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A sequential fractional differential problem of pantograph type: existence uniqueness and illustrations

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Abstract: In this study, a new class of sequential fractional differential problems of pantograph type is introduced. New existence and uniqueness criteria for the existence and uniqueness of solutions are discussed. Some existence results using Darbo's fixed point and measure of noncompactness are also studied. At the end, two illustrative examples are discussed.

Key words: Pantograph equation, Caputo and phi Caputo derivatives, sequential differential equation

1. Introduction

Mathematical models involving differential equations of classical or arbitrary order have been used significantly for describing many phenomena in engineering and scientific disciplines, such as physics, biophysics, chemistry, biology, electrodynamics, viscosity elasticity, see for example the research works in $[1-4, 6-8, 11, 15, 17, 19, 22, 23, 30, 34]$. The research works $[26, 29, 33, 37-39, 40]$ deal also with some applications.

The pantograph problem is one of the classical models. It is considered as a class of delay differential equations in which the derivative of the function, at any time, depends on the solution at previous time. Recently, an attention to the pantograph equations has considered $[9, 12, 16, 21, 29]$ due to their applications in modeling numerous processes of real world problems. For example, in $[10]$, it has been proposed a stage structured model of population growth. Then, the proposed model has been employed to study how the electric current is collected by the pantograph of an electric locomotive, see $[36]$. In the same sense, in $[18]$, a discretization of the following general pantograph equation has been investigated:

$$\begin{cases} y'(t) = ay(t) + by(\theta(t)) + cy(\varphi(t)), t \geq 0, \\ y(0) = y_0, \end{cases}$$

where a, b, c, y_0 are real numbers, θ and φ are strictly increasing functions on the nonnegative reals, with $\theta(0) = \varphi(0) = 0$, $\theta(t) < t$ and $\varphi(t) < t, t > 0$.

We cite also the paper $[27]$ where K. Guan et al. have studied the oscillatory behavior of solutions of the following pantograph problem:

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$$\begin{cases} x'(t) = P(t)\,x(t) - Q(t)\,x(\alpha t)\,, t \geq t_0, t = t_k, \\ x(t_k) = b_k x(t_k)\,, k = 1, 2, ..., \end{cases}$$

where $0 < \alpha < 1$ and $0 < t_0 < t_1 < ... < t_k < ...$ are fixed points, $P(t), Q(t) \in C([t_0, \infty), [0, \infty))$.

In [26], the authors have addressed and studied the following fractional pantograph equation:

$$\begin{cases} {}^cD^{\alpha}_{0+}\left(u(s) - P(s)\,u(\beta s)\right)(t) = f(t, u(t), u(\gamma t))\,, t \in [0, T]\,, \\ u(0) = u_0, \\ u'(0) = u_1, \end{cases}$$

where $1 < \alpha \leq 2$, ${}^cD^{\alpha}_{0+}$ is the Caputo derivative of order α, $0 < \beta, \gamma < 1$ and f, P are two functions that satisfy some imposed conditions.

Very recently, in [28] using the Ψ-Hilfer derivative, it has been investigated the existence and uniqueness as well as the stability for the following nonlinear neutral pantograph equation:

$$\begin{cases} D^{\alpha,\beta,\Psi} u(t) = g\left(t, u(t), u(\kappa t), D^{\alpha,\beta} u(\kappa t)\right), t \in J = [a, b]\,, \\ I^{1-\gamma,\Psi} u(a) = u_a, \end{cases}$$

where $D^{\alpha,\beta,\Psi}$ is the Ψ-Hilfer derivative of order $\alpha, 0 < \alpha < 1$ and $I^{1-\gamma,\Psi}$ is the Ψ-integral of order $1 - \gamma\,(\gamma = \alpha + \beta - \alpha\beta)$.

In [25], Fazli et al. have discussed the existence and uniqueness result for the fractional problem:

$$\begin{cases} D^{\beta}\left(D^{\alpha} x(t) + \lambda\right) = f(t, x(t))\,, 0 < t \leq 1, \\ x^{(i)}(0) = \mu_i, 0 \leq i < l, \\ x^{(i+\alpha)}(0) = \nu_i, 0 \leq i < n, \end{cases}$$

where $m - 1 < \alpha \leq m$, $n - 1 < \beta \leq n$, $l = \max(n, m)$, $m, n \in \mathbb{N}$, D^{α} is the Caputo derivative, $x(t)$ is the particle displacement, $\lambda \in \mathbb{R}$ is the friction coefficient and f is a noise term.

In the present work, in general, we are concerned with the study of a sequential pantograph problem involving the $\Phi-$Caputo derivatives [13]. The importance of this $\Phi-$theory is in its applications in real word phenomena [14, 16]. Also, the advantage of the $\Phi-$Caputo fractional derivative is its flexibility to combine all fractional derivatives introduced before (like for instance, Caputo, Hadamard, Hadamard–Caputo, and Caputo–Katugampola derivatives). The $\Phi-$Caputo operator is also important since it possesses the semigroup property which is crucial to obtain the structure of solutions.
So, in this work, we shall study the following $\Phi-$Caputo sequential pantograph fractional differential problem with integral conditions:

$$\begin{cases} {}^cD^{\beta,\Phi}\left({}^cD^{\alpha,\Phi} x(t) + g(t, x(t))\right) = f\left(t, x(t), x(\lambda t), {}^cD^{\alpha,\Phi} x(t)\right), t \in [0, 1]\,, \\ x(0) = 0, \\ x(1) = \int_0^1 h(s, x(s))\,ds. \end{cases} \tag{1.1}$$

We take into account the conditions that ${}^cD^{\alpha,\Phi}, {}^cD^{\beta,\Phi}$ are the $\Phi-$Caputo derivatives, such that $0 < \alpha, \beta \leq 1$, $f \in C([0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $g, h \in C([0, 1] \times \mathbb{R}, \mathbb{R})$, $\lambda \in \mathbb{R}^+$.

