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Two nonzero weak solutions for a quasilinear Kirchhoff type problem

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Abstract: We study the existence of two nonzero solutions for a class of quasilinear Kirchhoff problems. The approach is based on the variational methods. Our nonlinerity is contrast to some previous results is that superlinear growth at infinity.

Key words: p-Laplacian type operators, Dirichlet boundary value problem, variational methods

1. Introduction and main results

In this paper, we consider the following quasilinear Kirchhoff problem

$$\begin{cases} -\left(1+\int_\Omega |\nabla u|^p\right)\Delta_p u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $p > N \geq 3$, Δ_p is the p-Laplace operator defined as $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}u)$, $f \in C^0(\bar{\Omega}\times\mathbb{R})$ and λ is a positive parameter.

The problem (1.1) is related to the stationary analogue of the equation

$$u_{tt} - \left(1 + \int_\Omega |\nabla u(x)|^p dx\right)\Delta_p u(x) = g(x,t)$$

proposed by Kirchhoff as an extension of the classical d'Alembert's wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Similar nonlocal problems also model several physical and biological systems where u describes a process that depends on the average of itself, for example, the population density. It is worth mentioning that this problem received much attention after the work of Lions [8], where a functional analysis framework was proposed for the problem. Recently, the study of the Kirchhoff equation has been considered in the elliptic case and involving the p-Laplacian operator [1, 4–7, 9].

In this paper we consider the existence of nontrivial weak solutions for the problem (1.1). Precisely, using a variational approach, under conditions involving the antiderivative of f, that is $F(x,s) = \int_0^s f(x,t)dt$, we will

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obtain a precise interval of the parameter λ for which the problem (1.1) admits two nontrivial weak solutions (see Theorem 1.1).

We consider the space

$$X := W_0^{1,p}(\Omega)$$

endowed with the norm

$$\|u\| := \left(\int_\Omega |\nabla u|^p dx \right)^{1/p} \quad \forall u \in X.$$

The Rellich-Kondrachov theorem ensures that X is compactly embedded in $C^0(\bar{\Omega})$ when $p > N$ and so there exists a positive constant k such that

$$\|u\|_{C^0(\bar{\Omega})} \leq k\|u\| \tag{1.2}$$

for each $u \in X$. The following estimate of k was obtained by Talenti in [12] when $N \geq 3$, $\partial\Omega$ is of class $C^{1,1}$

$$k \leq \frac{1}{N^{1/p}\sqrt{\pi}} [\Gamma(1 + N/2)]^{1/N} \left(\frac{p-1}{p-N} \right)^{p'} (\text{meas}(\Omega))^{1/N-1/p},$$

where Γ denotes the gamma function, p' is the conjugate exponent of p and meas(Ω) is the Lebesgue measure of Ω.

Here and in the sequel we denote by D the radius of the greatest ball $B(x_0, D)$ with center x_0 contained in Ω. With $\alpha > 0$, we put

$$F^\alpha := \int_\Omega \max_{|\xi| \leq \alpha} F(x, \xi) dx,$$

where $F(x, \xi) = \int_0^\xi f(x, t)dt$ and we observe that $F^\alpha \geq 0$ for each $\alpha > 0$.

Finally, we put

$$l_D := \frac{\pi^{N/2}(D^N - (D/2)^N)}{2D^{2p}(\Gamma(1 + N/2))^2} \left[2D^p\Gamma(1 + N/2) + \pi^{N/2} \left(D^N - (D/2)^N \right) \right].$$

Theorem 1.1 *Assume that*

(f1) there exists $x_0 \in \Omega$, $D > 0$, δ, $\gamma \in \mathbb{R}$, *with* $0 < \delta < \gamma$, *such that*

$$\frac{F^\gamma}{\gamma^p} < \frac{1}{k^p l_D} \frac{\int_{B(x_0, D/2)} F(x, \delta)dx}{\max\{1, (2\delta)^{2p}\}}.$$

(f2) $F(x, t) \geq 0$ *for every* $x \in \Omega$ *and for all* $t \in [0, \delta]$.

(f3) there exist $m > 2p$, $s > 0$ *such that*

$$0 < mF(x, t) \leq tf(x, t)$$

for each $x \in \Omega$ *and* $|t| \geq s$.

