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# A discussion on the existence and uniqueness analysis for the coupled two-term fractional differential equations

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**Abstract:** This paper mainly concentrates on the study of a new boundary value problem of coupled nonlinear two-term fractional differential system. We make use of the theories on fractional calculus and fixed point approach to derive the existence and uniqueness results of the considered two-term fractional systems. To confirm the application of the stated outcomes, two examples are provided.

**Key words:** Multiterm fractional differential systems, coupled system, fixed point theorem

## 1. Introduction

In the last few decades, the principles of fractional calculus have played the main role in Mathematics. Differential equations of integer order cannot suit some physical problems, but these kind of problems fit in the differential equations of fractional order. Recently, many researchers have done valuable performances in electromagnetic, control theory, signal, porous media, viscoelasticity, biological, engineering problems, image processing, fluid flow, diffusion, theology, etc. For more specifics, [1–5, 10–47] and references therein. In some circumstances, we have to solve the equation containing more than one derivative term. This type of equation is known as a multiterm fractional differential equation. A standard example of a two-term fractional differential equation is the Bagley–Torvik equation [6]:

$$R\Psi''(t) + S\ {}^cD^{\frac{3}{2}}\Psi(t) + T\Psi(t) = \hbar(t),$$

where $\hbar$ is a given function and $R, S$ and $T$ are some constants. The above equation makes an appearance in motion modeling of a thin plate in a Newtonian fluid. Bagley–Torvik equation was discussed from a numerical point of view by Diethelm and Ford [9]. Another example of multiterm fractional differential equation is the Basset equation [7, 8].

$$\Psi'(t) + Q\ {}^cD^{\mu}\Psi(t) + R\Psi(t) = w(t),\ \Psi(0) = \Psi_0,\ 0 < \mu < 1.$$

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Basset equation is often applied with $\mu = \frac{1}{2}$. It represents the forces that arise if a spherical item is dropped into an incompressible viscous fluid.

Kaufmann and Yao [19] proved the existence of the two-term boundary value problem (BVP) having fractional order of the form

$$\begin{cases} (D^{\mu} - aD^{\nu})u(t) + f(t, u(t)) = 0, \ 1 < \mu < 2, \ 0 < \nu < \mu, \ t \in [0,1], \\ u(0) = 0, u(1) = 0, \end{cases}$$

where $D^{\mu}$ is the Riemann–Liouville (R-L) fractional derivative and $a \in \mathbb{R}$.

Ibrahim et al. [16] study the existence of solutions for a BVP of the form

$$\begin{cases} {}^{c}D^{\mu}u(t) - a\ {}^{c}D^{\nu}u(t) + f(t, u(t)) = 0, \ 0 < t < 1, \\ u(0) = u_0, u(1) = u_1, \end{cases}$$

where ${}^{c}D^{\mu}$ and ${}^{c}D^{\nu}$ are the Caputo fractional derivative with $1 < \mu \leq 2$ and $1 \leq \nu < \mu$, $a \in \mathbb{R}$ is a constant.

This paper concerns the existence and uniqueness of solutions of the following coupled system of two-term fractional differential systems of the form

$$\begin{cases} {}^{c}D^{\xi_1}\Psi_1(t) + \alpha_1\ {}^{c}D^{\sigma_1}\Psi_1(t) + \beta_1\hbar_1(t, \Psi_1(t), \Psi_2(t)) = 0, \\ {}^{c}D^{\xi_2}\Psi_2(t) + \alpha_2\ {}^{c}D^{\sigma_2}\Psi_2(t) + \beta_2\hbar_2(t, \Psi_1(t), \Psi_2(t)) = 0, \\ \Psi_1(0) = \Psi_1''(0) = \Psi_2(0) = \Psi_2''(0) = 0, \\ \rho_1(I^{\nu_1}\Psi_1)(\varepsilon_1) = \Psi_1(1), \ \rho_2(I^{\nu_2}\Psi_2)(\varepsilon_2) = \Psi_2(1), \end{cases} \tag{1.1}$$

where $t \in [0,1]$, $0 < \varepsilon_i < 1$, $2 < \xi_i \leq 3$, $2 \leq \sigma_i < \xi_i$, $\nu_i > 0, \alpha_i, \beta_i \in \mathbb{R}$, $i = 1, 2$, and ${}^{c}D^{\xi_i}$ stands for the Caputo fractional derivative of order $\xi_i$, $i = 1, 2$, $I^{\nu_i}$ are the R-L fractional integral of order $\nu_i$, $\hbar_i : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions and $\rho_i \in \mathbb{R}$, then

$$\rho_i \neq \frac{\Gamma(\nu_i + \xi_i - \sigma_i + 2) + a_i \frac{\Gamma(\nu_i + \xi_i - \sigma_i + 2)}{\Gamma(\xi_i - \sigma_i + 2)}}{a_i \varepsilon_i^{\nu_i + \xi_i - \sigma_i + 1} + \varepsilon_i^{\nu_i + 1} \frac{\Gamma(\nu_i + \xi_i - \sigma_i + 2)}{\Gamma(\nu_i + 2)}}, \ i = 1, 2.$$

Let $V = C([0,1], \mathbb{R})$, then obviously $(V, \|.\|_V)$ is a Banach space provided

$$\|\Psi\|_V = \{\sup |\Psi(t)| : t \in [0,1]\}.$$

## 2. Preliminaries

We provide fundamental realities, thoughts and lemmas wanted to arrange the primary consequences of our paper.

**Definition 2.1** *[35] The Caputo fractional derivative of order $q$ for a continuous function $f : [0, \infty) \to \mathbb{R}$ is given by*

$$^{c}D^{q}f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s) ds, \ n = [q] + 1,$$

*if $f^{(n)}(t)$ exists, where $[q]$ denotes the integer part of the real number $q$.*

**Definition 2.2** *[35] The Riemann–Liouville fractional integral of order $q$ for $f(t)$ is presented by*

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds, \;\; q > 0,$$

*if such integral exists.*

**Lemma 2.3** *[10] Let $m, n > 0$ and $h \in P_1[a,b]$ then $I^m I^n h = I^{m+n} h$.*

**Lemma 2.4** *[10] Let $\nu > -1$ and $n > 0$. Then*

$$I^n x^\nu = \frac{\Gamma(\nu+1)}{\Gamma(n+\nu+1)} x^{n+\nu}.$$

**Lemma 2.5** *[23] Let $\mu > 0$, then*

$$I^\mu \; {}^cD^\mu v(t) = v(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1},$$

*for some $c_i \in \mathbb{R}, \;\; i = 0, 1, 2, ..., n-1$, where $n = \lceil \mu \rceil$.*

## 3. Auxiliary result

For proving the existence of (1.1), we have to convert the system (1.1) into the equivalent integral equations.