It is to note that the problem (1.1) is important since, in one hand, it is more general than the above cited pantograph differential problems, and on the other hand, it is considered as a special type of delay differential problems with integral conditions. Such equations have various applications in chemical engineering, electrodynamics, underground water flow, plasma physics and population dynamics, see [5, 38]. In this sense, it has been confirmed that models of epidemics that lead to delay equations often have integral conditions that are imposed by the interpretation of these models. The neglect of these integral conditions may lead to solutions that behave in a radically different manner from solutions restricted to obey them, see [20].

Our investigation for the problem (1.1) is based on the application of Darbo's theorem. This investigation has two motivational reasons: the first one is the fact that Darbo's theorem extends both Schauder and Banach fixed point theorems, so it is better to apply Darbo's theorem instead of Schauder or Banach theorems. The second reason that motivates our application of Darbo's theorem is the abundance, by mathematicians, of this important theorem in proving the existence of solutions for a wide class of differential and integral equations. So, we fell motivated to present a contribution in this sense to fill the void and the lack in this filed of interest.

To the best of our knowledge, there are no papers devoted to the study of $\Phi-$Caputo sequential pantograph differential equations with integral boundary conditions using the techniques of measure of noncompactness.

The remainder of this paper is organized as follows: In Section 2, some preliminaries are presented. In Section 3, the main existence results for problem (1.1) are established, and in the fourth section, we give two examples to illustrate our results.

2. Preliminaries

2.1. Caputo derivatives

Definition 2.1 *The Riemann–Liouville fractional integral of order $\alpha > 0$, for a continuous function f on $[0,\infty)$ is defined as:*

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau; \alpha > 0, t > 0. \tag{2.1}$$

$$J^0 f(t) = f(t).$$

Definition 2.2 *The Caputo derivative of order α of $f \in C^n([0,\infty[)$ is defined as:*

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau, n-1 < \alpha \leq n, n \in N^*.$$

For more details, one can see the references [8, 17, 32, 35].

2.2. Φ-Caputo derivatives

In this section, we recall some notations, definitions and results of $\Phi-$Caputo derivatives [1, 9, 12, 13, 14, 30].

Definition 2.3 *Suppose $(0,b] \subset \mathbb{R}_+$ is a finite or infinite interval. Let $f \in L^1(0,b]$ and $\Phi(t) > 0$ be a monotone function on $(a,b]$ such that $\Phi \in C^n[(0,b],\mathbb{R}]$. Then, the operator*

$$J^{\alpha,\Phi} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \Phi'(\tau) (\Phi(t) - \Phi(\tau))^{\alpha-1} f(\tau)\, d\tau, n-1 < \alpha \leq n, n \in N^*, \tag{2.2}$$

is called the left sided Φ*-Riemann–Liouville integral of order* α *of* f *with respect to* Φ*.*

Definition 2.4 *Suppose* $(0,b] \subset \mathbb{R}_+$ *is a finite or infinite interval. Let also* $f \in L^1(0,b]$ *and* $\Phi(t) > 0$ *be a monotone function on* $(a,b]$ *such that* $\Phi(t) \in C^n[(a,b],\mathbb{R}]$*. Then the operator*

$$D^{\alpha,\Phi} f(t) = \left[\frac{1}{\Phi'(t)}\frac{d}{dt}\right]^n J^{n-\alpha,\Phi} f(t), n-1 < \alpha \leq n, n \in N^*,$$

is called the left sided Φ*-Riemann–Liouville derivative of order* α *of* f *with respect to* Φ*.*

Definition 2.5 *Let* $n-1 < \alpha \leq n$*,* $f \in C^n[(0,b],\mathbb{R}]$*. The left-sided* Φ*-Caputo fractional derivative of* f *of order* α *is determined as:*

$$\begin{aligned} {}^cD^{\alpha,\Phi} f(t) &= J^{n-\alpha,\Phi}\left[\Theta_\Phi^{[n]} f\right](t), n-1 < \alpha \leq n, n \in N^*, \\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^t \Phi'(\tau)(\Phi(t)-\Phi(\tau))^{n-\alpha-1}\left[\Theta_\Phi^{[n]} f\right](\tau)\, d\tau, n-1 < \alpha \leq n, n \in N^*, \end{aligned} \tag{2.3}$$

where, $\Theta_\Phi = \frac{1}{\Phi'(t)}\frac{d}{dt}$ *and* $\Theta_\Phi^{[n]} = \underbrace{\Theta_\Phi\Theta_\Phi\ldots\Theta_\Phi}_{n-times}$*.*

We given also the following lemma:

