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Mathematical analysis of local and global dynamics of a new epidemic model

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Abstract: In this paper, we construct a new $SEIR$ epidemic model reflecting the spread of infectious diseases. After calculating basic reproduction number $\mathcal{R}_0$ by the next generation matrix method, we examine the stability of the model. The model exhibits threshold behavior according to whether the basic reproduction number $\mathcal{R}_0$ is greater than unity or not. By using well-known Routh-Hurwitz criteria, we deal with local asymptotic stability of equilibrium points of the model according to $\mathcal{R}_0$. Also, we present a mathematical analysis for the global dynamics in the equilibrium points of this model using LaSalle's Invariance Principle associated with Lyapunov functional technique and Li-Muldowney geometric approach, respectively.

Key words: Lyapunov function, LaSalle's invariance principle, the second additive compound matrix, Li-Muldowney geometric approach, next generation matrix method, basic reproduction number, Jacobian matrix, Routh-Hurwitz criteria

1. Introduction

Differential, difference, integral or integro-differential equations are widely used to explain relevant phenomena in practical applications of areas such as physics, chemistry, biology, ecology, epidemiology, engineering and so on [5–7, 16]. Especially applications in mathematical biology have recently received considerable attention.

Population dynamics, one of the fundamental issues of mathematical biology, is interested in changes in population density caused by factors such as reproduction, mortality, and migration. The models reflected population dynamics are stated with time derivatives of components consisting the system to talk about the dynamic processes related to changes. By using system of differential equations, many authors describe the models on population dynamics and analyze the stability of its, [3, 27].

Population dynamics makes mathematically researches possible in interesting phenomena such as epidemic diseases, too. Models reflected the spread of infectious diseases and its analysis are attracted particular attention and so this branch is closely connected the theory of infectious diseases.

As is known, all creatures especially humans have been enormously influenced by infectious diseases during their lives. Millions of people have died of various infectious diseases so far in history. Mankind has striven to control the spread of infectious diseases, but this has not always been easy when considered in its entirety. In this context, mathematical modeling, which is one of the main tools in epidemiology, has an important role in understanding of the dynamics of spread of infectious diseases. In 1927, Kermack and McKendrick [19], by using a system of ODE constructed a mathematical model to describe the spread of infectious diseases in a

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population. In this model, known as the SIR epidemic model in the literature, they divided the population into nonintersecting three compartments; susceptibles (S), infectious (I) and removed individuals (R). Then many authors studied on SIR model (see [1, 25] and references therein). On the other hand, different compartmental epidemic models such as $SIRS, SEIR$ and $SVEIR$ have been introduced in lots of forms by many authors and various results about these epidemic models have been obtained [9, 10, 17, 18, 26, 28].

As an example of classical $SEIR$ models, in [26], the authors have considered the following model

$$\begin{aligned}
\frac{dS}{dt} &= \pi - \beta SI - \mu S, \\
\frac{dE}{dt} &= \beta SI - (\mu + \sigma + \tau) E, \\
\frac{dI}{dt} &= \sigma E - (\gamma + \mu + \delta) I, \\
\frac{dR}{dt} &= \tau E + \gamma I - \mu R.
\end{aligned}$$

In general, the authors focus on stability analysis of their proposing models according to reproduction number $\mathcal{R}_0$ defined as the number of secondary cases generated by an infectious. If $\mathcal{R}_0 < 1$ then invasion of individuals by the pathogen does not give rise to a widespread epidemic and so the disease gradually becomes extinct. Otherwise, that is for $\mathcal{R}_0 > 1$, the disease continues to spread in the population.

The Routh-Hurwitz criteria and LaSalle's invariance principle associated with Lyapunov functionals are among the common tools used in stability analysis of epidemic models.

In this paper we construct a new $SEIR$ model including distributed latent period. We assume that the population consists of nonintersecting four compartments; susceptible (S), exposed (E), infectious (I) and removed (R). In this model, the pathogen can be transmitted from the infectious to the susceptible when a susceptible comes into an effective contact with an infectious individual. In this case the susceptible individual becomes a candidate to be infectious with a certain probability changing according to some rates. But the susceptible individual may not be infectious immediately. The period of after effective contact before becoming infectious is defined as latent period. In other words, the latent period is the time between invasion of the body by a pathogenic organism and the time at which an individual is capable of transmitting the disease to susceptibles. Indeed, latent period varies from a few days to several months, depending on the causative organism and type of disease. But we should immediately note that this period is finite.

The most important difference between our model and other $SEIR$ models is that an individual leaving from S belongs to E through the latent period and becomes infectious himself with a certain rate (σ) after a latent period (τ). We use a distribute function to take into account the latent period changing according to individuals in order to add a more realistic structure to classical $SEIR$ epidemic models. The main difference in our model is especially due to this transition. This model approach can be considered for all infectious diseases such as measles, pox, dengue, seasonal or annual influenza, SARS, COVID-19, etc. which can be transmitted from person to person and have a latent period. After describing of the model, we present a mathematical analysis for local and global dynamics of it by considering appropriate methods and designing suitable techniques. Analyzing of the corresponding characteristic equation to Jacobian matrix at the equilibrium points, Routh-Hurwitz criteria, Lyapunov functional technique associated with LaSalle's invariance principle and Li-Muldowney geometric approach are the main techniques used in this study.

2. The model

In this paper, we constitute an $SEIR$ model including distributed delay given by the system of the following nonlinear ordinary integro-differential equations with the initial condition $E(t) = \widetilde{E}(t)$ for $t \in [-\tau, 0]$:

$$\begin{aligned}
\frac{dS}{dt} &= b - \beta S(t) I(t) - \mu S(t), \\
\frac{dE}{dt} &= \beta S(t) I(t) - \sigma \int_0^h f(\tau) E(t-\tau) d\tau - (\mu + \alpha) E(t), \\
\frac{dI}{dt} &= \sigma \int_0^h f(\tau) E(t-\tau) d\tau - (\gamma + \mu + \delta) I(t), \\
\frac{dR}{dt} &= \alpha E(t) + \gamma I(t) - \mu R(t).
\end{aligned} \tag{2.1}$$

Where $S(t)$, $E(t)$, $I(t)$ and $R(t)$ denote the numbers of the susceptibles, of exposed to the pathogen, of infectious individuals and of removed members at time t, respectively. Also all parameters and functions S, E, I, R are nonnegative. $N(t)$ shows the total number of the population at time t such that $S(t) + E(t) + I(t) + R(t) = N(t)$, $t \geq 0$. As a matter of course, the functions S, E, I, R and N are nonnegative.