**Lemma 3.1** *Let $\rho \neq \frac{\Gamma(\nu+\xi-\sigma+2) + a\frac{\Gamma(\nu+\xi-\sigma+2)}{\Gamma(\xi-\sigma+2)}}{a\varepsilon^{\nu+\xi-\sigma+1} + \varepsilon^{\nu+1}\frac{\Gamma(\nu+\xi-\sigma+2)}{\Gamma(\nu+2)}}$. Then for $\varsigma \in V$, the solution of*

$$\begin{cases} {}^cD^\xi \Psi(t) + a \; {}^cD^\sigma \Psi(t) + b\varsigma(t) = 0, \\ \Psi(0) = \Psi''(0) = 0, \\ \rho(I^\nu \Psi)(\varepsilon) = \Psi(1), 2 < \xi \leq 3, \;\; 2 \leq \sigma < \xi, \end{cases} \tag{3.1}$$

*satisfies the integral equation*

$$\begin{aligned} \Psi(t) = & \frac{-a}{\Gamma(\xi-\sigma)} \int_0^t (t-s)^{\xi-\sigma-1} \Psi(s)ds - \frac{b}{\Gamma(\xi)} \int_0^t (t-s)^{\xi-1} \varsigma(s)ds \\ & + Q\left(\frac{t\Gamma(\xi-\sigma+2) + at^{\xi-\sigma+1}}{\Gamma(\xi-\sigma+2)}\right)\left[\frac{\rho a}{\Gamma(\nu+\xi-\sigma)}\right. \\ & \times \int_0^\varepsilon (\varepsilon-s)^{\nu+\xi-\sigma-1} \Psi(s)ds + \frac{\rho b}{\Gamma(\nu+\xi)} \int_0^\varepsilon (\varepsilon-s)^{\nu+\xi-1} \varsigma(s)ds \\ & \left. - \frac{a}{\Gamma(\xi-\sigma)} \int_0^1 (1-s)^{\xi-\sigma-1} \Psi(s)ds - \frac{b}{\Gamma(\xi)} \int_0^1 (1-s)^{\xi-1} \varsigma(s)ds \right], \end{aligned} \tag{3.2}$$

*where*

$$Q = \frac{\gamma}{\rho a \varepsilon^{\nu+\xi-\sigma+1} + \rho\varepsilon^{\nu+1}\frac{\gamma}{\Gamma(\nu+2)} - a\frac{\gamma}{\Gamma(\xi-\sigma+2)} - \gamma},$$

*and $\gamma = \Gamma(\nu+\xi-\sigma+2)$.*

**Proof** Since $2 < \xi \leq 3$, by Lemma 2.5, we have

$$I^{\xi}\ {}^{c}D^{\xi}\Psi(t) = \Psi(t) + c_0 + c_1 t + c_2 t^2$$

for some constants $c_0, c_1, c_2$ and $t \in [0,1]$.

Now

$$\Psi(0) = 0 \Rightarrow c_0 = 0,$$

and

$$\Psi''(0) = 0 \Rightarrow c_2 = 0.$$

So, we have

$$I^{\xi}\ {}^{c}D^{\xi}\Psi(t) = \Psi(t) + c_1 t.$$

Implement $I^{\xi}$ on both sides of the given equation, we have

$$\begin{aligned} I^{\xi}\ {}^{c}D^{\xi}\Psi(t) &= -aI^{\xi}\ {}^{c}D^{\sigma}\Psi(t) - bI^{\xi}\varsigma(t) \\ \Rightarrow \Psi(t) + c_1 t &= -aI^{\xi}\ {}^{c}D^{\sigma}\Psi(t) - bI^{\xi}\varsigma(t). \end{aligned}$$

Now

$$\begin{aligned} I^{\xi}\ {}^{c}D^{\sigma}\Psi(t) &= I^{\xi-\sigma} I^{\sigma}\ {}^{c}D^{\sigma}\Psi(t) \\ &= I^{\xi-\sigma}(\Psi(t) + c_1 t) \\ &= I^{\xi-\sigma}\Psi(t) + \frac{c_1 t^{\xi-\sigma+1}}{\Gamma(\xi-\sigma+2)}. \end{aligned}$$

$$\Rightarrow \Psi(t) + c_1 t = -aI^{\xi-\sigma}\Psi(t) - \frac{ac_1 t^{\xi-\sigma+1}}{\Gamma(\xi-\sigma+2)} - bI^{\xi}\varsigma(t). \tag{3.3}$$

Further

$$\Psi(1) = -aI^{\xi-\sigma}\Psi(1) - c_1 - \frac{ac_1}{\Gamma(\xi-\sigma+2)} - bI^{\xi}\varsigma(1).$$

Applying $I^{\nu}$ to both sides of (3.3), we get

$$\begin{aligned} & I^{\nu}\Psi(t) + \frac{c_1 t^{1+\nu}}{\Gamma(\nu+2)} \\ &= -aI^{\nu+\xi-\sigma}\Psi(t) - \frac{ac_1 t^{\nu+\xi-\sigma+1}}{\Gamma(\nu+\xi-\sigma+2)} - bI^{\nu+\xi}\varsigma(t) \\ & \Rightarrow \rho I^{\nu}\Psi(\varepsilon) + \frac{\rho c_1 \varepsilon^{1+\nu}}{\Gamma(\nu+2)} \\ &= -\rho a I^{\nu+\xi-\sigma}\Psi(\varepsilon) - \frac{\rho a c_1 \varepsilon^{\nu+\xi-\sigma+1}}{\Gamma(\nu+\xi-\sigma+2)} - \rho b I^{\nu+\xi}\varsigma(\varepsilon). \end{aligned}$$

Using the boundary condition $\rho(I^{\nu}\Psi)(\varepsilon) = \Psi(1)$, we have

$$\begin{aligned} & -aI^{\xi-\sigma}\Psi(1) - c_1 - \frac{ac_1}{\Gamma(\xi-\sigma+2)} - bI^{\xi}\varsigma(1) + \frac{\rho c_1 \varepsilon^{\nu+1}}{\Gamma(\nu+2)} \\ &= -\rho a I^{\nu+\xi-\sigma}\Psi(\varepsilon) - \frac{\rho a c_1 \varepsilon^{\nu+\xi-\sigma+1}}{\Gamma(\nu+\xi-\sigma+2)} - \rho b I^{\nu+\xi}\varsigma(\varepsilon). \end{aligned}$$