Lemma 2.6 *Let* $\alpha > 0$ *and* $f : [0,b] \to \mathbb{R}$*. Then, we have:*

$${}^cD^{\alpha,\Phi} J^{\alpha,\Phi} f(t) = f(t), f \in C[0,b],$$

and

$$J^{\alpha,\Phi\,c} D^{\alpha,\Phi} f(t) = f(t) - \sum_{k=0}^{n-1} c_k (\Phi(t) - \Phi(0))^k,$$

where $f \in C^{n-1}[0,b], c_k = \frac{\Theta_\Phi^{[k]}(0)}{k!}$*.*

2.3. Measures of noncompactness

In this section, we present some results about measures of noncompactness.So, let $\mathbb{R}$ be the set of real numbers. Let also E be a real Banach space and $B(x,r)$ denotes the closed ball centered at x with radius r. The symbol B_r stands for the ball $B(0,r)$. For X a nonempty subset of E, we denote by $\overline{X}$ and $ConvX$ the closure and the closed convex hull of X, respectively. Furthermore, let us denote by $\wp_E$ the family of nonempty bounded subsets of E and by $\Re_E$ its subfamily consisting of all relatively compact subsets of E. For a given set W of functions $\varpi : [0,1]$, let us denote $W(t) = \{\varpi(t) : \varpi \in W\}, t \in [0,1]$ and $W([0,1]) = \{\varpi(t) : t \in [0,1]\}$.
Next, we recall the definition of the measure of noncompactness and some auxiliary results. For more details, see [24, 31, 41] and the references therein.

Definition 2.7 *The Kuratowski measure of noncompactness* μ_E *over the subset* X *of a Banach space* E *is given by*

$$\mu_E(X) = \inf\{\varepsilon > 0 : X \subseteq \cup_{i=1}^n X_i \text{ and } diam(X_i) \leq \varepsilon\}, \tag{2.4}$$

where

$$diam(X_i) = \sup\{\|x-y\| : x, y \in X_i\}.$$

We also present to the reader the following Darbo's theorem [24]:

Theorem 2.8 *Let X be a Banach space and C be a bounded, closed, convex and nonempty subset of X. Suppose that the continuous mapping $\Upsilon : C \to C$ is a μ_E-contraction. Then Υ has a fixed point in C.*

3. Main results

Let $E := \left\{x : x \in C([0,1]), {}^c D^{\alpha,\Phi}x \in C([0,1])\right\}$ be the Banach space endowed with the norm

$$\|x\|_E = \|x\| + \left\|{}^c D^{\alpha,\Phi}x\right\|,$$

where $\|x\| = \max_{t\in[0,1]}|x(t)|$ and $\left\|\left\|{}^c D^{\alpha,\Phi}x\right\|\right\| = \max_{t\in[0,1]}\left|\left\|{}^c D^{\alpha,\Phi}x\right\|\right|$.

Using E, we introduce the following hypotheses:

($\mathbf{H}_1$) The function f is continuous and there are two positive constants L_f, M_f satisfying

$$\left\|f\left(t,x,\tilde{x},{}^c D^{\alpha,\Phi}x\right) - f\left(t,y,\tilde{y},{}^c D^{\alpha,\Phi}y\right)\right\| \leq L_f\left(\|x-y\| + \|\tilde{x}-\tilde{y}\|\right) + M_f\left\|{}^c D^{\alpha,\Phi}x - {}^c D^{\alpha,\Phi}y\right\|.$$

($\mathbf{H}_2$) The function g is continuous and there exists a constant $L_g > 0$, such that

$$\|g(t,x) - g(t,y)\| \leq L_g\|x-y\|.$$

($\mathbf{H}_3$) The function h is continuous, and there exists a constant $L_h > 0$, such that

$$\|h(t,x) - h(t,y)\| \leq L_h\|x-y\|.$$

3.1. The pantograph integral representation

In this section, we present to the reader the following first result.

Lemma 3.1 *Let $y \in C([0,1])$. Then the problem*

$$\begin{aligned} {}^c D^{\beta,\Phi}\left({}^c D^{\alpha,\Phi}x(t) + z(t)\right) &= y(t), t \in [0,1], \\ x(0) &= 0, \\ x(1) &= \int_0^1 h(s,x(s))\,ds \end{aligned} \tag{3.1}$$

has a unique integral representation which is given by the expression:

$$\begin{aligned} x(t) &= \int_0^t \frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\alpha+\beta-1}y(\tau)}{\Gamma(\alpha+\beta)}d\tau - \int_0^t \frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\alpha-1}z(\tau)}{\Gamma(\alpha)}d\tau \\ &+ \left(\frac{(\Phi(t)-\Phi(0))}{(\Phi(1)-\Phi(0))}\right)^{\alpha}\left(\int_0^1 h(s,x(s))\,ds - \int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha+\beta-1}y(\tau)}{\Gamma(\alpha+\beta)}d\tau\right. \\ &\left.+ \int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha-1}z(\tau)}{\Gamma(\alpha)}d\tau\right). \end{aligned} \tag{3.2}$$

Proof By Lemma 2.6, we can reduce (1.1) to the equivalent equation:

$$^{c}D^{\alpha,\Phi}x(t) = J^{\beta,\Phi}y(t) - z(t) + c_0, c_0 \in \mathbb{R}. \tag{3.3}$$

Again, taking the integral operator $J^{\alpha,\Phi}$ on both sides of (3.3), we get

$$x(t) = J^{\alpha+\beta,\Phi}y(t) - J^{\alpha,\Phi}z(t) + c_0\frac{(\Phi(t)-\Phi(0))^{\alpha}}{\Gamma(\alpha+1)} + c_1, c_0, c_1 \in \mathbb{R}.$$