Then, for each $\lambda \in \Lambda_{\delta,\gamma} := \left] \frac{\max\{1,(2\delta)^{2p}\} l_D}{p \int_{B(x_0,D/2)} F(x,\delta)dx}, \frac{\gamma^p}{pk^pF^\gamma} \right[$, *problem* (1.1) *admits at least two nonzero solutions.*

Moreover we highlight that it is possible to obtain an existence result for the more general problem

$$\begin{cases} -\left(1+\int_\Omega |\nabla u|^p\right)\Delta_p u = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Precisely, we establish the existence of two precise intervals of the parameters λ and μ for which the problem (1.3) admits at least two nontrivial weak solutions.

Theorem 1.2 *Assume that f verifies conditions (f1)–(f3) of Theorem 1.1. Then, for each $\lambda \in \Lambda_{\delta,\gamma} := \left]\frac{\max\{1,(2\delta)^{2p}\}l_D}{p\int_{B(x_0,D/2)}F(x,\delta)dx}, \frac{\gamma^p}{pk^pF^\gamma}\right[$, and for each $g \in C^0(\bar{\Omega}\times\mathbb{R})$ such that*

(g1) $G(x,t) \geq 0$ for each $(x,t) \in \Omega \times [0,\delta]$

(g2) there exist a_1, $a_2 > 0$ and $0 < \alpha < p-1$ such that

$$|g(x,t)| \leq a_1|t|^\alpha + a_2$$

for each $x \in \Omega$ and $t \in \mathbb{R}$,

there exists $\eta_{\lambda,g} > 0$, where

$$\eta_{\lambda,g} = \left(\frac{\gamma^p \int_{B\left(x^0,\frac{D}{2}\right)} F(x,\delta)dx}{\max\{1,(2\delta)^{2p}\}l_D k^p} - F^\gamma\right)\frac{\lambda}{G^\gamma} \tag{1.4}$$

such that for each $\mu \in]0,\eta_{\lambda,g}[$ the problem (1.3) admits at least two nonzero weak solutions.

Slightly modifying the assumptions of Theorem 1.1 it is possible to obtain the following more applicable result:

Theorem 1.3 *Assume that f verifies condition (f3),*

(f1') $F(x,t) \geq 0$ for each $x \in \Omega$, $t \in \left[0, \left(\frac{p}{l_D}\right)^{\frac{1}{p}}\right]$;

(f2') $\limsup_{t\to 0^+} \frac{\inf_{x\in\Omega} F(x,t)}{t^p} = +\infty$.

Then, put $\gamma := k^{\frac{1}{p}}$, where k is defined by (1.2) and

$$\lambda^* := \begin{cases} \frac{1}{F^\gamma} & F^\gamma > 0 \\ +\infty & F^\gamma = 0 \end{cases}$$

for each $\lambda \in]0,\lambda^*[$ the problem (1.1) admits at least two nonzero weak solutions.

Last, we present a special case of Theorem 1.3 when f depends only on the second variable:

Theorem 1.4 *Let $f : \mathbb{R} \to [0,+\infty[$ such that*

(H1) $\limsup_{t\to 0^+} \frac{f(t)}{t^{p-1}} = +\infty$,

(H2) there exists $m > 2p, s > 0$ *such that* $0 < mF(t) \leq tf(t)$ *for each* $|t| \geq s$.

Then, put $\gamma := k(p)^{\frac{1}{p}}$ *and* $\lambda^* := \frac{1}{F(\gamma)|\Omega|}$, *for each* $\lambda \in]0, \lambda^*[$ *the problem*

$$\begin{cases} -\left(1 + \int_\Omega |\nabla u|^p\right) \Delta_p u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.5}$$

admits at least two nonzero weak solutions.

Now, we give an example to one of our theorem.