Solving for $c_1$, we get

$$c_1 = -\frac{\rho a Q}{\Gamma(\nu+\xi-\sigma)}\int_0^{\varepsilon}(\varepsilon-s)^{\nu+\xi-\sigma-1}\Psi(s)ds$$
$$-\frac{\rho b Q}{\Gamma(\nu+\xi)}\int_0^{\varepsilon}(\varepsilon-s)^{\nu+\xi-1}\varsigma(s)ds$$
$$+\frac{bQ}{\Gamma(\xi)}\int_0^1(1-s)^{\xi-1}\varsigma(s)ds+\frac{aQ}{\Gamma(\xi-\sigma)}\int_0^1(1-s)^{\xi-\sigma-1}\Psi(s)ds.$$

Using $c_1$ in (3.3), one can attain the solution (3.2). □

By referring Lemma 3.1, the solution of (1.1) is presented as the following integral equations:

$$\Psi_1(t) = -\frac{\alpha_1}{\Gamma(\xi_1-\sigma_1)}\int_0^t(t-s)^{\xi_1-\sigma_1-1}\Psi_1(s)ds$$
$$-\frac{\beta_1}{\Gamma(\xi_1)}\int_0^t(t-s)^{\xi_1-1}\hbar_1(s,\Psi_1(s),\Psi_2(s))ds$$
$$+Q_1\left(\frac{t\Gamma(\xi_1-\sigma_1+2)+\alpha_1 t^{\xi_1-\sigma_1+1}}{\Gamma(\xi_1-\sigma_1+2)}\right)$$
$$\left[\frac{\rho_1\alpha_1}{\Gamma(\nu_1+\xi_1-\sigma_1)}\int_0^{\varepsilon_1}(\varepsilon_1-s)^{\nu_1+\xi_1-\sigma_1-1}\Psi_1(s)ds\right.$$
$$+\frac{\rho_1\beta_1}{\Gamma(\nu_1+\xi_1)}\int_0^{\varepsilon_1}(\varepsilon_1-s)^{\nu_1+\xi_1-1}\hbar_1(s,\Psi_1(s),\Psi_2(s))ds$$
$$-\frac{\alpha_1}{\Gamma(\xi_1-\sigma_1)}\int_0^1(1-s)^{\xi_1-\sigma_1-1}\Psi_1(s)ds$$
$$\left.-\frac{\beta_1}{\Gamma(\xi_1)}\int_0^1(1-s)^{\xi_1-1}\hbar_1(s,\Psi_1(s),\Psi_2(s))ds\right],$$

and

$$\Psi_2(t) = -\frac{\alpha_2}{\Gamma(\xi_2-\sigma_2)}\int_0^t(t-s)^{\xi_2-\sigma_2-1}\Psi_2(s)ds$$
$$-\frac{\beta_2}{\Gamma(\xi_2)}\int_0^t(t-s)^{\xi_2-1}\hbar_2(s,\Psi_1(s),\Psi_2(s))ds$$
$$+Q_2\left(\frac{t\Gamma(\xi_2-\sigma_2+2)+a_2 t^{\xi_2-\sigma_2+1}}{\Gamma(\xi_2-\sigma_2+2)}\right)$$
$$\left[\frac{\rho_2\alpha_2}{\Gamma(\nu_2+\xi_2-\sigma_2)}\int_0^{\varepsilon_2}(\varepsilon_2-s)^{\nu_2+\xi_2-\sigma_2-1}\Psi_2(s)ds\right.$$
$$+\frac{\rho_2\beta_2}{\Gamma(\nu_2+\xi_2)}\int_0^{\varepsilon_2}(\varepsilon_2-s)^{\nu_2+\xi_2-1}\hbar_2(s,\Psi_1(s),\Psi_2(s))ds$$
$$-\frac{\alpha_2}{\Gamma(\xi_2-\sigma_2)}\int_0^1(1-s)^{\xi_2-\sigma_2-1}\Psi_2(s)ds$$
$$\left.-\frac{\beta_2}{\Gamma(\xi_2)}\int_0^1(1-s)^{\xi_2-1}\hbar_2(s,\Psi_1(s),\Psi_2(s))ds\right],$$

where

$$Q_i = \frac{\gamma_i}{\rho_i \alpha_i \varepsilon_i^{\nu_i+\xi_i-\sigma_i+1} + \rho_i \varepsilon_i^{\nu_i+1} \frac{\gamma_i}{\Gamma(\nu_i+2)} - \alpha_i \frac{\gamma_i}{\Gamma(\xi_i-\sigma_i+2)} - \gamma_i}$$

and $\gamma_i = \Gamma(\nu_i + \xi_i - \sigma_i + 2)$ for $i = 1, 2$.

Let $A = V \times V$. Then, $(A, \|.\|_A)$ is also a Banach space along with

$$\|(\Psi_1, \Psi_2)\|_A = \|\Psi_1\|_V + \|\Psi_2\|_V.$$

Let us define an operator $\chi : A \to A$ as

$$\chi(\Psi_1, \Psi_2)(t) = (\chi_1(\Psi_1, \Psi_2)(t), \chi_2(\Psi_1, \Psi_2)(t)), \tag{3.4}$$

where

$$\begin{aligned}
\chi_1(\Psi_1, \Psi_2)(t) = &-\frac{\alpha_1}{\Gamma(\xi_1 - \sigma_1)} \int_0^t (t-s)^{\xi_1-\sigma_1-1} \Psi_1(s) ds \\
&- \frac{\beta_1}{\Gamma(\xi_1)} \int_0^t (t-s)^{\xi_1-1} \hbar_1(s, \Psi_1(s), \Psi_2(s)) ds \\
&+ Q_1 \left( \frac{t\Gamma(\xi_1 - \sigma_1 + 2) + \alpha_1 t^{\xi_1-\sigma_1+1}}{\Gamma(\xi_1 - \sigma_1 + 2)} \right) \\
&\left[ \frac{\rho_1 \alpha_1}{\Gamma(\nu_1 + \xi_1 - \sigma_1)} \int_0^{\varepsilon_1} (\varepsilon_1 - s)^{\nu_1+\xi_1-\sigma_1-1} \Psi_1(s) ds \right. \\
&+ \frac{\rho_1 \beta_1}{\Gamma(\nu_1 + \xi_1)} \int_0^{\varepsilon_1} (\varepsilon_1 - s)^{\nu_1+\xi_1-1} \hbar_1(s, \Psi_1(s), \Psi_2(s)) ds \\
&- \frac{\alpha_1}{\Gamma(\xi_1 - \sigma_1)} \int_0^1 (1-s)^{\xi_1-\sigma_1-1} \Psi_1(s) ds \\
&\left. - \frac{\beta_1}{\Gamma(\xi_1)} \int_0^1 (1-s)^{\xi_1-1} \hbar_1(s, \Psi_1(s), \Psi_2(s)) ds \right],
\end{aligned}$$