Moreover, for $t \in [0,1]$, using the fact that $x(0) = 0$, we find:

$$c_1 = 0.$$

Also, since $x(1) = \int\limits_0^1 h(s, x(s))\,ds,$ then one can obtain

$$c_0 = \left(\frac{\Gamma(\alpha+1)}{(\Phi(1)-\Phi(0))^{\alpha}}\right)\left(\int\limits_0^1 h(s,x(s))\,ds - J^{\alpha+\beta,\Phi}y(1) + J^{\alpha,\Phi}z(1)\right).$$

Consequently,

$$x(t) = J^{\alpha+\beta,\Phi}y(t) - J^{\alpha,\Phi}z(t) + \left(\frac{(\Phi(t)-\Phi(0))}{(\Phi(1)-\Phi(0))}\right)^{\alpha}\left(\int\limits_0^1 h(s,x(s))\,ds - J^{\alpha+\beta,\Phi}y(1) + J^{\alpha,\Phi}z(1)\right),$$

which allows us to obtain the desired result. □

3.2. One pantograph solution via BCP principle

We begin this section, by defining the integral operator $\Im : E \to E$ by the following expression:

$$\begin{aligned}
\Im x(t) \quad = \quad & \int_0^t \frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\alpha+\beta-1} f\left(\tau, x(\tau), x(\lambda\tau), ^{c}D^{\alpha,\Phi}x(\tau)\right)}{\Gamma(\alpha+\beta)}d\tau \\
& - \int_0^t \frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\alpha-1} g(\tau, x(\tau))}{\Gamma(\alpha)}d\tau \\
& + \left(\frac{(\Phi(t)-\Phi(0))}{(\Phi(1)-\Phi(0))}\right)^{\alpha}(\int\limits_0^1 h(s,x(s))\,ds \\
& - \int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha+\beta-1} f\left(\tau, x(\tau), x(\lambda\tau), ^{c}D^{\alpha,\Phi}x(\tau)\right)}{\Gamma(\alpha+\beta)}d\tau \\
& + \int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha-1} g(\tau, x(\tau))}{\Gamma(\alpha)}d\tau).
\end{aligned} \tag{3.4}$$

Then, based on Banach contraction principle (BCP for short) [22, 23], we prove the following first main result.

Theorem 3.2 *Let* $(\mathbf{H}_1)-(\mathbf{H}_3)$ *hold. Suppose also that*

$$\Omega := (4L_f + 2M_f)\,\Delta_1 + 2L_g\Delta_2 + L_h\Delta_3 < 1. \tag{3.5}$$

Then, (1.1) *has a unique solution on* $[0,1]$.

Proof Let $t \in [0,1]$. So, we can write

$$\begin{aligned}
&\left|\Im x(t) - \Im y(t)\right| \\
= &\left|\int_0^t \frac{\Phi'(\tau)\left(\Phi(t)-\Phi(\tau)\right)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\right. \\
&\times\left[f\left(\tau, x(\tau), x(\lambda\tau), {}^cD^{\alpha,\Phi}x(\tau)\right) - f\left(\tau, y(\tau), y(\lambda\tau), {}^cD^{\alpha,\Phi}y(\tau)\right)\right]d\tau \\
&-\int_0^t \frac{\Phi'(\tau)\left(\Phi(t)-\Phi(\tau)\right)^{\alpha-1}\left[g(\tau,x(\tau)) - g(\tau,y(\tau))\right]}{\Gamma(\alpha)}d\tau \\
&+\left(\frac{(\Phi(t)-\Phi(0))}{(\Phi(1)-\Phi(0))}\right)^{\alpha} \times \Big(\int_0^1 \left[h(s,x(s)) - h(s,y(s))\right]ds \\
&-\int_0^1 \frac{\Phi'(\tau)\left(\Phi(1)-\Phi(\tau)\right)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \\
&\times\left[f\left(\tau, x(\tau), x(\lambda\tau), {}^cD^{\alpha,\Phi}x(\tau)\right) - f\left(\tau, y(\tau), y(\lambda\tau), {}^cD^{\alpha,\Phi}y(\tau)\right)\right]d\tau \\
&\left.+\int_0^1 \frac{\Phi'(\tau)\left(\Phi(1)-\Phi(\tau)\right)^{\alpha-1}\left[g(\tau,x(\tau)) - g(\tau,y(\tau))\right]}{\Gamma(\alpha)}d\tau\Big)\right|.
\end{aligned}$$

Using the Lipschitz assumption of f and the two hypotheses $(\mathbf{H}_2)-(\mathbf{H}_3)$, it yields that

$$\begin{aligned}
\left\|\Im x - \Im y\right\| \leq\; & \left(2L_f(\|x-y\|) + M_f\left\|{}^cD^{\alpha,\Phi}x - {}^cD^{\alpha,\Phi}y\right\|\right)\int_0^1 \frac{\Phi'(\tau)\left(\Phi(1)-\Phi(\tau)\right)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}d\tau \\
&+L_g\|x-y\|\int_0^t \frac{\Phi'(\tau)\left(\Phi(t)-\Phi(\tau)\right)^{\alpha-1}}{\Gamma(\alpha)}d\tau + L_h\|x-y\| \\
&+\left(2L_f(\|x-y\|) + M_f\left\|{}^cD^{\alpha,\Phi}x - {}^cD^{\alpha,\Phi}y\right\|\right)\int_0^1 \frac{\Phi'(\tau)\left(\Phi(1)-\Phi(\tau)\right)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}d\tau,
\end{aligned}$$

we obtain

$$\begin{aligned}
\left\|\Im x - \Im y\right\| \leq\; & \left(\frac{4L_f\left(\Phi(1)-\Phi(0)\right)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{2L_g\left(\Phi(1)-\Phi(0)\right)^{\alpha}}{\Gamma(\alpha+1)} + L_h\right)\|x-y\| \\
&+\left(\frac{2M_f\left(\Phi(1)-\Phi(0)\right)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right)\left\|{}^cD^{\alpha,\Phi}x - {}^cD^{\alpha,\Phi}y\right\|.
\end{aligned} \tag{3.6}$$