Example 1.5 *Example to Theorem 1.1. Let* $p = 6, N = 4$ *and* $D = 1$. *By* $meas(B(0,R)) = \frac{\pi^2}{2}R^4$ *we have* $l_1 = \frac{15\pi^2}{2^5} + \frac{15^2\pi^4}{2^{11}} = 15.328$. *By the Talenti's inequality* $k < \frac{1}{4^{1/6}\sqrt[2]{\pi}} 2^{1/4}\left(\frac{5}{2}\right)^{\frac{6}{5}}\left(\frac{\pi^2}{2}\right)^{\frac{1}{12}} = 1.8266$ *we choose* $k = 1$. *Let* $F(x,\xi) = F(\xi)$ *be an increasing function for* $\xi \geq 0, F(0) = 0$ *and* $F^\alpha(\xi) = F(\xi)\frac{\pi^2}{2}$. *Then, the inequality in* $(f1)$ *reduces to* $\frac{15.328 F(\gamma) 2^5}{\gamma^6} < F(\delta) < F(\gamma)$, *which implies* $2.808 = (15.3282^5)^{1/6} < \gamma$. *So, we can choose* $\delta = 2, \gamma = 3, F(2) = 2.5, F(3) = 3$.

Example to Theorem 1.4. Let $f : \mathbb{R} \to [0, +\infty[$ *be the function defined as follows:*

$$f(t) = \mu|t|^\kappa + |t|^q$$

with $\mu > 0$ *and* $0 < \kappa < 2p - 1 < q$. *Put* $\gamma := k(p)^{\frac{1}{p}}$ *and*

$$\mu^* := \frac{\kappa + 1}{\gamma^{\kappa+1}\left(\frac{1}{|\Omega|} - \frac{\gamma^{q+1}}{q+1}\right)}$$

owing to Theorem 1.4, for each $\mu \in]0, \mu^*[$ *the problem*

$$\begin{cases} -\left(1 + \int_\Omega |\nabla u|^p\right) \Delta_p u = \mu|t|^\kappa + |t|^q & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

admits at least two nonzero weak solutions.

The paper is organized as follows. In Section 2 we formulate the main results and the theorem of Bonanno and D'Agui [3]. In Section 3 we give the proofs of main results.

2. Variational framework

We introduce the functionals Φ, $\Psi : X \to \mathbb{R}$ defined by

$$\Phi(u) := \frac{1}{p}\|u\|^p + \frac{1}{2p}\|u\|^{2p}$$

and

$$\Psi(u) := \int_\Omega F(x,u)dx$$

for each $u \in X$ where $F(x,t) := \int_0^t f(x,\xi)d\xi$ for each $(x,t) \in \Omega \times \mathbb{R}$. By standard arguments, Φ and Ψ are in $C^1(X)$ and their Gâteaux derivatives are respectively

$$\langle \Phi'(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx + \int_\Omega |\nabla u|^p dx \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx$$

and

$$\langle \Psi'(u), v\rangle = \int_\Omega f(x,u) v dx$$

for each u, $v \in X$. In particular, for each $\lambda > 0$, the critical points of the functional

$$I_\lambda := \Phi - \lambda \Psi,$$

are weak solutions for problem (1.1).

In [10], Ricceri obtained a three critical points theorem and in [11] gave a general version of the theorem to extend the results for a class of more extensive equations. Later Bonanno and D'Aguì [3] developed Ricceri's result. In order to obtain solutions for the problem (1.1), we use a result obtained by Bonanno and D'Aguì in [3] which combines a local minimum theorem established in [2] with the mountain pass theorem.

Theorem 2.1 *Let X be a real Banach space and let Φ, $\Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_{x\in X}\Phi(x) = \Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\bar{x} \in X$, with $0 < \Phi(\bar{x}) < r$, such that:*

(a_1) $\frac{\sup_{\Phi(x)\leq r}\Psi(u)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})}$;

(a_2) *for each $\lambda \in \Lambda_r :=]\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\leq r}\Psi(u)}[$ the functional $I_\lambda : \Phi - \lambda\Psi$ satisfies the (PS)-condition and it is unbounded from below.*

Then, for each $\lambda \in \Lambda_r$, the functional I_λ admits at least two nonzero critical points $u_{\lambda,1}$, $u_{\lambda,2}$ such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$.

3. Proofs of main results

First we prove Theorem 1.1 and show that under Ambrosetti-Rabinowitz condition (f3) the functional I_λ satisfies Palais-Smale (PS) condition.