and

$$\chi_2(\Psi_1,\Psi_2)(t) = -\frac{\alpha_2}{\Gamma(\xi_2-\sigma_2)}\int_0^t (t-s)^{\xi_2-\sigma_2-1}\Psi_2(s)ds$$
$$-\frac{\beta_2}{\Gamma(\xi_2)}\int_0^t (t-s)^{\xi_2-1}\hbar_2(s,\Psi_1(s),\Psi_2(s))ds$$
$$+Q_2\left(\frac{t\Gamma(\xi_2-\sigma_2+2)+a_2t^{\xi_2-\sigma_2+1}}{\Gamma(\xi_2-\sigma_2+2)}\right)$$
$$\left[\frac{\rho_2\alpha_2}{\Gamma(\nu_2+\xi_2-\sigma_2)}\int_0^{\varepsilon_2}(\varepsilon_2-s)^{\nu_2+\xi_2-\sigma_2-1}\Psi_2(s)ds\right.$$
$$+\frac{\rho_2\beta_2}{\Gamma(\nu_2+\xi_2)}\int_0^{\varepsilon_2}(\varepsilon_2-s)^{\nu_2+\xi_2-1}\hbar_2(s,\Psi_1(s),\Psi_2(s))ds$$
$$-\frac{\alpha_2}{\Gamma(\xi_2-\sigma_2)}\int_0^1 (1-s)^{\xi_2-\sigma_2-1}\Psi_2(s)ds$$
$$\left.-\frac{\beta_2}{\Gamma(\xi_2)}\int_0^1 (1-s)^{\xi_2-1}\hbar_2(s,\Psi_1(s),\Psi_2(s))ds\right].$$

Observe that the fixed point of $\chi$ is the solution of (1.1).

For our comfort, we assume

$$P_i = \frac{|\alpha_i|}{\Gamma(\xi_i-\sigma_i+1)} + |Q_i|\left(\frac{\Gamma(\xi_i-\sigma_i+2)+|\alpha_i|}{\Gamma(\xi_i-\sigma_i+2)}\right)\left(\frac{|\rho_i\alpha_i|}{\Gamma(\nu_i+\xi_i-\sigma_i+1)}+\frac{|\alpha_i|}{\Gamma(\xi_i-\sigma_i+1)}\right), \tag{3.5}$$

and

$$J_i = \frac{|\beta_i|}{\Gamma(\xi_i+1)} + |Q_i|\left(\frac{\Gamma(\xi_i-\sigma_i+2)+|\alpha_i|}{\Gamma(\xi_i-\sigma_i+2)}\right)\left(\frac{|\rho_i\beta_i|}{\Gamma(\nu_i+\xi_i+1)}+\frac{|\beta_i|}{\Gamma(\xi_i+1)}\right), \tag{3.6}$$

for $i = 1, 2$.

## 4. Main results

**Theorem 4.1** *Assume that* $\exists\ p_i, j_i \geq 0\ (i = 1, 2)$ *and* $p_0, j_0 > 0$ *such that* $\forall\ \Psi_i \in \mathbb{R}$*, we have*

$$|\hbar_1(t,\Psi_1,\Psi_2)| \leq p_0 + p_1||\Psi_1|| + p_2||\Psi_2||,$$
$$|\hbar_2(t,\Psi_1,\Psi_2)| \leq j_0 + j_1||\Psi_1|| + j_2||\Psi_2||.$$

*with*

$$P_1 + P_2 + J_1p_1 + J_2j_1 < 1,$$
$$\textit{and } J_1p_2 + J_2j_2 < 1,$$

*therefore, the system* (1.1) *must have a solution.*

**Proof** Since $\hbar_1$ and $\hbar_2$ are continuous, therefore $\chi$ is also continuous. We claim $\chi$ is a completely continuous or compact operator.

Let $(\Psi_1, \Psi_2) \in \beta_\epsilon = \{(\Psi_1, \Psi_2) \in A; ||(\Psi_1, \Psi_2)||_A \leq \epsilon\}$ where $\epsilon > 0$

Then $||\Psi_i||_V \leq \mu_i$ for some $\mu_i,\ i = 1, 2$.

Let $t \in [0, 1]$, we have

$$\begin{aligned}
|\chi_1(\Psi_1, \Psi_2)(t)| \leq& \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1)} \int_0^t (t-s)^{\xi_1 - \sigma_1 - 1} |\Psi_1(s)| ds \\
&+ \frac{|\beta_1|}{\Gamma(\xi_1)} \int_0^t (t-s)^{\xi_1 - 1} |\hbar_1(s, \Psi_1(s), \Psi_2(s))| ds \\
&+ |Q_1| \left( \frac{\Gamma(\xi_1 - \sigma_1 + 2) + |\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 2)} \right) \left[ \frac{|\rho_1 \alpha_1|}{\Gamma(\nu_1 + \xi_1 - \sigma_1)} \right. \\
&\times \int_0^{\varepsilon_1} (\varepsilon_1 - s)^{\nu_1 + \xi_1 - \sigma_1 - 1} |\Psi_1(s)| ds \\
&+ \frac{|\rho_1 \beta_1|}{\Gamma(\nu_1 + \xi_1)} \int_0^{\varepsilon_1} (\varepsilon_1 - s)^{\nu_1 + \xi_1 - 1} |\hbar_1(s, \Psi_1(s), \Psi_2(s))| ds \\
&+ \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1)} \int_0^1 (1-s)^{\xi_1 - \sigma_1 - 1} |\Psi_1(s)| ds \\
&\left. + \frac{|\beta_1|}{\Gamma(\xi_1)} \int_0^1 (1-s)^{\xi_1 - 1} |\hbar_1(s, \Psi_1(s), \Psi_2(s))| ds \right] \\
&\leq \left[ \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 1)} + |Q_1| \left( \frac{\Gamma(\xi_1 - \sigma_1 + 2) + |\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 2)} \right) \right. \\
&\left. \left( \frac{|\rho_1 \alpha_1|}{\Gamma(\nu_1 + \xi_1 - \sigma_1 + 1)} + \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 1)} \right) \right] ||\Psi_1||_V \\
&+ \left[ \frac{|\beta_1|}{\Gamma(\xi_1 + 1)} + |Q_1| \left( \frac{\Gamma(\xi_1 - \sigma_1 + 2) + |\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 2)} \right) \right. \\
&\left. \left( \frac{|\rho_1 \beta_1|}{\Gamma(\nu_1 + \xi_1 + 1)} + \frac{|\beta_1|}{\Gamma(\xi_1 + 1)} \right) \right] [p_0 + p_1 ||\Psi_1||_V + p_2 ||\Psi_2||_V] \\
&\leq J_1 p_0 + (P_1 + J_1 p_1) \mu_1 + J_1 p_2 \mu_2.
\end{aligned}$$

Thus, we have

$$||\chi_1(\Psi_1, \Psi_2)||_V \leq J_1 p_0 + (P_1 + J_1 p_1) \mu_1 + J_1 p_2 \mu_2.$$

Similarly, we get

$$||\chi_2(\Psi_1, \Psi_2)||_V \leq J_2 j_0 + (P_2 + J_2 j_1) \mu_1 + J_2 j_2 \mu_2.$$

From the above inequalities, it follows that bounded sets of $A$ are mapped into bounded sets of $A$ under the mapping $\chi$.