On other hand, we have

$$
\begin{aligned}
{}^{c}D^{\alpha,\Phi}\Im x(t) &= \int_0^t \frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\beta-1} f\left(\tau, x(\tau), x(\lambda\tau), {}^{c}D^{\alpha,\Phi}x(\tau)\right)}{\Gamma(\beta)} d\tau - g(t,x(t)) \\
&+ \left(\frac{\Gamma(\alpha+1)}{(\Phi(1)-\Phi(0))^{\alpha}}\right) \times \Big(\int_0^1 h(s,x(s))\,ds \\
&- \int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha+\beta-1} f\left(\tau, x(\tau), x(\lambda\tau), {}^{c}D^{\alpha,\Phi}x(\tau)\right)}{\Gamma(\alpha+\beta)} d\tau \\
&+ \int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha-1} g(\tau,x(\tau))}{\Gamma(\alpha)} d\tau \Big),
\end{aligned}
$$

and

$$
\begin{aligned}
\left|{}^{c}D^{\alpha,\Phi}\Im x(t) - {}^{c}D^{\alpha,\Phi}\Im y(t)\right| &\leq \int_0^t \frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\beta-1}}{\Gamma(\beta)} \\
&\quad \left| f\left(\tau, x(\tau), x(\lambda\tau), {}^{c}D^{\alpha,\Phi}x(\tau)\right) - f\left(\tau, y(\tau), y(\lambda\tau), {}^{c}D^{\alpha,\Phi}y(\tau)\right)\right| d\tau \\
&\quad + |g(t,x(t)) - g(t,y(t))| + \left(\frac{\Gamma(\alpha+1)}{(\Phi(1)-\Phi(0))^{\alpha}}\right) \Big(\int_0^1 |h(s,x(s)) - h(s,y(s))|\,ds \\
&\quad + \int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \\
&\quad \left| f\left(\tau, x(\tau), x(\lambda\tau), {}^{c}D^{\alpha,\Phi}x(\tau)\right) - f\left(\tau, y(\tau), y(\lambda\tau), {}^{c}D^{\alpha,\Phi}y(\tau)\right)\right| d\tau \\
&\quad + \int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha-1} |g(\tau,x(\tau)) - g(\tau,y(\tau))|}{\Gamma(\alpha)} d\tau \Big).
\end{aligned}
$$

Thanks to $(\mathbf{H}_1)-(\mathbf{H}_3)$, we get

$$
\begin{aligned}
&\left\|{}^{c}D^{\alpha,\Phi}\Im x - {}^{c}D^{\alpha,\Phi}\Im y\right\| \leq \left(2L_f(\|x-y\|) + M_f\left\|{}^{c}D^{\alpha,\Phi}x - {}^{c}D^{\alpha,\Phi}y\right\|\right) \int_0^t \frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\beta-1}}{\Gamma(\beta)} d\tau \\
&+ L_g\|x-y\| + \left(\frac{\Gamma(\alpha+1)}{(\Phi(1)-\Phi(0))^{\alpha}}\right) \times \\
&\left(L_h\|x-y\| + \left(2L_f(\|x-y\|) + M_f\left\|{}^{c}D^{\alpha,\Phi}x - {}^{c}D^{\alpha,\Phi}y\right\|\right) \int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} d\tau\right. \\
&\left. + L_g\|x-y\| \int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha-1}}{\Gamma(\alpha)} d\tau\right).
\end{aligned}
\tag{3.7}
$$

Consequently,

$$\begin{aligned} \left\| {}^cD^{\alpha,\Phi}\Im x - {}^c D^{\alpha,\Phi}\Im y \right\| &\leq \left(\frac{4L_f (\Phi(1)-\Phi(0))^{\beta}}{\Gamma(\beta+1)} + 2L_g + \left(\frac{\Gamma(\alpha+1)}{(\Phi(1)-\Phi(0))^{\alpha}} \right) L_h \right) \|x-y\| \\ &\quad + \frac{2M_f (\Phi(1)-\Phi(0))^{\beta}}{\Gamma(\beta+1)} \left\| {}^cD^{\alpha,\Phi}x - {}^c D^{\alpha,\Phi} y \right\|. \end{aligned} \tag{3.8}$$