Lemma 3.1 *Assume (f3) satisfies, then functional I_λ satisfies the (PS)-condition and it is unbounded from below.*

Proof Let $\{u_n\}$ is a (PS)-sequence, that is,

$$\{I_\lambda(u_n)\} \text{ is bounded in } \mathbb{R} \text{ and } I'_\lambda(u_n) \to 0 \text{ in } X^*.$$

Fixed $1 > 0$, there exists $n_1 \in \mathbb{N}$, such that $\forall n \geq n_1$

$$\|I'_\lambda(u_n)\|_* \leq 1.$$

This implies that, $\forall n \geq n_1$

$$|\langle I'_\lambda(u_n), u_n\rangle| \leq \|I'_\lambda(u_n)\|_* \|u_n\| \leq \|u_n\|.$$

that is

$$-\|u_n\| \leq \langle I'_\lambda(u_n), u_n\rangle \leq \|u_n\|.$$

In particular, $\forall n \geq n_1$, we have

$$m^{-1}\|u_n\| \geq -m^{-1}\langle I'_\lambda(u_n), u_n\rangle.$$

Now, using the previous inequality, one has

$$\begin{aligned} I_\lambda(u_n) - m^{-1}\langle I'_\lambda(u_n), u_n\rangle &= \left(\frac{1}{p} - \frac{1}{m}\right)\|u\|^p + \left(\frac{1}{2p} - \frac{1}{m}\right)\|u\|^{2p} - \lambda \int_\Omega \left[\frac{1}{m} f(x, u_n)u_n - F(x, u_n)\right] dx \\ &= \left(\frac{1}{p} - \frac{1}{m}\right)\|u\|^p + \left(\frac{1}{2p} - \frac{1}{m}\right)\|u\|^{2p} \\ &\quad - \lambda \int_{|u_n|<s} \left[\frac{1}{m} f(x, u_n)u_n - F(x, u_n)\right] dx \\ &\quad - \lambda \int_{|u_n|\geq s} \left[\frac{1}{m} f(x, u_n)u_n - F(x, u_n)\right] dx, \end{aligned}$$

where s is defined in (f3). Due to f is continuous. One has

$$\begin{aligned} \left| \int_{|u_n|<s} \left[\frac{1}{m} f(x, u_n)u_n - F(x, u_n)\right] dx \right| &\leq \int_{|u_n|<s} \left|\frac{1}{m} f(x, u_n)u_n - F(x, u_n)\right| dx \\ &\leq \int_{|u_n|<s} \frac{1}{m}|f(x, u_n)||u_n| dx + \int_{|u_n|<s} |F(x, u_n)| dx \\ &\leq \frac{1}{m} s \int_{|u_n|<s} \max_{|\xi|\leq s} |f(x, \xi)| dx + \int_{|u_n|<s} \max_{|\xi|\leq s} |F(x, \xi)| dx \\ &:= A \in [0. + \infty[. \end{aligned}$$

This also implies that

$$-A < \int_{|u_n|<s} \left[\frac{1}{m} f(x, u_n)u_n - F(x, u_n)\right] dx < A.$$

Furthermore, $\{I_\lambda(u_n)\}$ is bounded, there exists $M > 0$, $n_2 \in \mathbb{N}$, $\forall n \geq n_2$

$$|I_\lambda(u_n)| \leq M.$$

Let $\bar{n} := \max\{n_1, n_2\}$. Then by (f3) for all $n \geq \bar{n}$, it follows

$$\begin{aligned} M + \frac{1}{m}\|u_n\| &\geq I_\lambda(u_n) - \frac{1}{m}\langle I'_\lambda(u_n), u_n\rangle \\ &= \left(\frac{1}{p} - \frac{1}{m}\right)\|u\|^p + \left(\frac{1}{2p} - \frac{1}{m}\right)\|u\|^{2p} - \lambda \int_\Omega \left[\frac{1}{m} f(x, u_n)u_n - F(x, u_n)\right] dx \\ &\geq \left(\frac{1}{p} - \frac{1}{m}\right)\|u\|^p + \left(\frac{1}{2p} - \frac{1}{m}\right)\|u\|^{2p} - \lambda A. \end{aligned}$$

So, we get $\{u_n\}$ is bounded on X.