Next, we show that $\chi(\beta_\epsilon)$ is equicontinuous. Let $0 \leq t_1 < t_2 \leq 1$ and $(\Psi_1, \Psi_2) \in \beta_\epsilon$. Then we have

$$
\begin{aligned}
&|\chi_1(\Psi_1,\Psi_2)(t_2) - \chi_1(\Psi_1,\Psi_2)(t_1)| \\
&\leq \frac{|\alpha_1|}{\Gamma(\xi_1-\sigma_1)} \bigg| \int_0^{t_1} [(t_2-s)^{\xi_1-\sigma_1-1} - (t_1-s)^{\xi_1-\sigma_1-1}] \Psi_1(s) ds \\
&+ \int_{t_1}^{t_2} (t_2-s)^{\xi_1-\sigma_1-1} \Psi_1(s) ds \bigg| \\
&+ \frac{|\beta_1|}{\Gamma(\xi_1)} \bigg| \int_0^{t_1} [(t_2-s)^{\xi_1-1} - (t_1-s)^{\xi_1-1}] \hbar_1(s,\Psi_1(s),\Psi_2(s)) ds \\
&+ \int_{t_1}^{t_2} (t_2-s)^{\xi_1-1} \hbar_1(s,\Psi_1(s),\Psi_2(s)) ds \bigg| \\
&+ |Q_1| \bigg( \frac{|t_2-t_1|\Gamma(\xi_1-\sigma_1+2) + |\alpha_1||t_2^{\xi_1-\sigma_1+1} - t_1^{\xi_1-\sigma_1+1}|}{\Gamma(\xi_1-\sigma_1+2)} \bigg) \\
&\bigg[ \frac{|\rho_1\alpha_1|}{\Gamma(\nu_1+\xi_1-\sigma_1)} \int_0^{\varepsilon_1} (\varepsilon_1-s)^{\nu_1+\xi_1-\sigma_1-1} |\Psi_1(s)| ds \\
&+ \frac{|\rho_1\beta_1|}{\Gamma(\nu_1+\xi_1)} \int_0^{\varepsilon_1} (\varepsilon_1-s)^{\nu_1+\xi_1-1} |\hbar_1(s,\Psi_1(s),\Psi_2(s))| ds \\
&+ \frac{|\alpha_1|}{\Gamma(\xi_1-\sigma_1)} \int_0^1 (1-s)^{\xi_1-\sigma_1-1} |\Psi_1(s)| ds \\
&+ \frac{|\beta_1|}{\Gamma(\xi_1)} \int_0^1 (1-s)^{\xi_1-1} |\hbar_1(s,\Psi_1(s),\Psi_2(s))| ds \bigg] \\
&\leq \frac{|\alpha_1|}{\Gamma(\xi_1-\sigma_1+1)} |t_2^{\xi_1-\sigma_1} - t_1^{\xi_1-\sigma_1}| \mu_1 \\
&+ \frac{|\beta_1|}{\Gamma(\xi_1+1)} |t_2^{\xi_1} - t_1^{\xi_1}| [p_0 + p_1\mu_1 + p_2\mu_2] \\
&+ |Q_1| \bigg( \frac{|t_2-t_1|\Gamma(\xi_1-\sigma_1+2) + |\alpha_1||t_2^{\xi_1-\sigma_1+1} - t_1^{\xi_1-\sigma_1+1}|}{\Gamma(\xi_1-\sigma_1+2)} \bigg) \\
&\bigg[ \frac{|\rho_1\alpha_1|}{\Gamma(\nu_1+\xi_1-\sigma_1+1)} \mu_1 + \frac{|\rho_1\beta_1|}{\Gamma(\nu_1+\xi_1+1)} [p_0 + p_1\mu_1 + p_2\mu_2] \\
&+ \frac{|\alpha_1|}{\Gamma(\xi_1-\sigma_1+1)} \mu_1 + \frac{|\beta_1|}{\Gamma(\xi_1+1)} [p_0 + p_1\mu_1 + p_2\mu_2] \bigg].
\end{aligned}
$$

Now the right-hand side approaches to zero when $t_1$ approaches to $t_2$. Thus

$$\|\chi_1(\Psi_1,\Psi_2)(t_2) - \chi_1(\Psi_1,\Psi_2)(t_1)\|_V \to 0 \text{ as } t_1 \to t_2.$$

Similarly

$$\|\chi_2(\Psi_1,\Psi_2)(t_2) - \chi_2(\Psi_1,\Psi_2)(t_1)\|_V \to 0 \text{ as } t_1 \to t_2.$$

Thus

$$\|\chi(\Psi_1,\Psi_2)(t_2) - \chi(\Psi_1,\Psi_2)(t_1)\|_A \to 0, \text{ as } t_1 \to t_2.$$

Combining steps I to III and using the consequence of Arzelá–Ascoli theorem, $\chi : A \to A$ is a completely continuous operator.

Let $\Theta = \{(\Psi_1, \Psi_2) \in A : (\Psi_1, \Psi_2) = \theta\chi(\Psi_1, \Psi_2) \text{ for some } \theta \in (0, 1)\}$.

We will show that the set $\Theta$ is bounded.

Let $(\Psi_1, \Psi_2) \in \Theta \Rightarrow (\Psi_1, \Psi_2)(t) = \theta\chi(\Psi_1, \Psi_2)(t)$ for some $\theta \in (0, 1)$.