In the model, we assume that all new members of the population get involved in S at the constant rate b. The parameter μ represents the natural death rate of all compartments. Also β is the effective contact rate between susceptibles and infectious. α represents the rate of the members "exposed but is not be infectious" and so the number of individuals transferred to R from E is $\alpha E(t)$ at every time t. On the other hand γ shows the recovery rate of infectious and δ is the death rate due to the infection in compartment I.

Also the function f is a distribute function showing density of the exposed individuals whose latent period is τ. It is assumed that $f : [0, h] \to [0, 1]$ is continuous and satisfies $\int_0^h f(\tau) d\tau = 1$ such that h is the superior limit of latent periods in the class E. Additionally, we assume that every members of exposed does not become infectious. On behalf of reflecting this fact we write σ which denotes the progression rate to I from E. The term $f(\tau) E(t-\tau)$ represents the number of exposed individuals entered in latent process of the latent period with exposure age τ (i.e. time elapsed since exposure to the pathogen). So these individuals complete their latent periods at time t. Therefore, taking all these assumptions into account, the total number of individuals transferred from E to I at each time t is

$$\sigma \int_0^h f(\tau) E(t-\tau) d\tau.$$

3. Some basic results related to the model

Now we consider the equilibrium points, basic reproduction number and suitable region for the model (2.1).

3.1. Equilibrias

Firstly we note that, since the function R is not involved by other equations of (2.1), it is sufficient to consider the system (2.1) without dR/dt.

The system has two equilibrium points. It can be easily seen that DFEP (the disease-free equilibrium point) of the system is

$$\epsilon_0 = (S_0, E_0, I_0) = \left(\frac{b}{\mu}, 0, 0\right).$$

On the other hand, to determine EEP (the endemic equilibrium point) of the system (2.1), we have to solve the following system of algebraic equations with $I^* \neq 0$:

$$\begin{aligned} 0 &= b - \beta S^* I^* - \mu S^*, \\ 0 &= \beta S^* I^* - (\sigma + \mu + \alpha) E^*, \\ 0 &= \sigma E^* - (\gamma + \mu + \delta) I^*. \end{aligned}$$

Then EEP is obtained as

$$\epsilon_* = (S^*, E^*, I^*) = \left(\frac{(\sigma + \mu + \alpha)(\gamma + \mu + \delta)}{\beta\sigma}, \frac{(b - \mu S^*)(\gamma + \mu + \delta)}{\beta\sigma S^*}, \frac{b - \mu S^*}{\beta S^*}\right).$$

We should immediately note that the meaningfulness of ϵ_* will be discussed after determining $\mathcal{R}_0$.

3.2. Reproduction number for the model

In mathematical epidemiology, the dynamics of models of infectious diseases are generally established by a threshold known as the basic reproduction number $\mathcal{R}_0$. Characteristically, if $\mathcal{R}_0 < 1$ then an infectious individual can not even create averagely one new case during his/her infectiousness period and so the disease cannot continue to spread. If $\mathcal{R}_0 > 1$ then each infectious produces more than one new cases and as a result of this the disease increasingly continues to spread in the population. Since $\mathcal{R}_0$ allows to determine the amount of effort which is necessary either to prevent an epidemic or to eliminate the disease in a population, estimation of $\mathcal{R}_0$ is vital for infectious diseases.

Now, let us calculate $\mathcal{R}_0$ by using the next generation matrix method [11, 12].

Let $X = (E, I, S)^T$. So model (2.1) can be written as

$$\frac{dX}{dt} = \mathcal{P}(X) - \mathcal{V}(X),$$

such that

$$\frac{dX}{dt} = \begin{bmatrix} \dot{E} \\ \dot{I} \\ \dot{S} \end{bmatrix}, \qquad \mathcal{P}(X) = \begin{bmatrix} \beta S(t) I(t) \\ 0 \\ 0 \end{bmatrix},$$

$$\mathcal{V}(X) = \begin{bmatrix} \sigma \int_0^h f(\tau) E(t-\tau) d\tau + (\mu + \alpha) E(t) \\ (\gamma + \mu + \delta) I(t) - \sigma \int_0^h f(\tau) E(t-\tau) d\tau \\ \beta S(t) I(t) + \mu S(t) - b \end{bmatrix}.$$

In this splitting, $\mathcal{P}(X)$ is the matrix formed with writing of the partitionings in which new infections appear in compartments E, I and S, respectively; and $\mathcal{V}(X)$ is the matrix formed with writing of the partitionings in which other transitions between compartments E, I and S, and other compartments, respectively.

By differentiating $\mathcal{P}(X)$ and $\mathcal{V}(X)$ at DFEP $\epsilon_0 = \left(\frac{b}{\mu}, 0, 0\right)$ with respect to E, I, S respectively, we get

$$d\mathcal{P}(\epsilon_0) = \begin{bmatrix} 0 & \beta S_0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$d\mathcal{V}(\epsilon_0) = \begin{bmatrix} \sigma+\mu+\alpha & 0 & 0 \\ -\sigma & \gamma+\mu+\delta & 0 \\ 0 & \beta S_0 & \mu \end{bmatrix}.$$

Considering the infection can be only existed in E and I, let us create the matrices P and V in the form of

$$P = d\mathcal{P}_{2x2} = \begin{bmatrix} 0 & \beta S_0 \\ 0 & 0 \end{bmatrix}$$

and

$$V = d\mathcal{V}_{2x2} = \begin{bmatrix} \sigma+\mu+\alpha & 0 \\ -\sigma & \gamma+\mu+\delta \end{bmatrix}.$$