Finally, from (3.6) and (3.8), it is easy to see that

$$\begin{aligned} \|\Im x - \Im y\|_E &= \max\left\{ \|\Im x - \Im y\| + \left\| {}^cD^{\alpha,\Phi}\Im x - {}^c D^{\alpha,\Phi}\Im y \right\| \right\} \\ &\leq \left(\frac{4L_f (\Phi(1)-\Phi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{2L_g (\Phi(1)-\Phi(0))^{\alpha}}{\Gamma(\alpha+1)} + L_h \right. \\ &\quad \left. + \frac{4L_f (\Phi(1)-\Phi(0))^{\beta}}{\Gamma(\beta+1)} + 2L_g + \left(\frac{\Gamma(\alpha+1)}{(\Phi(1)-\Phi(0))^{\alpha}} \right) L_h \right) + \|x-y\| \\ &\quad + \left(\frac{2M_f (\Phi(1)-\Phi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{2M_f (\Phi(1)-\Phi(0))^{\beta}}{\Gamma(\beta+1)} \right) \left\| {}^cD^{\alpha,\Phi}x - {}^c D^{\alpha,\Phi} y \right\|, \\ &\leq \left((4L_f + 2M_f) \left(\frac{(\Phi(1)-\Phi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{(\Phi(1)-\Phi(0))^{\beta}}{\Gamma(\beta+1)} \right) \right. \\ &\quad \left. + (2L_g) \left(\frac{(\Phi(1)-\Phi(0))^{\alpha} + \Gamma(\alpha+1)}{\Gamma(\alpha+1)} \right) + L_h \left(\frac{\Gamma(\alpha+1) + (\Phi(1)-\Phi(0))^{\alpha}}{(\Phi(1)-\Phi(0))^{\alpha}} \right) \right) \|x-y\|_E, \\ &\leq \left((4L_f + 2M_f)\Delta_1 + 2L_g\Delta_2 + L_h\Delta_3 \right) \|x-y\|_E, \end{aligned}$$

where

$$\Delta_1 = \frac{(\Phi(1)-\Phi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{(\Phi(1)-\Phi(0))^{\beta}}{\Gamma(\beta+1)},$$

$$\Delta_2 = \frac{(\Phi(1)-\Phi(0))^{\alpha} + \Gamma(\alpha+1)}{\Gamma(\alpha+1)}$$

and

$$\Delta_3 = \frac{\Gamma(\alpha+1) + (\Phi(1)-\Phi(0))^{\alpha}}{(\Phi(1)-\Phi(0))^{\alpha}}.$$

Since $\Omega < 1$, then $\Im$ is contraction mapping. Hence, by the BCP principle, we state that $\Im$ has a unique fixed point which is the unique solution of (1.1). □

3.3. One pantograph solution via BCP principle and Holder inequality

The following main result deals with the existence of a unique solution of the studied problem by using both the BCP principle and Holder inequality [6, 34]. To prove that result, we need the following hypothesis:

($\mathbf{H}_4$) The function f is continuous, and there exists a function φ, such that

$$\begin{aligned}&\left\|f\left(t,x,\tilde{x},{}^cD^{\alpha,\Phi}x\right)-f\left(t,y,\tilde{y},{}^cD^{\alpha,\Phi}y\right)\right\|\\ \leq\ &\varphi(t)\left(\|x-y\|+\|\tilde{x}-\tilde{y}\|+\left\|{}^cD^{\alpha,\Phi}x-{}^cD^{\alpha,\Phi}y\right\|\right),\end{aligned}$$

where $t\in[0,1]\,,x,y\in E,\varphi\in L^{\frac{1}{p}}\left([0,1]\,,\mathbb{R}^+\right),p\in(0,1)$ and $\|\varphi\|=\left(\int_0^1(\varphi(\tau))^{\frac{1}{p}}\,d\tau\right)^p$.

Theorem 3.3 *If the hypotheses* $(\mathbf{H}_2),(\mathbf{H}_3)$ *and* $(\mathbf{H}_4)$ *are satisfied and*

$$F:=\frac{6\|\varphi\|(\Phi(1)-\Phi(0))^{\alpha+\beta-p}}{\Gamma(\alpha+\beta)}\left(\frac{1-p}{\alpha+\beta-p}\right)^{1-p}+\frac{6\|\varphi\|(\Phi(1)-\Phi(0))^{\beta-p}}{\Gamma(\beta)}\left(\frac{1-p}{\beta-p}\right)^{1-p}+2L_g\Delta_2+L_h\Delta_3<1, \tag{3.9}$$

then, the problem (1.1) *has a unique solution on* $[0,1]$.

Proof Let $t\in[0,1]$. Then, we have

$$\begin{aligned}|\Im x(t)-\Im y(t)|\leq&\int_0^t\frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\\ &\times\left|f\left(\tau,x(\tau),x(\lambda\tau),{}^cD^{\alpha,\Phi}x(\tau)\right)-f\left(\tau,y(\tau),y(\lambda\tau),{}^cD^{\alpha,\Phi}y(\tau)\right)\right|d\tau\\ &+\int_0^t\frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\alpha-1}|g(\tau,x(\tau))-g(\tau,y(\tau))|}{\Gamma(\alpha)}d\tau\\ &+\left(\frac{(\Phi(t)-\Phi(0))}{(\Phi(1)-\Phi(0))}\right)^{\alpha}\Big(\int_0^1|h(s,x(s))-h(s,y(s))|\,ds\\ &+\int_0^1\frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\\ &\times\left|f\left(\tau,x(\tau),x(\lambda\tau),{}^cD^{\alpha,\Phi}x(\tau)\right)-f\left(\tau,y(\tau),y(\lambda\tau),{}^cD^{\alpha,\Phi}y(\tau)\right)\right|d\tau\\ &+\int_0^1\frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha-1}|g(\tau,x(\tau))-g(\tau,y(\tau))|}{\Gamma(\alpha)}d\tau\Big).\end{aligned}$$