Then we have

$$\Psi_1(t) = \theta\chi_1(\Psi_1, \Psi_2)(t),\ \Psi_2(t) = \theta\chi_2(\Psi_1, \Psi_2)(t),\ \forall\ t \in [0, 1].$$

$$\begin{aligned}
&|\Psi_1(t)| = |\theta\chi_1(\Psi_1, \Psi_2)(t)| \\
&\leq \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1)} \int_0^t (t-s)^{\xi_1-\sigma_1-1}|\Psi_1(s)|ds \\
&+ \frac{|\beta_1|}{\Gamma(\xi_1)} \int_0^t (t-s)^{\xi_1-1}|\hbar_1(s, \Psi_1(s), \Psi_2(s))|ds \\
&+ |Q_1|\left(\frac{\Gamma(\xi_1 - \sigma_1 + 2) + |\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 2)}\right)\left[\frac{|\rho_1\alpha_1|}{\Gamma(\nu_1 + \xi_1 - \sigma_1)}\right. \\
&\times \int_0^{\varepsilon_1} (\varepsilon_1 - s)^{\nu_1+\xi_1-\sigma_1-1}|\Psi_1(s)|ds \\
&+ \frac{|\rho_1\beta_1|}{\Gamma(\nu_1 + \xi_1)} \int_0^{\varepsilon_1} (\varepsilon_1 - s)^{\nu_1+\xi_1-1}|\hbar_1(s, \Psi_1(s), \Psi_2(s))|ds \\
&+ \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1)} \int_0^1 (1-s)^{\xi_1-\sigma_1-1}|\Psi_1(s)|ds \\
&\left.+ \frac{|\beta_1|}{\Gamma(\xi_1)} \int_0^1 (1-s)^{\xi_1-1}|\hbar_1(s, \Psi_1(s), \Psi_2(s))|ds\right] \\
&\leq \left[\frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 1)} + |Q_1|\left(\frac{\Gamma(\xi_1 - \sigma_1 + 2) + |\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 2)}\right)\right. \\
&\left.\left(\frac{|\rho_1\alpha_1|}{\Gamma(\nu_1 + \xi_1 - \sigma_1 + 1)} + \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 1)}\right)\right]||\Psi_1||_V \\
&+ \left[\frac{|\beta_1|}{\Gamma(\xi_1 + 1)} + |Q_1|\left(\frac{\Gamma(\xi_1 - \sigma_1 + 2) + |\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 2)}\right)\right. \\
&\left.\left(\frac{|\rho_1\beta_1|}{\Gamma(\nu_1 + \xi_1 + 1)} + \frac{|\beta_1|}{\Gamma(\xi_1 + 1)}\right)\right][p_0 + p_1||\Psi_1||_V + p_2||\Psi_2||_V].
\end{aligned}$$

Thus, we have

$$||\Psi_1||_V \leq J_1p_0 + (P_1 + J_1p_1)||\Psi_1||_V + J_1p_2||\Psi_2||_V.$$

Similarly, we get

$$||\Psi_2||_V \leq J_2j_0 + (P_2 + J_2j_1)||\Psi_1||_V + J_2j_2||\Psi_2||_V.$$

i.e.

$$\begin{aligned}
\|(\Psi_1, \Psi_2)\|_A \leq\ & J_1p_0 + J_2j_0 + (P_1 + J_1p_1 + P_2 + J_2j_1)||\Psi_1||_V \\
&+ (J_1p_2 + J_2j_2)||\Psi_2||_V.
\end{aligned}$$

which implies that

$$\|(\Psi_1, \Psi_2)\|_A \leq \frac{J_1 p_0 + J_2 j_0}{N},$$

where

$$N = \min\{1 - P_1 - J_1 p_1 - P_2 - J_2 j_1, 1 - J_1 p_2 - J_2 j_2\},$$

$\Rightarrow \Theta$ is a bounded set.

In view of Schaefer's fixed point theorem, $\chi$ should have at least one fixed point and that is a solution of (1.1). □

**Theorem 4.2** *Assume that* $\exists$ *constants* $\Upsilon_i > 0, \ i = 1, 2$ *such that* $\forall t \in [0, 1]$ *and* $\forall \ \Psi_i, \Psi_i' \in \mathbb{R}, \ i = 1, 2,$

$$|\hbar_i(t, \Psi_1, \Psi_2) - \hbar_i(t, \Psi_1', \Psi_2')| \leq \Upsilon_i(|\Psi_1 - \Psi_1'| + |\Psi_2 - \Psi_2'|),$$

*and*

$$P_1 + P_2 + \Upsilon_1 J_1 + \Upsilon_2 J_2 < 1,$$

*therefore, the system* (1.1) *has a unique solution defined on* $[0, 1]$*.*

**Proof** Let $(\Psi_1, \Psi_2), (\Psi_1', \Psi_2') \in A$ and $t \in [0, 1]$, then we have

$$\begin{aligned}
&|\chi_1(\Psi_1, \Psi_2)(t) - \chi_1(\Psi_1', \Psi_2')(t)| \\
&\leq \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1)} \int_0^t (t - s)^{\xi_1 - \sigma_1 - 1} |\Psi_1(s) - \Psi_1'(s)| ds \\
&+ \frac{|\beta_1|}{\Gamma(\xi_1)} \int_0^t (t - s)^{\xi_1 - 1} |\hbar_1(s, \Psi_1(s), \Psi_2(s)) - \hbar_1(s, \Psi_1'(s), \Psi_2'(s))| ds \\
&+ |Q_1| \left( \frac{\Gamma(\xi_1 - \sigma_1 + 2) + |\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 2)} \right) \left[ \frac{|\rho_1 \alpha_1|}{\Gamma(\nu_1 + \xi_1 - \sigma_1)} \right. \\
&\times \int_0^{\varepsilon_1} (\varepsilon_1 - s)^{\nu_1 + \xi_1 - \sigma_1 - 1} |\Psi_1(s) - \Psi_1'(s)| ds \\
&+ \frac{|\rho_1 \beta_1|}{\Gamma(\nu_1 + \xi_1)} \int_0^{\varepsilon_1} (\varepsilon_1 - s)^{\nu_1 + \xi_1 - 1} |\hbar_1(s, \Psi_1(s), \Psi_2(s)) - \hbar_1(s, \Psi_1'(s), \Psi_2'(s))| ds \\
&+ \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1)} \int_0^1 (1 - s)^{\xi_1 - \sigma_1 - 1} |\Psi_1(s) - \Psi_1'(s)| ds \\
&\left. + \frac{|\beta_1|}{\Gamma(\xi_1)} \int_0^1 (1 - s)^{\xi_1 - 1} |\hbar_1(s, \Psi_1(s), \Psi_2(s)) - \hbar_1(s, \Psi_1'(s), \Psi_2'(s))| ds \right] \\
&\leq [P_1 + \Upsilon_1 J_1][||\Psi_1 - \Psi_1'||_V + ||\Psi_2 - \Psi_2'||_V],
\end{aligned}$$

and consequently, we obtain

$$\begin{aligned}
&||\chi_1(\Psi_1, \Psi_2) - \chi_1(\Psi_1', \Psi_2')||_V \\
&\leq [P_1 + \Upsilon_1 J_1](||\Psi_1 - \Psi_1'||_V + ||\Psi_2 - \Psi_2'||_V).
\end{aligned} \tag{4.1}$$

Similarly

$$\begin{aligned}&||\chi_2(\Psi_1,\Psi_2)-\chi_2(\Psi_1',\Psi_2')||_V\\&\leq [P_2+\Upsilon_2 J_2](||\Psi_1-\Psi_1'||_V+||\Psi_2-\Psi_2'||_V).\end{aligned} \tag{4.2}$$

