs^2) \Bigg(\Big[\left(p^2+pq+q^2\right) ((1-t)(p+q-1)+s) \Big] \\
&\qquad + \Big[p^2 + (p+q)(q-1) \Big] \Bigg) \left| f\left(s, u(p^2 s)\right) \right| d_{p,q}s \\
&\leq \psi\left(||u||\right) ||\varphi||_{L^1} \mathcal{A},
\end{aligned}
$$

and as a consequence of this last equality, we have

$$
\frac{||u||}{\psi(||u||) ||\varphi||_{L^1} \mathcal{A}} \leq 1.
$$

Under the assumption (H_4), there exists M such that $||u|| \neq \mathrm{M}$. Define

$$
U = \{ u \in C([0,1], \mathbb{R}) : ||u|| < \mathrm{M} \}.
$$

Observe that the operator $\mathcal{F} : U \to C([0,1], \mathbb{R})$ is continuous and completely continuous (which is well known to be compact restricted to bounded sets).

From the choice of U, there is no $u \in \partial U$ such that $u = \lambda \mathcal{F} u$ for some $\lambda \in (0,1)$. We conclude from nonlinear alternative of Leray–Schauder type that $\mathcal{F}$ has a fixed point $u \in U$, which is a solution of problem (1.3). This finishes the proof.

□

The third result is derived from the following Boyed and Wong fixed point theorem.

Definition 3.4 *[29] Let set $\mathbb{E}$ be a Banach space and set $\mathfrak{A} : \mathbb{E} \to \mathbb{E}$ be a mapping. $\mathfrak{A}$ is called a nonlinear contraction if there exists a continuous nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\Psi(0) = 0$ and $\Psi(\rho) < \rho$ for all $\rho > 0$ has the following property:*

$$
||\mathfrak{A}x - \mathfrak{A}y|| \leq \Psi(||x - y||), \ \forall x, y \in \mathbb{E}.
$$

Lemma 3.5 *(Boyed and Wong) [5] Assume that $\mathbb{E}$ is a Banach space and set $\mathfrak{A} : \mathbb{E} \to \mathbb{E}$ is a nonlinear contraction. Then, $\mathfrak{A}$ has a unique fixed point in $\mathbb{E}$.*

Theorem 3.6 *One supposes that*

(H_5) *there exists a continuous function $h : [0,1] \to \mathbb{R}^+$ with the property that*

$$
|f(t,x) - f(t,y)| \leq h(t) \frac{|x - y|}{G + |x - y|}
$$

$\forall t \in [0,1]$ *and* $x, y \geq 0$, *where*

$$G = \frac{1+2(p+q)}{p(p+q-1)} \int_0^1 (1-qs)h(s)d_{p,q}s$$
$$+ \frac{(p^2-q^2)}{p^3(p+q-1)(p^2+pq+q^2)} \int_0^1 (s-qs^2)\Bigg(\Big[(p^2+pq+q^2)((p+q-1)+s)\Big]$$
$$+ \Big[p^2+(p+q)(q-1)\Big]\Bigg)h(s)d_{p,q}s.$$

Then, problem (1.3) *has a unique solution.*

Proof Consider the operator $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ defined by (2.25). We set a continuous nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\Psi(\rho) = \frac{G\rho}{G+\rho}, \ \forall \rho \geq 0,$$

with $\Psi(0) = 0$ and $\Psi(\rho) < \rho$, $\forall \rho > 0$. Let $u, v \in \mathcal{C}$. Then, we get

$$|f(s,u(s)) - f(s,v(s))| \leq h(s)\frac{|u-v|}{G+|u-v|}.$$

Therefore, we have

$$|\mathcal{F}u(t) - \mathcal{F}v(t)|$$
$$\leq \Bigg\{\frac{1}{p}\int_0^1 (1-qs)h(s)d_{p,q}s + \frac{1}{p}\frac{p+q}{p+q-1}\int_0^1 (1-qs)h(s)d_{p,q}s$$
$$+ \frac{(p^2-q^2)}{p^3(p+q-1)(p^2+pq+q^2)} \int_0^1 (s-qs^2)\Bigg(\Big[(p^2+pq+q^2)((p+q-1)+s)\Big]$$
$$+ \Big[p^2+(p+q)(q-1)\Big]\Bigg)h(s)d_{p,q}s\Bigg\} \times \frac{||u-v||}{G+||u-v||}.$$