We have the last task to do, that is prove $\{u_n\}$ is convergence in X. Since $\{u_n\}$ is bounded, up to a subsequence, one has

$$u_n \rightharpoonup u \text{ in } X$$

$$u_n \to u \text{ in } C(\bar{\Omega})$$

that is Ψ' is compact. We have

$$\int_\Omega f(x,u_n)(u_n-u)dx \to 0 \text{ as } n\to\infty.$$

Finally, we prove $\|u_n-u\|\to 0$ in X. In fact

$$\begin{aligned}(1+\|u_n\|^p)&\int_\Omega \left\langle |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u, \nabla u_n-\nabla u\right\rangle dx\\ &= \langle I'_\lambda(u_n), u_n-u\rangle + \lambda\int_\Omega f(x,u_n)(u_n-u)dx\\ &-(1+\|u_n\|^p)\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla u_n-\nabla u\rangle dx.\end{aligned}$$

In view of $u_n\rightharpoonup u$, we have

$$\int_\Omega |\nabla u|^{p-2}\langle\nabla u,\nabla u_n-\nabla u\rangle dx\to 0 \quad \text{as } n\to\infty.$$

Thus

$$(1+\|u_n\|^p)\int_\Omega \left\langle |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u, \nabla u_n-\nabla u\right\rangle dx\to 0 \quad \text{as } n\to\infty.$$

Using the standard inequality given by

$$\left\langle |x|^{p-2}x-|y|^{p-2}y, x-y\right\rangle \geq C_p|x-y|^p \quad \text{if } p\geq 2$$

or

$$\left\langle |x|^{p-2}x-|y|^{p-2}y, x-y\right\rangle \geq \frac{C_p|x-y|^2}{(|x|+|y|)^{2-p}} \quad \text{if } 2>p>1,$$

we obtain

$$C_p\int_\Omega |\nabla u_n-\nabla u|^p dx \leq (1+\|u_n\|^p)\int_\Omega \left\langle |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u, \nabla u_n-\nabla u\right\rangle dx.$$

For $p\geq 2$ the strong convergence is clear, that is, $\|u_n-u\|\to 0$ in X.

For $1<p<2$ one can use that

$$B_n := \langle |\nabla u_n|^{p-2}|\nabla u_n| - |\nabla u|^{p-2}|\nabla u|, \nabla u_n-\nabla u\rangle \to 0,$$

as $n\to\infty$ and by Hölder inequality

$$B_n \geq (\|u_n\|^{p-1}-\|u\|^{p-1})(\|u_n\|-\|u\|)\geq 0$$

which implies that $\|u_n\| \to \|u\|$, as $n \to \infty$. Then by uniform convexity of X and $u_n \rightharpoonup u$ weakly follows that $u_n \to u$ strongly in X.

Last, we show the functional $\Phi - \lambda\Psi$ is unbounded from below. Let

$$k(t) = t^{-m}F(x,tu) - F(x,u), \quad t \geq 1;$$

then, we get

$$k'(t) = t^{-m-1}(f(x,tu)tu - mF(x,tu)) \geq 0$$

for all $t \geq 1$ by (f3). Hence, it follows that $k(t) \geq k(1) = 0$ for all $t \geq 1$, that is

$$F(x,tu) \geq t^m F(x,u)$$

for all $x \in \mathbb{R}^N$, $|u| \geq s$ and $t \geq 1$. Thus, by $m > 2p$ we have

$$\begin{aligned} I_\lambda(tu) &= \frac{1}{p}\|tu\|^p + \frac{1}{2p}\|tu\|^{2p} - \lambda \int_\Omega F(x,tu)dx \\ &\leq \frac{t^p}{p}\|u\|^p + \frac{t^{2p}}{2p}\|u\|^{2p} - \lambda t^m \int_\Omega F(x,u)dx \\ &\to -\infty \end{aligned}$$