It follows from (4.1) and (4.2) that

$$\begin{aligned}&||\chi(\Psi_1,\Psi_2)-\chi(\Psi_1',\Psi_2')||_A\\&\leq (P_1+P_2+\Upsilon_1 J_1+\Upsilon_2 J_2)(||\Psi_1-\Psi_1'||_V+||\Psi_2-\Psi_2'||_V).\end{aligned}$$

Since $P_1+P_2+\Upsilon_1 J_1+\Upsilon_2 J_2<1$, then $\chi$ is a contraction. Therefore, by referring Banach fixed point theorem, $\chi$ must have a unique fixed point i.e. (1.1) has a unique solution. □

## 5. Examples

### Example 1

$$\begin{cases} {}^cD^{\frac{13}{5}}\Psi_1(t)+\frac{{}^cD^{\frac{11}{5}}\Psi_1(t)}{1993}+5\left(\frac{(t+8)^2|\Psi_1(t)+\Psi_2(t)|}{(1+|\Psi_1(t)+\Psi_2(t)|)}+\sin(\frac{\Psi_1(t)}{65}+\frac{\Psi_2(t)}{78})\right)=0,\\ {}^cD^{\frac{17}{7}}\Psi_2(t)+\frac{{}^cD^{\frac{15}{7}}\Psi_2(t)}{1989}+6\left((t+6)^2+\ln(1+|\frac{\Psi_1(t)}{56}+\frac{\Psi_2(t)}{87}|)\right)=0,\\ \Psi_1(0)=\Psi_1''(0)=\Psi_2(0)=\Psi_2''(0)=0,\\ \frac{1}{11}(I^{\frac{8}{5}}\Psi_1)(\frac{1}{4})=\Psi_1(1),\ \ \frac{1}{13}(I^{\frac{19}{7}}\Psi_2)(\frac{1}{5})=\Psi_2(1). \end{cases} \tag{5.1}$$

*Here* $\xi_1=\frac{13}{5},\alpha_1=\frac{1}{1993},\ \sigma_1=\frac{11}{5},\ \beta_1=5,\ \xi_2=\frac{17}{7},\alpha_2=\frac{1}{1989},\ \sigma_2=\frac{15}{7},\ \beta_2=6,\ \rho_1=\frac{1}{11},\ \nu_1=\frac{8}{5},\ \varepsilon_1=\frac{1}{4},\ \rho_2=\frac{1}{13},\ \nu_2=\frac{19}{7},\ \varepsilon_2=\frac{1}{5},\ \hbar_1(t,\Psi_1,\Psi_2)=\frac{(t+8)^2|\Psi_1+\Psi_2|}{(1+|\Psi_1+\Psi_2|)}+\sin(\frac{\Psi_1}{65}+\frac{\Psi_2}{78})$ *and* $\hbar_2(t,\Psi_1,\Psi_2)=(t+6)^2+\ln(1+|\frac{\Psi_1}{56}+\frac{\Psi_2}{87}|)$.

*For all* $\Psi_1,\Psi_2\in\mathbb{R}$ *and* $t\in[0,1]$, *we have*

$$\begin{aligned}|\hbar_1(t,\Psi_1,\Psi_2)|&\leq 81+\left|\frac{\Psi_1}{65}+\frac{\Psi_2}{78}\right|\\&\leq 81+\frac{1}{65}|\Psi_1|+\frac{1}{78}|\Psi_2|,\end{aligned}$$

*and*

$$\begin{aligned}|\hbar_2(t,\Psi_1,\Psi_2)|&\leq 49+\left|\frac{\Psi_1}{56}+\frac{\Psi_2}{87}\right|\\&\leq 49+\frac{1}{56}|\Psi_1|+\frac{1}{87}|\Psi_2|.\end{aligned}$$

*Here* $p_0=81,p_1=\frac{1}{65},p_2=\frac{1}{78},j_0=49,j_1=\frac{1}{56}$ *and* $j_2=\frac{1}{87}$.

*Also*

$$Q_1=-1.00026,\ \ Q_2=-0.999578.$$

*Further*

$$
\begin{aligned}
&P_1 + P_2 + p_1 J_1 + j_1 J_2 \\
&= \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 1)} + |Q_1|\left(\frac{\Gamma(\xi_1 - \sigma_1 + 2) + |\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 2)}\right) \\
&\left(\frac{|\rho_1 \alpha_1|}{\Gamma(\nu_1 + \xi_1 - \sigma_1 + 1)} + \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 1)}\right) \\
&+ \frac{|\alpha_2|}{\Gamma(\xi_2 - \sigma_2 + 1)} + |Q_2|\left(\frac{\Gamma(\xi_2 - \sigma_2 + 2) + |\alpha_2|}{\Gamma(\xi_2 - \sigma_2 + 2)}\right) \\
&\left(\frac{|\rho_2 \alpha_2|}{\Gamma(\nu_2 + \xi_2 - \sigma_2 + 1)} + \frac{|\alpha_2|}{\Gamma(\xi_2 - \sigma_2 + 1)}\right) \\
&+ \frac{p_1|\beta_1|}{\Gamma(\xi_1 + 1)} + p_1|Q_1|\left(\frac{\Gamma(\xi_1 - \sigma_1 + 2) + |\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 2)}\right) \\
&\left(\frac{|\rho_1 \beta_1|}{\Gamma(\nu_1 + \xi_1 + 1)} + \frac{|\beta_1|}{\Gamma(\xi_1 + 1)}\right) \\
&+ \frac{j_1|\beta_2|}{\Gamma(\xi_2 + 1)} + j_1|Q_2|\left(\frac{\Gamma(\xi_2 - \sigma_2 + 2) + |\alpha_2|}{\Gamma(\xi_2 - \sigma_2 + 2)}\right) \\
&\left(\frac{|\rho_2 \beta_2|}{\Gamma(\nu_2 + \xi_2 + 1)} + \frac{|\beta_2|}{\Gamma(\xi_2 + 1)}\right) \\
&= 0.113658 < 1,
\end{aligned}
$$

*and*

$$
\begin{aligned}
&p_2 J_1 + j_2 J_2 \\
&= \frac{p_2|\beta_1|}{\Gamma(\xi_1 + 1)} + p_2|Q_1|\left(\frac{\Gamma(\xi_1 - \sigma_1 + 2) + |\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 2)}\right) \\
&\left(\frac{|\rho_1 \beta_1|}{\Gamma(\nu_1 + \xi_1 + 1)} + \frac{|\beta_1|}{\Gamma(\xi_1 + 1)}\right) \\
&+ \frac{j_2|\beta_2|}{\Gamma(\xi_2 + 1)} + j_2|Q_2|\left(\frac{\Gamma(\xi_2 - \sigma_2 + 2) + |\alpha_2|}{\Gamma(\xi_2 - \sigma_2 + 2)}\right) \\
&\left(\frac{|\rho_2 \beta_2|}{\Gamma(\nu_2 + \xi_2 + 1)} + \frac{|\beta_2|}{\Gamma(\xi_2 + 1)}\right) \\
&= 0.079586 < 1.
\end{aligned}
$$