This gives that $||\mathcal{F}u - \mathcal{F}v|| \leq \Psi(||u-v||)$. Hence, we see that $\mathcal{F}$ is a nonlinear contraction. Therefore, by Lemma 3.5, the operator $\mathcal{F}$ has a unique fixed point in $\mathcal{C}$, which is a unique solution of problem (1.3). □

As a next result, we prove the existence of solutions of (1.3) by using Krasnoselskii's fixed point theorem below.

Theorem 3.7 *[23] Let* $\mathcal{K}$ *be a bounded closed convex and nonempty subset of a Banach space* $\mathbb{X}$*. Let* $\mathfrak{A}$, $\mathfrak{B}$ *be operators such that:*

(1) $\mathfrak{A}x + \mathfrak{B}y \in \mathcal{K}$ *whenever* $x, y \in \mathcal{K}$.

(2) $\mathfrak{A}$ *is compact and continuous.*

(3) $\mathfrak{B}$ *is a contraction mapping.*

Then, there exists $z \in \mathcal{K}$ *such that* $z = \mathfrak{A}z + \mathfrak{B}z$.

Theorem 3.8 *Let* $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ *be a continuous function satisfying* (H_1), *also the following assumption holds:*

(H_6) $|f(t,u)|\leq\mu(t),\ \forall(t,u)\in[0,1]\times\mathbb{R}$, *and* $\mu\in L^1([0,1],\mathbb{R}^+)$.

If

$$\mathcal{L}\left\{\frac{1}{p+q}+\frac{(p^2-q^2)}{p(p+q-1)(p^2+pq+q^2)^2}\right.$$
$$\left.+\frac{p-q}{(p+q-1)(p^2+q^2)(p^2+pq+q^2)}+\frac{(p-q)(1+p+q)}{p(p+q-1)(p^2+pq+q^2)}\right\}<1, \tag{3.1}$$

then problem (1.3) *has at least one solution on* $[0,1]$.

Proof Defining $\max\limits_{t\in[0,1]}|\mu(t)|=||\mu||$ and fixing a constant $\mathrm{R}\geq||\mu||\mathcal{A}$, where $\mathcal{A}$ is defined by (2.25), and consider $B_{\mathrm{R}}=\{u\in\mathcal{C}:||\mu||\leq\mathrm{R}\}$.

We regard the operators $\mathcal{F}_1$ and $\mathcal{F}_2$ on the ball B_{R} as

$$(\mathcal{F}_1u)(t)=-\frac{1}{p}\int_0^t(t-qs)f(s,u(s))d_{pq}s,$$

$$(\mathcal{F}_2u)(t)=\frac{1}{p}\frac{p+q}{p+q-1}\int_0^1(1-qs)f(s,u(ps))d_{p,q}s$$
$$-\frac{(p^2-q^2)}{p^3\,(p+q-1)\,(p^2+pq+q^2)}\int_0^1(s-qs^2)\Bigg(\Big[\left(p^2+pq+q^2\right)((1-t)(p+q-1)+s)\Big]$$
$$+\Big[p^2+(p+q)(q-1)\Big]\Bigg)f\left(s,u(p^2s)\right)d_{p,q}s.$$

For $u,v\in B_{\mathrm{R}}$, we have

$$||\mathcal{F}_1u+\mathcal{F}_2v||\leq||\mu||\mathcal{A}\leq\mathrm{R}.$$

This implies that $\mathcal{F}_1u+\mathcal{F}_2v\in B_{\mathrm{R}}$. In view of condition (3.1), it follows that $\mathcal{F}_2$ is a contraction mapping.