The matrix PV^{-1} whose spectral radius will give the formula of the basic reproduction number is obtained as

$$PV^{-1} = \begin{bmatrix} \frac{\beta\sigma S_0}{(\sigma+\mu+\alpha)(\gamma+\mu+\delta)} & \frac{\beta S_0}{\gamma+\mu+\delta} \\ 0 & 0 \end{bmatrix}.$$

From the biological meanings of P and V, it follows that P is entrywise nonnegative and V is a nonsingular M-matrix, so V^{-1} is entrywise nonnegative. Let $\breve{i}\breve{i}\breve{i}(0)$ show the number of **i**nitially **i**nfected **i**ndividuals. Then $PV^{-1}\breve{i}\breve{i}\breve{i}(0)$ is an entrywise nonnegative vector giving the expected number of new infections. Matrix PV^{-1} has $(1;1)$ entry equal to the expected number of secondary infections in compartments E and I produced by an infected individual introduced in compartments E and I. Thus PV^{-1} is the next generation matrix and $\mathcal{R}_0 = \rho\left(PV^{-1}\right)$; where ρ denotes the spectral radius.

Thus, considering the characteristic polynomial of PV^{-1}, the spectral radius of the next generation matrix is

$$\rho\left(PV^{-1}\right) = \frac{\beta\sigma S_0}{(\sigma+\mu+\alpha)(\gamma+\mu+\delta)}.$$

Therefore $\mathcal{R}_0$ is found as

$$\mathcal{R}_0 = \frac{b\beta\sigma}{\mu(\sigma+\mu+\alpha)(\gamma+\mu+\delta)} \tag{3.1}$$

for the model (2.1).

Besides that EEP ϵ_* can be rewritten as

$$\epsilon_* = (S^*, E^*, I^*) = \left(\frac{b}{\mu\mathcal{R}_0}, \frac{\mu(\gamma+\mu+\delta)(\mathcal{R}_0-1)}{\beta\sigma}, \frac{\mu(\mathcal{R}_0-1)}{\beta}\right).$$

Conclusion 3.1 *Model (2.1) has always the DFEP ϵ_0. Particularly, if $\mathcal{R}_0 < 1$ the ϵ_0 is unique equilibria. If $\mathcal{R}_0 > 1$, there exist two equilibrias; DFEP ϵ_0 and EEP ϵ_*. Also ϵ_* is meaningful only when $\mathcal{R}_0 > 1$.*

3.3. Positively invariant region

As is known, a set Ω is invariant with respect to

$$\frac{dN}{dt} = g(N)$$

if $N(0) \in \Omega$ requires $N(t) \in \Omega$ for all $t \in \mathbb{R}$. Especially if $N(0) \in \Omega$ requires $N(t) \in \Omega$ for all $t \in \mathbb{R}_+ = [0, \infty)$ then it is said that Ω is positively invariant.

Theorem 3.2 *The set*

$$\Omega = \left\{ (S, E, I, R) : S, I, R \in C(\mathbb{R}_+, \mathbb{R}_+),\ E \in C([-\tau, \infty), \mathbb{R}_+) \text{ and } N(t) \leq \frac{b}{\mu} \right\}$$

is positively invariant for the model (2.1).

Proof Adding the all equations of system (2.1), we get

$$\frac{dN}{dt} + \mu N(t) = b - \delta I,$$

and so

$$\frac{dN}{dt} + \mu N(t) \leq b. \tag{3.2}$$

Taking into account that

$$N(t) = \frac{b}{\mu} + ce^{-\mu t}$$

is solution of

$$\frac{d}{dt}\left(N(t)\, e^{\mu t}\right) = be^{\mu t}$$

then for the initial condition $t = 0$, we obtain

$$c = N(0) - \frac{b}{\mu}.$$

Thus we have

$$N(t) = N(0)\, e^{-\mu t} + \frac{b}{\mu}\left(1 - e^{-\mu t}\right). \tag{3.3}$$

By the Standard Comparison Theorem [21], we deduce that $N(t)$ given by (3.3) is the maximal solution of inequality (3.2). Hence

$$N(t) \leq N(0)\, e^{-\mu t} + \frac{b}{\mu}\left(1 - e^{-\mu t}\right)$$

for all $t \geq 0$. Particularly, $N(t) \leq b/\mu$ if $N(0) \leq b/\mu$. So Ω is positively invariant for system (2.1).

On the other hand, if $N(0) > b/\mu$ then either the solution enters Ω infinite time or $N(t)$ approaches b/μ asymptotically. Hence, Ω attracts all solutions of (2.1). Thus the model (2.1) can be just evaluated in Ω mathematically and epidemiologically. □

4. Local and global stability of DFEP

In this section, we present stability results for DFEP ϵ_0 by analyzing the corresponding characteristic equation and using LaSalle's invariance principle associated with Lyapunov functional technique.

Theorem 4.1 *If $\mathcal{R}_0 < 1$, DFEP ϵ_0 is locally asymptotically stable in Ω.*

Proof The Jacobian matrix of system (2.1) at DFEP ϵ_0 is

$$J(\epsilon_0) = \begin{bmatrix} -\mu & 0 & -\beta S_0 \\ 0 & -(\sigma+\mu+\alpha) & \beta S_0 \\ 0 & \sigma & -(\gamma+\mu+\delta) \end{bmatrix}.$$

Thus, the corresponding characteristic equation of $J(\epsilon_0)$ is described by

$$(-\mu-\lambda)\left(\lambda^2 + [(\sigma+\mu+\alpha)+(\gamma+\mu+\delta)]\lambda + (\sigma+\mu+\alpha)(\gamma+\mu+\delta) - \beta\sigma S_0\right) = 0. \tag{4.1}$$

This equation always has negative root $\lambda_1 = -\mu$. For the other roots (λ_2 and λ_3) of Equation (4.1), we have

$$\lambda_2 + \lambda_3 = -(\sigma+\mu+\alpha) - (\gamma+\mu+\delta) < 0$$

and

$$\begin{aligned} \lambda_2\lambda_3 &= \frac{\mu(\sigma+\mu+\alpha)(\gamma+\mu+\delta) - b\beta\sigma}{\mu} \\ &= (\sigma+\mu+\alpha)(\gamma+\mu+\delta)(1-\mathcal{R}_0). \end{aligned}$$