Using $(\mathbf{H}_2)-(\mathbf{H}_3)-(\mathbf{H}_4)$, we can write

$$\begin{aligned}
&\left|\Im x(t)-\Im y(t)\right| \\
\leq\ & \left(2\|x-y\|+\left\|{}^cD^{\alpha,\Phi}x-{}^cD^{\alpha,\Phi}y\right\|\right)\int_0^t \frac{\Phi'(\tau)\left(\Phi(t)-\Phi(\tau)\right)^{\alpha+\beta-1}\varphi(\tau)}{\Gamma(\alpha+\beta)}d\tau \\
&+L_g\|x-y\|\int_0^t \frac{\Phi'(\tau)\left(\Phi(t)-\Phi(\tau)\right)^{\alpha-1}}{\Gamma(\alpha)}d\tau+L_g\|x-y\| \\
&+\left(2\|x-y\|+\left\|{}^cD^{\alpha,\Phi}x-{}^cD^{\alpha,\Phi}y\right\|\right)\int_0^1 \frac{\Phi'(\tau)\left(\Phi(1)-\Phi(\tau)\right)^{\alpha+\beta-1}\varphi(\tau)}{\Gamma(\alpha+\beta)}d\tau \\
&+L_g\|x-y\|\int_0^1 \frac{\Phi'(\tau)\left(\Phi(1)-\Phi(\tau)\right)^{\alpha-1}}{\Gamma(\alpha)}d\tau.
\end{aligned}$$

Thanks to Holder inequality, it yields that

$$\begin{aligned}
\|\Im x-\Im y\| \leq\ & \frac{\left(2\|x-y\|+\left\|{}^cD^{\alpha,\Phi}x-{}^cD^{\alpha,\Phi}y\right\|\right)}{\Gamma(\alpha+\beta)}\left(\int_0^t\left(\Phi'(\tau)\left(\Phi(t)-\Phi(\tau)\right)^{\alpha+\beta-1}\right)^{\frac{1}{1-p}}d\tau\right)^{1-p} \\
&\times\left(\int_0^t\left(\varphi(\tau)\right)^{\frac{1}{p}}d\tau\right)^p \\
&+\frac{2L_g\left(\Phi(1)-\Phi(0)\right)^{\alpha}}{\Gamma(\alpha+1)}\|x-y\|+L_h\|x-y\| \qquad (3.10)\\
&+\frac{\left(2\|x-y\|+\left\|{}^cD^{\alpha,\Phi}x-{}^cD^{\alpha,\Phi}y\right\|\right)}{\Gamma(\alpha+\beta)}\left(\int_0^1\left(\Phi'(\tau)\left(\Phi(1)-\Phi(\tau)\right)^{\alpha+\beta-1}\right)^{\frac{1}{1-p}}d\tau\right)^{1-p} \\
&\times\left(\int_0^t\left(\varphi(\tau)\right)^{\frac{1}{p}}d\tau\right)^p.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\Im x-\Im y\| \leq\ & \frac{\left(2\|x-y\|+\left\|{}^cD^{\alpha,\Phi}x-{}^cD^{\alpha,\Phi}y\right\|\right)\left(\Phi(1)-\Phi(0)\right)^{\alpha+\beta-p}}{\Gamma(\alpha+\beta)}\left(\frac{1-p}{\alpha+\beta-p}\right)^{1-p}\|\varphi\| \\
&+\frac{2L_g\left(\Phi(1)-\Phi(0)\right)^{\alpha}}{\Gamma(\alpha+1)}\|x-y\|+L_h\|x-y\| \qquad (3.11)\\
&\frac{\left(2\|x-y\|+\left\|{}^cD^{\alpha,\Phi}x-{}^cD^{\alpha,\Phi}y\right\|\right)\left(\Phi(1)-\Phi(0)\right)^{\alpha+\beta-p}}{\Gamma(\alpha+\beta)}\left(\frac{1-p}{\alpha+\beta-p}\right)^{1-p}\|\varphi\|.
\end{aligned}$$

On other hand, thanks to $(\mathbf{H}_2)-(\mathbf{H}_3)-(\mathbf{H}_4)$ and using the same arguments as in the proof of Theorem

2, we can write

$$\begin{aligned}
& \left\|{}^{c}D^{\alpha,\Phi}\Im x-{}^{c}D^{\alpha,\Phi}\Im y\right\| \\
\leq\ & \left(2\left\|x-y\right\|+\left\|{}^{c}D^{\alpha,\Phi}x-{}^{c}D^{\alpha,\Phi}y\right\|\right)\int_{0}^{t}\frac{\Phi'(\tau)\left(\Phi(t)-\Phi(\tau)\right)^{\beta-1}}{\Gamma(\beta)}\varphi(\tau)\,d\tau \\
& +L_{g}\left\|x-y\right\|+\left(\frac{\Gamma(\alpha+1)}{\left(\Phi(1)-\Phi(0)\right)^{\alpha}}\right)(L_{h}\left\|x-y\right\|+\left(2\left\|x-y\right\|+{}^{c}\left\|D^{\alpha,\Phi}x-{}^{c}D^{\alpha,\Phi}y\right\|\right)\times \\
& \int_{0}^{1}\frac{\Phi'(\tau)\left(\Phi(1)-\Phi(\tau)\right)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\varphi(\tau)\,d\tau+L_{g}\left\|x-y\right\|\int_{0}^{1}\frac{\Phi'(\tau)\left(\Phi(1)-\Phi(\tau)\right)^{\alpha-1}}{\Gamma(\alpha)}d\tau).
\end{aligned}$$