At present, we will show below that $\mathcal{F}_1$ is compact and continuous. The continuity of f together with the assumption (H_6) yields that the operator $\mathcal{F}_1$ is continuous and uniformly bounded on B_{R}. Setting

$\sup\limits_{(t,u)\in[0,1]\times B_{\mathrm{R}}} |f(t,u)| = f_{\max} < \infty$, also for $t_1, t_2 \in [0,1]$ with $t_1 \le t_2$ and $u \in B_{\mathrm{R}}$, we obtain that

$$\begin{aligned}
|F_1u(t_2) - F_1u(t_1)| &= \left| -\frac{1}{p}\int_0^{t_2}(t_2 - qs)f(s,u(s))d_{pq}s + \frac{1}{p}\int_0^{t_1}(t_1 - qs)f(s,u(s))d_{pq}s \right| \\
&= \left| \frac{1}{p}\int_0^{t_2}(t_2 - qs)f(s,u(s))d_{pq}s - \frac{1}{p}\int_0^{t_1}(t_1 - qs)f(s,u(s))d_{pq}s \right| \\
&= \left| \frac{1}{p}\left(\int_0^{t_1}(t_2 - t_1)f(s,u(s))d_{pq}s + \int_{t_1}^{t_2}(t_2 - qs)f(s,u(s))d_{pq}s \right) \right| \\
&\le \frac{1}{p}f_{\max}\left(\int_0^{t_1}(t_2 - t_1)d_{pq}s + \int_{t_1}^{t_2}(t_2 - qs)d_{pq}s \right).
\end{aligned}$$

It is clear that as $(t_2 - t_1) \to 0$, the right-hand side of the above inequality tends to be zero. Thus, $\mathcal{F}_1$ is relatively compact on B_{R}. Hence, by the Arzela-Ascoli Theorem, $\mathcal{F}_1$ is compact on B_{R}. Since all the assumptions of Theorem 3.7 are satisfied and by conclusion of Theorem 3.7, we deduce that problem (1.3) has at least one solution on $[0,1]$. This completes the proof. □

4. Illustrative examples

Example 4.1 *Consider the following boundary value problem for second-order quantum* (p,q)*-difference equation of the form:*

$$\begin{cases} D^2_{\frac{7}{9},\frac{5}{7}}u(t) + \dfrac{te^{-\sin(t^2)}}{5(t+e^{t\cos(t)})}u(t) = 0, \ t \in [0,1], \\ u(0) = \displaystyle\int_0^1 u(t)d_{p,q}t, \quad u(1) = \int_0^1 tu(t)d_{p,q}t. \end{cases} \tag{4.1}$$

Setting constants $p = \dfrac{7}{9}, q = \dfrac{5}{7}$*, and function* $f(t,u(t)) = \dfrac{te^{-\sin(t^2)}}{5(t+e^{t\cos(t)})}u(t)$*, we have* $\mathcal{A} \approx 3.853$*. Hence,* $|f(t,u) - f(t,v)| \le \dfrac{4}{25}|u-v|$*, then the condition* (H_1) *is satisfied with* $\mathrm{L} = \dfrac{4}{25}$*. Indeed, we get* $\mathrm{L}\mathcal{A} \approx 0.6165 < 1$*. Consequently, by Theorem 3.1, we conclude that problem 4.1 has a unique solution on* $[0,1]$*.*

Example 4.2 *Consider the following boundary value problem for second-order quantum* (p,q)*-difference equation of the form:*

$$\begin{cases} D^2_{\frac{7}{8},\frac{5}{6}}u(t) + \left(\dfrac{2t}{3} + 1\right)\dfrac{|u|}{1+|u|} = 0, \ t \in [0,1], \\ u(0) = \displaystyle\int_0^1 u(t)d_{p,q}t, \quad u(1) = \int_0^1 tu(t)d_{p,q}t. \end{cases} \tag{4.2}$$