For $\mathcal{R}_0 < 1$, since $\lambda_2\lambda_3 > 0$, we can say that all roots of Equation (4.1) have negative real parts. Hence DFEP ϵ_0 is locally asymptotically stable for $\mathcal{R}_0 < 1$. □

By the way let us focus on derivation of the expression $\sigma\int_0^h f(\tau)E(t-\tau)d\tau$ with respect to E, used in constructing the Jacobian matrices in proof of Theorems 4.1 and 5.1. If we define operator T as

$$T(f,E)(t,\tau) = f(\tau)E(t-\tau)$$

then we write

$$\frac{\partial}{\partial E}\left(\sigma\int_0^h f(\tau)E(t-\tau)d\tau\right) = \sigma\int_0^h \frac{\partial}{\partial E}T(f,E)d\tau.$$

Indeed, if we choose $\widetilde{t} = t-\tau$ then $E(t-\tau) = E(\widetilde{t})$ and so we can say that even though $E(t)$ and $E(\widetilde{t})$ have different independent variables, it represent the same function. Since

$$\frac{\partial E(\widetilde{t})}{\partial E} = 1,$$

we can say

$$\frac{\partial T(f,E)}{\partial E} = f(\tau).$$

So we obtain

$$\frac{\partial}{\partial E}\left(\sigma\int_0^h f(\tau)\,E(t-\tau)\,d\tau\right)=\sigma\int_0^h f(\tau)\,d\tau=\sigma.$$

Theorem 4.2 *If $\mathcal{R}_0<1$, DFEP ϵ_0 is globally asymptotically stable in Ω.*

Proof Let us consider the following nonnegative function that we prepared in accordance with model (2.1):

$$W(t)=\sigma E(t)+(\sigma+\mu+\alpha)\,I(t)+\sigma(\mu+\alpha)\int_0^h f(\tau)\left(\int_{t-\tau}^t E(z)\,dz\right)d\tau. \tag{4.2}$$

Differentiating with respect to time yields we get

$$\begin{aligned}
\frac{dW}{dt} &= \sigma\beta S(t)\,I(t)-\sigma^2\int_0^h f(\tau)\,E(t-\tau)\,d\tau-\sigma(\mu+\alpha)\,E(t)\\
&\quad+\sigma^2\int_0^h f(\tau)\,E(t-\tau)\,d\tau-\sigma(\gamma+\mu+\delta)\,I(t)\\
&\quad+\sigma(\mu+\alpha)\int_0^h f(\tau)\,E(t-\tau)\,d\tau-(\mu+\alpha)(\gamma+\mu+\delta)\,I(t)\\
&\quad+\sigma(\mu+\alpha)\int_0^h f(\tau)\,\frac{d}{dt}\left(\int_{t-\tau}^t E(z)\,dz\right)d\tau\\
&= \sigma\beta S(t)\,I(t)-\sigma(\mu+\alpha)\,E(t)-\sigma(\gamma+\mu+\delta)\,I(t)\\
&\quad+\sigma(\mu+\alpha)\int_0^h f(\tau)\,E(t-\tau)\,d\tau-(\mu+\alpha)(\gamma+\mu+\delta)\,I(t)\\
&\quad+\sigma(\mu+\alpha)\underbrace{\int_0^h f(\tau)\,E(t)\,d\tau}_{E(t)}-\sigma(\mu+\alpha)\int_0^h f(\tau)\,E(t-\tau)\,d\tau\\
&= \sigma\beta S(t)\,I(t)-\sigma(\gamma+\mu+\delta)\,I(t)-(\mu+\alpha)(\gamma+\mu+\delta)\,I(t)\\
&= I(t)\left[\sigma\beta S(t)-(\sigma+\mu+\alpha)(\gamma+\mu+\delta)\right]\\
&\leq I(t)\left(\frac{b\beta\sigma}{\mu}-(\sigma+\mu+\alpha)(\gamma+\mu+\delta)\right)\\
&= I(t)(\sigma+\mu+\alpha)(\gamma+\mu+\delta)\left(\frac{b\beta\sigma}{\mu(\sigma+\mu+\alpha)(\gamma+\mu+\delta)}-1\right)\\
&= I(t)(\sigma+\mu+\alpha)(\gamma+\mu+\delta)(\mathcal{R}_0-1)\\
&\leq 0.
\end{aligned}$$

This result shows that W is a Lyapunov function in Ω for system (2.1). According to LaSalle's invariance principle [22] the limit set of each solution is contained in the largest invariant subset of

$$\left\{(S,E,I):(S,E,I) \text{ is a solution of } \frac{dW}{dt}=0\right\}.$$

Also this largest invariant subset consists only singleton ϵ_0 for $\mathcal{R}_0<1$. Thus DFEP ϵ_0 is globally asymptotically stable. □

5. Local and global stability of EEP

In this part, we present stability results for EEP by using the Routh-Hurwitz criteria and Li-Muldowney geometric approach, respectively.

Theorem 5.1 *If $\mathcal{R}_0>1$, EEP ϵ_* is locally asymptotically stable in Ω.*

Proof The Jacobian matrix at EEP ϵ_* of model (2.1) is

$$J(\epsilon_*)=\begin{bmatrix} -\beta I^* - \mu & 0 & -\beta S^* \\ \beta I^* & -(\sigma+\mu+\alpha) & \beta S^* \\ 0 & \sigma & -(\gamma+\mu+\delta) \end{bmatrix}.$$