*Therefore, in view of Theorem 4.1, there exists at least one solution for* (5.1)*.*

**Example 2**

$$
\begin{cases}
{}^cD^{\frac{5}{2}}\Psi_1(t) + \frac{{}^cD^{\frac{7}{3}}\Psi_1(t)}{499} + 3\left(\frac{|\Psi_1(t)|}{(t+9)^2(1+|\Psi_1(t)|)} + \frac{|\Psi_2(t)|}{(t+79)(1+|\Psi_2(t)|)}\right) = 0, \\
{}^cD^{\frac{9}{4}}\Psi_2(t) + \frac{{}^cD^{\frac{11}{5}}\Psi_2(t)}{1947} + 4\left(\frac{\tan^{-1}\Psi_1(t)}{89+t} + \frac{\sin(2\pi\Psi_2(t))}{192\pi}\right) = 0, \\
\Psi_1(0) = \Psi_1''(0) = \Psi_2(0) = \Psi_2''(0) = 0, \\
\frac{1}{7}(I^{\frac{11}{6}}\Psi_1)(\frac{1}{3}) = \Psi_1(1), \ \frac{1}{9}(I^{\frac{19}{20}}\Psi_2)(\frac{1}{2}) = \Psi_2(1).
\end{cases}
\tag{5.2}
$$

*In this case,* $\xi_1 = \frac{5}{2}, \alpha_1 = \frac{1}{499},\ \sigma_1 = \frac{7}{3},\ \beta_1 = 3,\ \xi_2 = \frac{9}{4}, \alpha_2 = \frac{1}{1947},\ \sigma_2 = \frac{11}{5},\ \beta_2 = 4,\ \rho_1 = \frac{1}{7},\ \nu_1 = \frac{11}{6},\ \varepsilon_1 = \frac{1}{3},\ \rho_2 = \frac{1}{9},\ \nu_2 = \frac{19}{20},\ \varepsilon_2 = \frac{1}{2},\ \hbar_1(t,\Psi_1,\Psi_2) = \frac{|\Psi_1|}{(t+9)^2(1+|\Psi_1|)} + \frac{|\Psi_2|}{(t+79)(1+|\Psi_2|)}$ *and* $\hbar_2(t,\Psi_1,\Psi_2) = \frac{\tan^{-1}\Psi_1}{89+t} + \frac{\sin(2\pi\Psi_2)}{192\pi}$.

*For all* $\Psi_1, \Psi_2, \Psi_1', \Psi_2' \in \mathbb{R}$ *and* $t \in [0,1]$, *we have*

$$\begin{aligned}|\hbar_1(t,\Psi_1,\Psi_2) - \hbar_1(t,\Psi_1',\Psi_2')| &\leq \frac{1}{81}|\Psi_1 - \Psi_1'| + \frac{1}{79}|\Psi_2 - \Psi_2'| \\ &\leq \frac{1}{79}(|\Psi_1 - \Psi_1'| + |\Psi_2 - \Psi_2'|),\end{aligned}$$

*and*

$$\begin{aligned}|\hbar_2(t,\Psi_1,\Psi_2) - \hbar_2(t,\Psi_1',\Psi_2')| &\leq \frac{1}{89}|\Psi_1 - \Psi_1'| + \frac{1}{96}|\Psi_2 - \Psi_2'| \\ &\leq \frac{1}{89}(|\Psi_1 - \Psi_1'| + |\Psi_2 - \Psi_2'|).\end{aligned}$$

*Here* $\Upsilon_1 = \frac{1}{79}$ *and* $\Upsilon_2 = \frac{1}{89}$.

*Also*

$$Q_1 = -0.9995,\ Q_2 = -1.0148.$$

*Further*

$$\begin{aligned}&P_1 + P_2 + \Upsilon_1 J_1 + \Upsilon_2 J_2 \\ &= \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 1)} + |Q_1|\left(\frac{\Gamma(\xi_1 - \sigma_1 + 2) + |\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 2)}\right) \\ &\left(\frac{|\rho_1\alpha_1|}{\Gamma(\nu_1 + \xi_1 - \sigma_1 + 1)} + \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 1)}\right) \\ &+ \frac{|\alpha_2|}{\Gamma(\xi_2 - \sigma_2 + 1)} + |Q_2|\left(\frac{\Gamma(\xi_2 - \sigma_2 + 2) + |\alpha_2|}{\Gamma(\xi_2 - \sigma_2 + 2)}\right) \\ &\left(\frac{|\rho_2\alpha_2|}{\Gamma(\nu_2 + \xi_2 - \sigma_2 + 1)} + \frac{|\alpha_2|}{\Gamma(\xi_2 - \sigma_2 + 1)}\right) \\ &+ \frac{\Upsilon_1|\beta_1|}{\Gamma(\xi_1 + 1)} + \Upsilon_1|Q_1|\left(\frac{\Gamma(\xi_1 - \sigma_1 + 2) + |\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 2)}\right) \\ &\left(\frac{|\rho_1\beta_1|}{\Gamma(\nu_1 + \xi_1 + 1)} + \frac{|\beta_1|}{\Gamma(\xi_1 + 1)}\right) \\ &+ \frac{\Upsilon_2|\beta_2|}{\Gamma(\xi_2 + 1)} + \Upsilon_2|Q_2|\left(\frac{\Gamma(\xi_2 - \sigma_2 + 2) + |\alpha_2|}{\Gamma(\xi_2 - \sigma_2 + 2)}\right) \\ &\left(\frac{|\rho_2\beta_2|}{\Gamma(\nu_2 + \xi_2 + 1)} + \frac{|\beta_2|}{\Gamma(\xi_2 + 1)}\right) \\ &= 0.0647757 < 1.\end{aligned}$$

*Thus, in view of Theorem 4.2, we deduce that (5.2) has a unique solution.*

## Conclusion

Our paper mainly concentrated on the study of a new BVP of coupled nonlinear two-term fractional differential systems. We employed the theories on fractional calculus and fixed point approach to derive the existence and uniqueness results of the considered two-term fractional systems. To confirm the application of the stated outcomes, two examples are provided.

In future work, we focus on the existence results for the BVP of coupled nonlinear two-term fractional impulsive differential systems.

## Conflict of interest

The authors have no conflict of interest.

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