Setting constants $p = \dfrac{7}{8}, q = \dfrac{5}{6}$*, and function* $f(t,u(t)) = h(t)\dfrac{|u|}{1+|u|}$ *where* $h(t) = \left(\dfrac{2t}{3} + 1\right)$*, we get*

$$
\begin{aligned}
G =& \frac{1+2(p+q)}{p(p+q-1)} \int_0^1 (1-qs)\left(\frac{2s}{3}+1\right) d_{p,q}s \\
&+ \frac{(p^2-q^2)}{p^3(p+q-1)(p^2+pq+q^2)} \int_0^1 (s-qs^2)\Bigg(\left[(p^2+pq+q^2)((p+q-1)+s)\right] \\
&\qquad + \left[p^2+(p+q)(q-1)\right]\Bigg)\left(\frac{2s}{3}+1\right) d_{p,q}s \\
\approx& 4,6896.
\end{aligned}
$$

Hence, $|f(t,u)-f(t,v)| \leq \left(\frac{2t}{3}+1\right)\frac{|u-v|}{4.6896+|u-v|}$, *by Theorem 3.6, we conclude that problem 4.2 has a unique solution on* $[0,1]$.

Example 4.3 *Consider the following boundary value problem for second-order quantum* (p,q)*-difference equation of the form:*

$$
\begin{cases}
D^2_{\frac{8}{9},\frac{15}{18}} u(t) + \left(\frac{e^{-\sin t}}{3(1+t)}\frac{|u(t)|}{2+|u(t)|}\right) = 0, \; t \in [0,1], \\
u(0) = \int_0^1 u(t) d_{p,q}t, \quad u(1) = \int_0^1 t u(t) d_{p,q}t.
\end{cases}
\tag{4.3}
$$

Setting constants $p = \frac{8}{9}, q = \frac{15}{18}$, *and function* $f(t,u(t)) = \left(\frac{e^{-\sin t}}{3(1+t)}\frac{|u(t)|}{2+|u(t)|}\right)$, *we have* $|f(t,u)| \leq \frac{e^{-\sin t}}{3(1+t)} := \mu_0(t)$. *Then the function* f *satisfies condition* (H_6). *Moreover, the condition* (H_1) *is satisfied with* $\mathrm{L} = \frac{1}{3}$ *and the condition* (3.1) *equals* 0.7398 *for values of* p *and* q *are given above.*

Hence, by Theorem 3.8, we conclude that problem 4.3 has a unique solution on $[0,1]$.

References

[1] Adams CR. On the linear ordinary q-difference equation. Annals of Mathematics 1928; 195-205.

[2] Ahmad B. Boundary-value problems for nonlinear third-order q-difference equations. Electronic Journal of Differential Equations 2011; 94: 1-7.

[3] Ahmad B, Alsaedi A, Ntouyas SK. A study of second-order q-difference equations with boundary conditions. Advances in Difference Equations 2012; 2012 (1): 1-10. doi: 10.1186/1687-1847-2012-35

[4] Bangerezako G. Variational q-calculus. Journal of Mathematical Analysis and Applications 2004; 289 (2): 650-665.

[5] Boyd DW, Wong JS. On nonlinear contractions. Proceedings of the American Mathematical Society 1969; 20(2): 458-464.

[6] Burban I. Two-parameter deformation of the oscillator algebra and (p, q)–analog of two-dimensional conformal field theory. Journal of Nonlinear Mathematical Physics 1995; 2 (3-4): 384-391.

[7] Carmichael RD. The general theory of linear q-difference equations. American Journal of Mathematics 1912; 34 (2): 147-168.