Considering that

$$S^*=\frac{(\sigma+\mu+\alpha)(\gamma+\mu+\delta)}{\beta\sigma}$$

and

$$\begin{aligned} I^* &= \frac{b-\mu S^*}{\beta S^*} \\ &= \frac{b\beta\sigma-\mu(\sigma+\mu+\alpha)(\gamma+\mu+\delta)}{\beta(\sigma+\mu+\alpha)(\gamma+\mu+\delta)} \\ &= \frac{\mu(\mathcal{R}_0-1)}{\beta} \end{aligned}$$

the corresponding characteristic equation of $J(\epsilon_*)$ is found as

$$\lambda^3+C_1\lambda^2+C_2\lambda+C_3=0, \tag{5.1}$$

where

$$C_1=\frac{(\sigma+\mu+\alpha)^2(\gamma+\mu+\delta)+(\sigma+\mu+\alpha)(\gamma+\mu+\delta)^2+b\beta\sigma}{(\sigma+\mu+\alpha)(\gamma+\mu+\delta)},$$

$$C_2=\frac{b\beta\sigma((\sigma+\mu+\alpha)+(\gamma+\mu+\delta))}{(\sigma+\mu+\alpha)(\gamma+\mu+\delta)}$$

and

$$\begin{aligned} C_3 &= b\beta\sigma-\mu(\sigma+\mu+\alpha)(\gamma+\mu+\delta) \\ &= \mu(\sigma+\mu+\alpha)(\gamma+\mu+\delta)(\mathcal{R}_0-1). \end{aligned}$$

Since all parameters are positive we say that $C_1,\ C_2,\ C_3 > 0$ for $\mathcal{R}_0 > 1$.

According to Routh-Hurwitz criteria we calculate as

$$H_1 = C_1 > 0,$$

$$\begin{aligned} H_2 &= \frac{C_1C_2 - C_3}{C_1} \\ &= \frac{\begin{array}{c}(\sigma+\mu+\alpha)(\gamma+\mu+\delta) \\ \times\left[\mu(\sigma+\mu+\alpha)(\gamma+\mu+\delta) - b\beta\sigma + \frac{b\beta\sigma((\sigma+\mu+\alpha)+(\gamma+\mu+\delta))\left[(\sigma+\mu+\alpha)^2(\gamma+\mu+\delta)+(\sigma+\mu+\alpha)(\gamma+\mu+\delta)^2+b\beta\sigma\right]}{(\sigma+\mu+\alpha)^2(\gamma+\mu+\delta)^2}\right]\end{array}}{(\sigma+\mu+\alpha)^2(\gamma+\mu+\delta)+(\sigma+\mu+\alpha)(\gamma+\mu+\delta)^2+b\beta\sigma} \\ &> 0 \text{ (after the simplifications)} \end{aligned}$$

and

$$H_3 = C_3 > 0.$$

Hence we conclude that all the roots of Equation (5.1) have negative real parts. Therefore, EEP $\epsilon_* = (S^*, E^*, I^*)$ is locally asymptotically stable. □

To examine the global dynamics of ϵ_*, we use the geometric approach that can be applied in the proofs of global stability of dynamical systems, proposed by Li and Muldowney [24]. A general theoretical summary for relevant details is provided for readers in Appendix.

Firstly we will focus that system (2.1) is uniformly persistent. System (2.1) is uniformly persistent [4, 29] if there exists a constant $c > 0$, independent of initial data in $\mathring{\Omega}$, such that, any solution $(S(t), E(t), I(t))$ of (2.1) satisfies

$$\liminf_{t\to\infty} S(t) > c,\ \liminf_{t\to\infty} E(t) > c \text{ and } \liminf_{t\to\infty} I(t) > c$$

provided $(S(0), E(0), I(0)) \in \mathring{\Omega}$.

On the other hand, when $\mathcal{R}_0 > 1$ by utilizing Lyapunov function created in (4.2) one can easily seen that ϵ_0 is unstable. Indeed, if $\mathcal{R}_0 > 1$, $dW/dt > 0$ for S sufficiently close to b/μ except when $I = 0$. Solutions starting sufficiently close to ϵ_0 leave from the neighborhood of ϵ_0 after a certain part. By using the result about uniformly persistence in [15] and the similar argument to the proof of Proposition 3.3 in [23], it can be shown that, when $\mathcal{R}_0 > 1$, the instability of ϵ_0 implies the uniform persistence of (2.1). For this reason, the proof of the following result is omitted in order to avoid repetition.

Theorem 5.2 *If $\mathcal{R}_0 > 1$, system (2.1) is uniformly persistent.*

Theorem 5.3 *If $\mathcal{R}_0 > 1$, EEP ϵ_* is globally asymptotically stable in Ω.*

Proof In accordance with Theorem 5.2 which is said that system (2.1) is uniformly persistent together with the boundedness of solutions, we can say that there exists a compact set Φ in the interior of Ω which is absorbing for (2.1), [4]. Thus the assumption (LM2) given in Appendix is satisfied. Based on Li and Muldowney's technique outlined in Appendix, the proof of the theorem is created by choosing a suitable vector norm $|\cdot|$ in $\mathbb{R}^3$ and a 3×3 matrix-valued function $A(x)$ so that the quantity $\overline{q_2}$ defined by (7.3) in Appendix part is negative.

Let $x=(S,E,I)$ and $f(x)$ denote the vector field of (2.1). Then the Jacobian matrix $J=Df(x)$ along each solution (2.1) is

$$J=\begin{bmatrix} -\beta I-\mu & 0 & -\beta S \\ \beta I & -(\sigma+\mu+\alpha) & \beta S \\ 0 & \sigma & -(\gamma+\mu+\delta) \end{bmatrix}$$

and its corresponding second additive compound matrix $J^{[2]}$ is obtained as

$$J^{[2]}=\begin{bmatrix} -\beta I-\mu-(\sigma+\mu+\alpha) & \beta S & \beta S \\ \sigma & -\beta I-\mu-(\gamma+\mu+\delta) & 0 \\ 0 & \beta I & -(\sigma+\mu+\alpha)-(\gamma+\mu+\delta) \end{bmatrix}.$$

Let us establish matrix A as

$$A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{E}{I} & 0 \\ 0 & 0 & \frac{E}{I} \end{bmatrix}.$$

It can be easily seen that

$$A_f=\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{E'}{I}-\frac{I'E}{I^2} & 0 \\ 0 & 0 & \frac{E'}{I}-\frac{I'E}{I^2} \end{bmatrix}$$

and so

$$A_fA^{-1}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{E'}{E}-\frac{I'}{I} & 0 \\ 0 & 0 & \frac{E'}{E}-\frac{I'}{I} \end{bmatrix}.$$