as $t \to +\infty$ for $u \in X$, $u \neq 0$. □

Proof [Proof of Theorem 1.1] We consider the space X and apply Theorem 2.1 to the functionals Φ, Ψ defined above by choosing $r = \frac{\gamma^p}{pk^p}$. Lemma 3.1 implies that, for each $\lambda > 0$, the functional $I_\lambda : \Phi - \lambda\Psi$ satisfies the (PS)-condition and it is unbounded from below. Fix $\lambda \in \Lambda_{\delta,\gamma}$, and the properties of the functionals Φ and Ψ ensure that the functional $I_\lambda = \Phi - \lambda\Psi$ verifies the regularities requested in Theorem 2.1. We denote by $\bar{v}$ the function of X defined by,

$$\bar{v}(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0, D), \\ \frac{2\delta}{D}(D - |x - x_0|) & \text{if } x \in B(x_0, D) \setminus B(x_0, D/2), \\ \delta & \text{if } x \in B(x_0, D/2). \end{cases} \tag{3.1}$$

One has

$$\begin{aligned} \Phi(\bar{v}) =& \frac{1}{p}\|\bar{v}\|^2 + \frac{1}{2p}\|\bar{v}\|^{2p} \\ =& \frac{1}{p}\int_\Omega |\nabla\bar{v}(x)|^p dx + \frac{1}{2p}\left(\int_\Omega |\nabla\bar{v}(x)|^p dx\right)^2 \\ =& \frac{1}{p}\int_{B(x_0,D)\setminus B(x_0,D/2)} \frac{(2\delta)^p}{D^p} dx + \frac{1}{2p}\left(\int_{B(x_0,D)\setminus B(x_0,D/2)} \frac{(2\delta)^p}{D^p} dx\right)^2 \\ =& \frac{1}{p}\frac{(2\delta)^p}{D^p}[\text{meas}(B(x_0,D)) - \text{meas}(B(x_0,D/2))] + \frac{1}{2p}\frac{(2\delta)^{2p}}{D^{2p}}[\text{meas}(B(x_0,D)) - \text{meas}(B(x_0,D/2))]^2 \\ =& \frac{(2\delta)^p\pi^{N/2}(D^N - (D/2)^N)}{2pD^{2p}(\Gamma(1+N/2))^2}\left[2D^p\Gamma(1+N/2) + (2\delta)^p\pi^{N/2}\left(D^N - (D/2)^N\right)\right]. \end{aligned}$$

Therefore, one has

$$\Phi(\bar{v}) \leq \frac{\max\{1,(2\delta)^{2p}\}}{p} l_D. \tag{3.2}$$

Taking into account that $\bar{v}(x) \in [0,\delta]$ for each $x \in \Omega$, condition (f2) ensures that

$$\Psi(\bar{v}) = \int_\Omega F(x,\bar{v}(x))dx \geq \int_{B(x_0,D/2)} F(x,\delta)dx.$$

It follows that

$$\frac{\Psi(\bar{v})}{\Phi(\bar{v})} \geq \frac{p\int_{B(x_0,D/2)} F(x,\delta)dx}{\max\{1,(2\delta)^{2p}\} l_D}. \tag{3.3}$$

For each $u \in \Phi^{-1}(]-\infty,r])$ one has

$$\|u\| \leq (pr)^{1/p}$$

and so, thanks to (1.2),

$$\|u\|_{C^0(\bar{\Omega})} \leq k(pr)^{1/p} = \gamma.$$

Moreover, we have

$$\Psi(u) = \int_\Omega F(x,u(x))dx \leq F^\gamma$$

for each $u \in \Phi^{-1}(]-\infty,r[)$. This leads to

$$\frac{1}{r} \sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) \leq \frac{k^p p}{\gamma^p} F^\gamma. \tag{3.4}$$

Now, taking into account condition (f1) one has

$$\frac{k^p p}{\gamma^p} F^\gamma < \frac{p\int_{B(x_0,D/2) F(x,\delta)dx}}{l_D \max\{1,(2\delta)^{2p}\}} \leq \frac{\Psi(\bar{v})}{\Phi(\bar{v})}$$

and so condition (a1) of Theorem 2.1 is verified.