[8] Chakrabarti R, Jagannathan R. A (p, q)-oscillator realization of two-parameter quantum algebras. Journal of Physics A: Mathematical and General 1991; 24 (13): L711.

[9] El-Shahed M, Hassan H. Positive solutions of q-difference equation. Proceedings of the American Mathematical Society 2010; 138 (5): 1733-1738.

[10] Finkelstein RJ. q-field theory. Letters in Mathematical Physics 1995; 34 (2): 169-176.

[11] Finkelstein RJ. q-deformation of the lorentz group. Journal of Mathematical Physics 1996; 37 (2): 953-964.

[12] Floreanini R, Vinet L. q-gamma and q-beta functions in quantum algebra representation theory. Journal of Computational and Applied Mathematics 1996; 68 (1-2): 57-68.

[13] Freund PG, Zabrodin AV. The spectral problem for the q-knizhnik-zamolodchikov equation and continuous q-jacobi polynomials. Communications in Mathematical Physics 1995; 173 (1): 17-42.

[14] Gasper G, Rahman M. Basic Hypergeometric Series. 2nd ed. Cambridge, UK: Cambridge University Press, 2004.

[15] Gholami Y. Second order two-parametric quantum boundary value problems. Differential Equations Applications 2019; 11 (2): 243-265.

[16] Granas A, Dugundji J. Elementary fixed point theorems. In: Fixed Point Theory. New York, NY, USA: Springer, 2003, pp. 9-84.

[17] Guezane-Lakoud A, Hamidane N, Khaldi R. Existence and uniqueness of solution for a second order boundary value problem. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics 2013; 62: 121-129.

[18] Hounkonnou MN, Désiré J, Kyemba B. R(p, q)-calculus: differentiation and integration. SUT Journal of Mathematics 2013; 49 (2): 145-167.

[19] Ismail MEH, Simeonov P. q-difference operators for orthogonal polynomials. Journal of Computational and Applied Mathematics 2009; 233 (3): 749-761.

[20] Jackson FH. q-difference equations. American Journal of Mathematics 1910: 32 (4): 305-314.

[21] Jagannathan R, Rao KS. Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series. In: Proceedings of the International Conference on Number Theory and Mathematical Physics; Kumbakonam, India; 2005.

[22] Kamsrisuk N, Promsakon C, Ntouyas SK, Tariboon J. Nonlocal boundary value problems for (p, q)-difference equations. Differential Equations Applications 2018; 10 (2); 183-195.

[23] Krasnoselsky MA. Two remarks on the method of successive approximations. Uspekhi Matematicheskikh Nauk 1955; 10: 123-127.

[24] Kulish PP, Damaskinsky EV. On the q oscillator and the quantum algebra suq (1, 1). Journal of Physics A: Mathematical and General 1990; 23 (9): L415.

[25] Mason TE. On properties of the solutions of linear q-difference equations with entire function coefficients. American Journal of Mathematics 1915; 37 (4): 439-444.

[26] Promsakon C, Kamsrisuk N, Ntouyas SK, Tariboon J. On the second-order quantum (p,q)-difference equations with separated boundary conditions. Advances in Mathematical Physics 2018; Article ID 9089865.

[27] Sadjang PN. On the fundamental theorem of (p,q)-calculus and some (p,q)-Taylor formulas. Results in Mathematics 2018; 73 (1): 1-21.

[28] Sahai V, Yadav S. Representations of two parameter quantum algebras and p, q-special functions. Journal of Mathematical Analysis and Applications 2007; 335 (1): 268-279.

[29] Sitthiwirattham T, Tariboon J, Ntouyas SK. Three-point boundary value problems of nonlinear second-order q-difference equations involving different numbers of q. Journal of Applied Mathematics 2013; Article ID 763786.

[30] Zhuang H, Liu W. Existence results for a second-order q-difference equation with only integral conditions. University Politehnica of Bucharest Scientific Bulletin-Series A-Applied Mathematics and Physics 2017; 79 (4): 221-234.