Therefore the matrix $B=A_fA^{-1}+AJ^{[2]}A^{-1}$ is obtained as

$$B=\begin{bmatrix} -\beta I-\mu-(\sigma+\mu+\alpha) & \frac{I}{E}\beta S & \frac{I}{E}\beta S \\ \frac{E}{I}\sigma & \frac{E'}{E}-\frac{I'}{I}-\beta I-\mu-(\gamma+\mu+\delta) & 0 \\ 0 & \beta I & \frac{E'}{E}-\frac{I'}{I}-(\sigma+\mu+\alpha)-(\gamma+\mu+\delta) \end{bmatrix}.$$

Also B can be written as

$$B=\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where

$$B_{11}=\left[-\beta I-\mu-(\sigma+\mu+\alpha)\right],$$

$$B_{12}=\begin{bmatrix} \frac{I}{E}\beta S & \frac{I}{E}\beta S \end{bmatrix},$$

$$B_{21}=\begin{bmatrix} \frac{E}{I}\sigma \\ 0 \end{bmatrix}$$

and

$$B_{22}=\begin{bmatrix} \frac{E'}{E}-\frac{I'}{I}-\beta I-\mu-(\gamma+\mu+\delta) & 0 \\ \beta I & \frac{E'}{E}-\frac{I'}{I}-(\sigma+\mu+\alpha)-(\gamma+\mu+\delta) \end{bmatrix}.$$

Let us consider the vector norm defined with

$$|(a_1,a_2,a_3)|=\max\{|a_1|,|a_2|+|a_3|\}$$

in $\mathbb{R}^3$ with $(a_1,a_2,a_3)\in\mathbb{R}^3$.

On the other hand, the Lozinskiĭ measure μ_L with respect to this norm can be estimated as follows:

$$\mu_L(B)\leq\max\{g_1,g_2\} \tag{5.2}$$

such that

$$\begin{aligned} g_1 &= \mu_L(B_{11})+|B_{12}|, \\ g_2 &= |B_{21}|+\mu_L(B_{22}). \end{aligned}$$

Note that $\mu_L(B_{22})$ is the Lozinskiĭ measure of the matrix B_{22} with respect to l_1 norm in $\mathbb{R}^2$. Also $|B_{12}|$ and $|B_{21}|$ are the operator norms of B_{12} and B_{21} with mappings from $\mathbb{R}^2$ to $\mathbb{R}$ and from $\mathbb{R}$ to $\mathbb{R}^2$, respectively. Where we consider that $\mathbb{R}^2$ is endowed with l_1 norm. Also note that since B_{11} is a scalar, its Lozinskiĭ measure with respect to any vector norm in $\mathbb{R}$ is equal to B_{11}.

Therefore, the matrix norms $|B_{12}|$ and $|B_{21}|$ with respect to the vector norm are obtained as

$$|B_{12}|=\frac{I}{E}\beta S \qquad \text{and} \qquad |B_{21}|=\frac{E}{I}\sigma.$$

Also

$$\mu_L(B_{11})=-\beta I-\mu-(\sigma+\mu+\alpha),$$

and to calculate $\mu_L(B_{22})$, the absolute value of the off-diagonal elements in each column of matrix B_{22} are added to the diagonal one and after is taken the maximum one among this two sums. Hence

$$\begin{aligned} \mu_L(B_{22}) &= \max\left\{\frac{E'}{E}-\frac{I'}{I}-\beta I-\mu-(\gamma+\mu+\delta)+\beta I,\frac{E'}{E}-\frac{I'}{I}-(\sigma+\mu+\alpha)-(\gamma+\mu+\delta)\right\} \\ &= \max\left\{\frac{E'}{E}-\frac{I'}{I}-\mu-(\gamma+\mu+\delta),\frac{E'}{E}-\frac{I'}{I}-(\sigma+\mu+\alpha)-(\gamma+\mu+\delta)\right\} \\ &= \frac{E'}{E}-\frac{I'}{I}-\mu-(\gamma+\mu+\delta). \end{aligned}$$

Therefore

$$g_1=\mu_L(B_{11})+|B_{12}|=-\beta I-\mu-(\sigma+\mu+\alpha)+\beta S\frac{I}{E} \tag{5.3}$$

and

$$g_2=|B_{21}|+\mu_L(B_{22})=\frac{E'}{E}-\frac{I'}{I}-\mu-(\gamma+\mu+\delta)+\frac{E}{I}\sigma. \tag{5.4}$$

From second and third equation of (2.1), we write

$$\beta S\frac{I}{E}=\frac{E'}{E}+\sigma+\mu+\alpha \tag{5.5}$$

and

$$\frac{E}{I}\sigma=\frac{I'}{I}+\gamma+\mu+\delta, \tag{5.6}$$

respectively.

Substituting the equalities (5.5) and (5.6) into (5.3) and (5.4), respectively, we obtain

$$\begin{aligned} g_1 &= \frac{E'}{E}-\beta I-\mu \\ &\leq \frac{E'}{E}-\mu \end{aligned}$$

and

$$g_2=\frac{E'}{E}-\mu.$$

So by (5.2) we write

$$\mu_L(B)\leq \max\{g_1,g_2\}=\frac{E'}{E}-\mu.$$

Since system (2.1) is uniformly persistent by Proposition 5.2, there exist $c>0$ and $T>0$ such that $t>T$ implies

$$E(t)>c \qquad \text{and} \qquad \frac{1}{t}\log E(t)<\frac{\mu}{2}$$

for all $(S(0),E(0),I(0))\in\Phi$. As a result, we have

$$\overline{q_2}=\limsup_{t\to\infty}\left(\sup\frac{1}{t}\int_0^t \mu_L(B)\,ds\right)<\log E(t)-\mu\leq-\frac{\mu}{2}<0.$$

Hereby, the conditions of Theorem 7.3 given in Appendix are satisfied and therefore EEP ϵ_* is globally asymptotically stable in Ω.