We observe that conditions $\delta < \gamma$ and (f1) ensure that

$$\max\{1,(2\delta)^{2p}\} l_D < \frac{\gamma^p}{k^p}.$$

In fact, if $\max\{1,(2\delta)^{2p}\} l_D \geq \frac{\gamma^p}{k^p}$, taking into account that $\max_{|\xi|\leq\gamma} F(x,\xi) \geq F(x,\delta)$ for each $x \in \Omega$, we obtain

$$\frac{F^\gamma}{\gamma^p} \geq \frac{F^\gamma}{k^p l_D \max\{1,(2\delta)^{2p}\}} \geq \frac{1}{k^p l_D} \frac{\int_{B(x_0,D/2)} F(x,\delta)dx}{\max\{1,(2\delta)^{2p}\}}$$

and this is absurd because of (f1). In this way condition $\Phi(\bar{v}) < r$ requested in Theorem 2.1 is satisfied.

Since $\lambda \in \Lambda_{\delta,\gamma} := \left] \frac{\max\{1,(2\delta)^{2p}\} l_D}{p\int_{B(x_0,D/2)} F(x,\delta)dx}, \frac{\gamma^p}{pk^p F^\gamma} \right[$, Theorem 2.1 guarantees the existence of at least two nonzero critical points for the functional I_λ which are nontrivial weak solutions of problem (1.1). □

Proof [Proof of Theorem 1.2] In order to obtain this result, fix $\lambda \in \Lambda_{\delta,\gamma}$, $g \in C^0(\overline{\Omega} \times \mathbb{R})$ verifying (g1) and (g2) and $\mu \in]0, \eta_{\lambda,g}[$ and we consider the space X and apply Theorem 2.1 to the functional Φ, defined as above and $\Psi_{\lambda,\mu}$ defined by

$$\Psi_{\lambda,\mu}(u) := \int_{\Omega} \Big[F(x,u(x)) + \frac{\mu}{\lambda} G(x,u(x)) \Big] \, dx.$$

for each $u \in X$.

By arguing as in proof of Theorem 1.1 and by choosing $r = \frac{\gamma^p}{pk^p}$ we have

$$\frac{1}{r} \sup_{u \in \Phi^{-1}(1-\infty,r])} \Psi_{\lambda,\mu}(u) \leq \frac{k^p p}{\gamma^p} F^\gamma + \frac{k^p p}{\gamma^p} \frac{\mu}{\lambda} G^\gamma \tag{3.5}$$

and

$$\begin{aligned} \frac{k^p p}{\gamma^p} F^\gamma + \frac{k^p p}{\gamma^p} \frac{\mu}{\lambda} G^\gamma &< \frac{k^p p}{\gamma^p} F^\gamma + \frac{k^p p}{\gamma^p} \frac{\eta_{\lambda,g}}{\lambda} G^\gamma \\ &= \frac{p \int_{B\left(x^0, \frac{D}{2}\right)} F(x,\delta) dx}{l_D \max\{1, (2\delta)^{2p}\}} \\ &\leq \frac{p \int_{B\left(x^0, \frac{D}{2}\right)} F(x,\delta) dx}{l_D \max\{1, (2\delta)^{2p}\}} + \frac{p\mu}{\lambda} \frac{\int_{B\left(x^0, \frac{D}{2}\right)} G(x,\delta) dx}{l_D \max\{1, (2\delta)^{2p}\}} \\ &\leq \frac{\Psi_{\lambda,\mu}(\overline{v})}{\Phi(\overline{v})} \end{aligned}$$

where $\bar{v}$ is the test function introduced in (3.1). Moreover arguing as in Lemma 3.1, conditions (f3) and (g2) imply that, for each $\lambda > 0$ and $\mu > 0$, the functional

$$I_{\lambda,\mu} := \Phi - \lambda \Psi_{\lambda,\mu}$$

satisfies the (PS)-condition and it is unbounded from below. At this point Theorem 2.1 provides the existence of at least two nonzero critical points for the functional $I_{\lambda,\mu}$ which are weak solutions for problem (1.3). □

Proof [Proof of Theorem 1.3] Fix $\lambda \in]0, \lambda^*[$, and because of assumption (f2'), there exists $\delta \in]0, \left(\frac{p}{l_D}\right)^{\frac{1}{p}}[$ such that

$$\frac{\inf_{x\in\Omega} F(x,\delta)}{\max\{1,(2\delta)^{2p}\}} > \frac{l_D}{pm\left(B\left(x_0, \frac{D}{2}\right)\right)\lambda}. \tag{3.6}$$