□

6. Conclusion

In this paper we construct a new $SEIR$ model including distributed latent period. The most important difference between our model and other $SEIR$ models is that an individual leaving from S belongs to E through the latent period and becomes infectious himself with a certain rate (σ) after a latent period (τ). We use a distribute function to take into account the latent period changing according to individuals in order to add a more realistic structure to classical $SEIR$ epidemic models.

The presented model has always the disease-free equilibrium point. Particularly, if $\mathcal{R}_0<1$ it is unique equilibria. Also the model has an endemic equilibrium point in addition to disease-free equilibrium when $\mathcal{R}_0>1$.

As one of the main results, we focus on stability analysis of the model according to $\mathcal{R}_0$. The mathematical results we obtained from the stability analysis of the model epidemiologically mean that if $\mathcal{R}_0 < 1$ then invasion of individuals by the pathogen does not give rise to a widespread epidemic and so the disease gradually becomes extinct. Otherwise, that is for $\mathcal{R}_0 > 1$, the disease continues to spread in the population.

7. Appendix

In this section, brief summary information about the Routh-Hurwitz criteria, LaSalle invariance principle associated with Lyapunov's direct method and Li-Muldowney technique, which have a quite wide and toilsome theory, are given enough to remind the reader in this article without going into details.

7.1. Routh-Hurwitz criteria

The Routh-Hurwitz criteria is generally used to determine local asymptotic stability of an equilibrium for nonlinear systems of differential equations. This criterion is a method showing the stability of a nonlinear system by taking into account the coefficients of characteristic equation of Jacobian matrix of the system at equilibrium points. This important method that gives necessary and sufficient conditions for all of the roots of the characteristic polynomial to lie in the left half of the complex plane takes its name from E. J. Routh and A. Hurwitz, who contributed to the formulation of this criteria. In general the Routh stability criterion states a polynomial has all roots in the open left half plane if and only if all first-column elements of the Routh array have the same sign.

A tabular method (Routh-Hurwitz table) can be used to determine the stability when the roots of a high order characteristic polynomial are difficult to obtain. For an nth-degree polynomial in the form $P(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$ the table has $n+1$ rows and the following structure:

$$H_n = \begin{bmatrix} a_n & a_{n-2} & a_{n-4} & \cdots & 0 \\ a_{n-1} & a_{n-3} & a_{n-5} & \cdots & 0 \\ b_1 & b_2 & b_3 & \cdots & 0 \\ c_1 & c_2 & c_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & 0 & 0 & \cdots & 0 \end{bmatrix}$$

where the elements b_i and c_i can be computed as follows:

$$b_i = \frac{a_{n-1}a_{n-2i} - a_n a_{n-(2i+1)}}{a_{n-1}},$$

and

$$c_i = \frac{b_1 a_{n-(2i+1)} - a_{n-1} b_{i+1}}{b_1}.$$

Then the number of sign changes in the first column gives the number of nonnegative roots. For stability, all the elements in the first column of the Routh array must be positive.

For example, for the system whose its characteristic polynomial is given by $P(s) = a_4 s^4 + a_3 s^3 + a_2 s^2 +$

$a_1 s + a_0$, we have

$$H_4 = \begin{bmatrix} a_4 & a_2 & a_0 & 0 \\ a_3 & a_1 & 0 & 0 \\ \frac{a_3 a_2 - a_4 a_1}{a_3} & \frac{a_3 a_0 - a_4 \times 0}{a_3} = a_0 & 0 & 0 \\ \frac{(a_3 a_2 - a_4 a_1) a_1 - a_3^2 a_0}{a_3 a_2 - a_4 a_1} & 0 & 0 & 0 \\ a_0 & 0 & 0 & 0 \end{bmatrix}.$$

So the conditions that must be satisfied for stability of the given system are as follows:

$$a_3 > 0,$$

$$a_3 a_2 - a_4 a_1 > 0,$$

$$(a_3 a_2 - a_4 a_1)\, a_1 - a_3^2 a_0 > 0$$

and

$$a_0 > 0.$$

If these inequalities are satisfied, the system, which the characteristic polynomial P corresponds, is locally asymptotically stable.

Now let us talk more briefly about the Routh coefficients obtained for the characteristic equation being in our study.

As in the proof of Theorem 5.1, for the characteristic polynomial $P(\lambda) = \lambda^3 + C_1 \lambda^2 + C_2 \lambda + C_3$, the Hurwitz coefficients are obtained as

$$H_1 = C_1,$$

$$H_2 = \frac{C_1 C_2 - C_3}{C_1},$$

and

$$H_3 = \frac{\frac{C_1 C_2 - C_3}{C_1} - C_1 \times 0}{\frac{C_1 C_2 - C_3}{C_1}} = C_3,$$

where the coefficients C_i are real constants for $i = 1, 2, 3$.

The Hurwitz coefficients obtained the Routh table for the characteristic polynomial in the form $P(\lambda) = \lambda^4 + C_1 \lambda^3 + C_2 \lambda^2 + C_3 \lambda + C_4$ are as follows:

$$H_1 = C_1,$$

$$H_2 = \frac{C_1 C_2 - C_3}{C_1},$$

$$H_3 = \frac{\frac{(C_1 C_2 - C_3) C_3}{C_1} - C_1 C_4}{\frac{C_1 C_2 - C_3}{C_1}} = \frac{C_3 H_2 - C_1^2 C_4}{H_2}$$

and

$$H_4 = \frac{C_4 H_3 - 0 \times H_3}{H_3} = C_4.$$

For stability of the relevant systems firstly the coefficients C_i and after H_i must be positive.

7.2. Lyapunov functional technique and LaSalle invariance principle

Lyapunov functions are among the methods that may be used to prove the global stability of a system of ordinary differential equations. This method given by A. M. Lyapunov does not show how to find a Lyapunov function V. Moreover the tricky part is that there is no systematic way to construct Lyapunov functions and it generally requires deep efforts. The reader can review the references in [20, 22] for details concerning Lyapunov functional technique, LaSalle invariance principle and the other methods related to the stability of nonlinear systems.

Consider the autonomous system

$$\dot{x} = f(x), \tag{7.1}$$

where $f : D \to \mathbb{R}^n$ is a locally Lipschitz map from a domain $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$. Suppose $\overline{x} \in D$ is an equilibrium point of 7.1; that is $f(\overline{x}) = 0$. Our goal is to characterize and study the stability of $\overline{x}$.

Theorem 7.1 *(Lyapunov's direct method) Let $\overline{x}$ be an equilibrium point for 7.1 and $D \subset \mathbb{R}^n$ be a domain containing $\overline{x}$. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that*

$$V(\overline{x}) = 0 \text{ and } V(x) > 0 \text{ in } D \setminus \{\overline{x}\}.$$

Then

(a) if $\dot{V}(x) \leq 0$ for all $x \in D$, $\overline{x}$ is stable;

(b) if $\dot{V}(x) < 0$ for all $x \in D \setminus \{\overline{x}\}$, $\overline{x}$ is asymptotically stable;

(c) if $\dot{V}(x) > 0$ for all $x \in D \setminus \{\overline{x}\}$, $\overline{x}$ is unstable.

A function V satisfying the conditions of this theorem is called Lyapunov function. Theorem 7.1 allows to determine the stability of the equilibrium point without explicitly solving the mentioned system.

Theorem 7.2 *(LaSalle invariance principle) Let $\Omega \subset D$ be a compact set that is positively invariant with respect to 7.1. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(x) \leq 0$ in Ω. Let K be the set of all points in Ω where $\dot{V}(x) = 0$. Let M be the largest invariant set in K. Then every solution starting in Ω approaches M as $t \to \infty$.*

7.3. Li-Muldowney technique

The definitions presented in the following part, which are well-known in the literature, are taken from [24, 30], which we have benefit a lot in this study.

7.3.1. Lozinskiĭ measure

Let $|\cdot|$ denote a vector norm in $\mathbb{R}^n$ as well as the matrix norm which it induces for $n \times n$ matrices. The Lozinskiĭ measure μ_L of a $n \times n$ matrix M with respect to the norm $|\cdot|$ is defined as

$$\mu_L(M) = \lim_{h \to 0^+} \frac{|I + hM| - 1}{h}.$$

For more details about Lozinskiĭ measure, the reader can examine to the reference [8].

7.3.2. The second additive compound matrix

Let M be a linear operator on $\mathbb{R}^n$ and denote its matrix representation with respect to the standard basis of $\mathbb{R}^n$. Let $\Lambda^2\mathbb{R}^n$ denote the exterior product of $\mathbb{R}^n$. M induces canonically a linear operator $M^{[2]}$ on $\Lambda^2\mathbb{R}^n$ for $u_1,\ u_2 \in \mathbb{R}^n$, define $M^{[2]}(u_1 \Lambda u_2) := M(u_1)\Lambda u_2 + u_1 \Lambda M(u_2)$ and extend the definition over $\Lambda^2\mathbb{R}^n$ by linearity.

The matrix representation of $M^{[2]}$ with respect to the canonical basis in $\Lambda^2\mathbb{R}^n$ is called the second additive compound matrix of M. This is an $\binom{n}{2} \times \binom{n}{2}$ matrix and for $n=2$ and $n=3$ we define, respectively, as

$$M^{[2]}_{2\times 2} = trM$$

and

$$M^{[2]}_{3\times 3} = \begin{bmatrix} m_{11}+m_{22} & m_{23} & -m_{13} \\ m_{32} & m_{11}+m_{33} & m_{12} \\ -m_{31} & m_{21} & m_{22}+m_{33} \end{bmatrix}.$$

7.3.3. A short brief of the approach suggested by Li and Muldowney by utilizing Lozinskiĭ measure and the second additive compound matrix

Let the map $x \to f(x)$ from an open subset $\Psi \subset \mathbb{R}^n$ to $\mathbb{R}^n$ be such that each solution $x(t)$ to the differential equation

$$\dot{x} = f(x) \tag{7.2}$$

is uniquely determined by its initial value $x(0) = x_0$ and denote this solution $x(t, x_0)$.

An equilibrium point $\overline{x} \in \Psi$ of (7.2) is said to be globally asymptotically stable or simply globally stable in Ψ, if it is locally asymptotically stable and all trajectories in Ψ converge to $\overline{x}$.

Let $J = Df(x)$ be the Jacobian matrix of f at x and assume that following two conditions are satisfied:

(LM1) System (7.2) has a unique equilibrium $\overline{x}$ in Ψ,

(LM2) System (7.2) has a compact absorbing set $\breve{\Psi} \subset \Psi$.

Consider a nonsingular $\binom{n}{2} \times \binom{n}{2}$ matrix-valued function $x \to A(x)$ which is a continuously differentiable function in Ψ and a vector norm $|\cdot|$ on $\mathbb{R}^{\binom{n}{2}}$. Also assume that $A^{-1}(x)$ exists and is continuous for $x \in \breve{\Psi}$. Let μ_L be the Lozinskiĭ measure with respect to $|\cdot|$ and the quantity $\overline{q_2}$ is defined as

$$\overline{q_2} = \limsup_{t\to\infty} \left(\sup_{x_0 \in \breve{\Psi}} \frac{1}{t} \int_0^t \mu_L \left(B\left(x(s, x_0)\right)\right) ds \right), \tag{7.3}$$

where

$$B = A_f A^{-1} + A J^{[2]} A^{-1}$$

and $J^{[2]} = Df(x)^{[2]} = \frac{\partial f}{\partial x}^{[2]}$ is the second additive compound matrix of $J = Df(x) = \frac{\partial f}{\partial x}$.

Under the preparations given in general terms here, the following important result for global stabilities has been proved in [24] with theoretical details.

Theorem 7.3 *Let Ψ be a simply connected region. Under assumptions (LM1) and (LM2), the unique equilibrium $\overline{x}$ of nonlinear dynamical system $\dot{x} = f(x)$ is globally stable in Ψ if there exists a function $A(x)$ and a Lozinskiĭ measure μ_L such that $\overline{q_2}$ defined in (7.3) satisfies $\overline{q_2} < 0$.*

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