We define X, Φ, Ψ and $\bar{v}$ as in Theorem 1.1 and apply Theorem 2.1 by choosing $r = 1$. Taking into account (3.6), we obtain

$$\sup_{u \in \Phi^{-1}(1-\infty,1)} \Psi(u) \leq F^\gamma < \frac{1}{\lambda} < \frac{p \int_{B\left(x^0, \frac{D}{2}\right)} F(x,\delta) dx}{l_D \max\{1, (2\delta)^{2p}\}} \leq \frac{\Psi_{\lambda,\mu}(\overline{v})}{\Phi(\overline{v})}.$$

Moreover we observe that condition $\delta \in]0, \left(\frac{p}{l_D}\right)^{\frac{1}{p}}[$ leads to $\Phi(\overline{v}) < 1$.

Since all the assumptions of Theorem 2.1 are verified the functional $I_\lambda = \Phi - \lambda\Psi$ admits at least two nonzero critical points which are nontrivial weak solutions of problem (1.1). □

Proof [Proof of Theorem 1.4] Because of nonnegativity of f the function F is increasing and so condition (f1') of Theorem 1.3 is satisfied for each $t \geq 0$. Moreover since $F'(t) = f(t)$, (H1) implies that

$$\lim_{t\to 0^+} \frac{F(t)}{t^p} = \lim_{t\to 0^+} \frac{f(t)}{pt^{p-1}} = +\infty.$$

Taking into account the fact that $F^\gamma = F(\gamma)|\Omega|$, the conclusion follows from Theorem 1.3. □

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References

[1] Autuori G, Colasuonno F, Pucci P. On the existence of stationary solutions for higher-order p-Kirchhoff problems. Communications in Contemporary Mathematics 2014; 16 (5): 1450002. doi: 10.1142/S0219199714500023

[2] Bonanno G. A critical point theorem via the Ekeland variational principle. Nonlinear Analysis 2012; 75 (5): 2992-3007. doi: 10.1016/j.na.2011.12.003

[3] Bonanno G, D'AguìG. Two nonzero solutions for elliptic Dirichlet problems. Zeitschrift für Analysis und ihre Anwendungen 2016; 35 (4): 449-464. doi: 10.4171/ZAA/1573

[4] Chung NT. Existence of positive solutions for a nonlocal problem with dependence on the gradient. Applied Mathematics Letters 2015; 41: 28-34. doi: 10.1016/j.aml.2014.10.011

[5] Corrêa FJSA, Figueiredo GM. On an elliptic equation of p-Kirchhoff type via variational methods. Bulletin of the Australian Mathematical Society 2006; 74 (2): 263-277. doi: 10.1017/S000497270003570X

[6] Corrêa FJSA, Figueiredo GM. On a p-Kirchhoff equation via Krasnoselskii's genus. Applied Mathematics Letters 2009; 22 (6): 819-822. doi: 10.1016/j.aml.2008.06.042

[7] Han W, Yao J. The sign-changing solutions for a class of p-Laplacian Kirchhoff type problem in bounded domains. Computers & Mathematics with Applications 2018; 76 (7): 1779-1790. doi: 10.1016/j.camwa.2018.07.029

[8] Lions JL. On some questions in boundary value problems of mathematical physics. Contemporary developments in continuum mechanics and partial differential equations. North-Holland, Amsterdam-New York, 1978, pp. 284-346

[9] Ourraoui A. On a p-Kirchhoff problem involving a critical nonlinearity. Comptes Rendus Mathematique 2014; 352 (4): 295–298. doi: 10.1016/j.crma.2014.01.015

[10] Ricceri B. On a three critical points theorem. Archiv der Mathematik 2000; 75 (3): 220-226. doi: 10.1007/s000130050496

[11] Ricceri B. A three critical points theorem revisited. Nonlinear Analysis 2009; 70 (9): 3084-3089. doi: 10.1016/j.na.2008.04.010

[12] Talenti G. Best constant in Sobolev inequality. Annali di Matematica Pura ed Applicata (1923 -) 1976; 110: 353-372. doi: 10.1007/BF02418013