The Holder inequality allows us to obtain

$$\begin{aligned}
& \left\|{}^{c}D^{\alpha,\Phi}\Im x-{}^{c}D^{\alpha,\Phi}\Im y\right\|\leq\left(2\left\|x-y\right\|+\left\|{}^{c}D^{\alpha,\Phi}x-{}^{c}D^{\alpha,\Phi}y\right\|\right) \\
& \times\frac{1}{\Gamma(\beta)}\left(\int_{0}^{t}\left(\Phi'(\tau)\left(\Phi(t)-\Phi(\tau)\right)^{\beta-1}\right)^{\frac{1}{1-p}}d\tau\right)^{1-p}\left(\int_{0}^{t}\left(\varphi(\tau)\right)^{\frac{1}{p}}d\tau\right)^{p} \\
& +L_{g}\left\|x-y\right\|+\left(\frac{\Gamma(\alpha+1)}{\left(\Phi(1)-\Phi(0)\right)^{\alpha}}\right)(L_{h}\left\|x-y\right\|+\left(2\left\|x-y\right\|+\left\|{}^{c}D^{\alpha,\Phi}x-{}^{c}D^{\alpha,\Phi}y\right\|\right) \\
& \times\frac{1}{\Gamma(\alpha+\beta)}\left(\int_{0}^{1}\left(\Phi'(\tau)\left(\Phi(1)-\Phi(\tau)\right)^{\alpha+\beta-1}\right)^{\frac{1}{1-p}}d\tau\right)^{1-p}\left(\int_{0}^{1}\left(\varphi(\tau)\right)^{\frac{1}{p}}d\tau\right)^{p} \\
& +L_{g}\left\|x-y\right\|\int_{0}^{1}\frac{\Phi'(\tau)\left(\Phi(1)-\Phi(\tau)\right)^{\alpha-1}}{\Gamma(\alpha)}d\tau).
\end{aligned}$$

Therefore, we can state that

$$\begin{aligned}
\left\|{}^{c}D^{\alpha,\Phi}\Im x-{}^{c}D^{\alpha,\Phi}\Im y\right\| \leq\ & \left(2\left\|x-y\right\|+\left\|{}^{c}D^{\alpha,\Phi}x-{}^{c}D^{\alpha,\Phi}y\right\|\right)\frac{\left(\Phi(1)-\Phi(0)\right)^{\beta-p}}{\Gamma(\beta)}\left(\frac{1-p}{\beta-p}\right)^{1-p}\left\|\varphi\right\| \\
& +L_{g}\left\|x-y\right\|+\left(\frac{\Gamma(\alpha+1)}{\left(\Phi(1)-\Phi(0)\right)^{\alpha}}\right)(L_{h}\left\|x-y\right\| \\
& -\left(2\left\|x-y\right\|+\left\|{}^{c}D^{\alpha,\Phi}x-{}^{c}D^{\alpha,\Phi}y\right\|\right)\frac{\left(\Phi(1)-\Phi(0)\right)^{\alpha+\beta-p}}{\Gamma(\alpha+\beta)}\left(\frac{1-p}{\alpha+\beta-p}\right)^{1-p}\left\|\varphi\right\| \\
& +\frac{L_{g}\left(\Phi(1)-\Phi(0)\right)^{\alpha}}{\Gamma(\alpha+1)}\left\|x-y\right\|).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\left\|{}^{c}D^{\alpha,\Phi}\Im x-{}^{c}D^{\alpha,\Phi}\Im y\right\| \leq\ & \left(2\left\|x-y\right\|+\left\|{}^{c}D^{\alpha,\Phi}x-{}^{c}D^{\alpha,\Phi}y\right\|\right)\frac{2\left\|\varphi\right\|\left(\Phi(1)-\Phi(0)\right)^{\beta-p}}{\Gamma(\beta)}\left(\frac{1-p}{\beta-p}\right)^{1-p} \\
& +2L_{g}\left\|x-y\right\|+\left(\frac{\Gamma(\alpha+1)}{\left(\Phi(1)-\Phi(0)\right)^{\alpha}}\right)L_{h}\left\|x-y\right\|.
\end{aligned} \tag{3.12}$$

In view (3.11) and (3.12), we have

$$\begin{aligned}\|\Im x-\Im y\|_{E} \leq & \left(\frac{4\left(\Phi\left(1\right)-\Phi\left(0\right)\right)^{\alpha+\beta-p}}{\Gamma\left(\alpha+\beta\right)}\left(\frac{1-p}{\alpha+\beta-p}\right)^{1-p}\|\varphi\|+\frac{2L_{g}\left(\Phi\left(1\right)-\Phi\left(0\right)\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}+L_{h}\right)\|x-y\| \\ & +\frac{2\left(\Phi\left(1\right)-\Phi\left(0\right)\right)^{\alpha+\beta-p}}{\Gamma\left(\alpha+\beta\right)}\left(\frac{1-p}{\alpha+\beta-p}\right)^{1-p}\|\varphi\|\left\|{}^{c}D^{\alpha,\Phi}x-{}^{c}D^{\alpha,\Phi}y\right\| \\ & +\left(\frac{4\|\varphi\|\left(\Phi\left(1\right)-\Phi\left(0\right)\right)^{\beta-p}}{\Gamma\left(\beta\right)}\left(\frac{1-p}{\beta-p}\right)^{1-p}+2L_{g}+\left(\frac{\Gamma\left(\alpha+1\right)}{\left(\Phi\left(1\right)-\Phi\left(0\right)\right)^{\alpha}}\right)L_{h}\right)\|x-y\| \\ & +\frac{2\|\varphi\|\left(\Phi\left(1\right)-\Phi\left(0\right)\right)^{\beta-p}}{\Gamma\left(\beta\right)}\left(\frac{1-p}{\beta-p}\right)^{1-p}\left\|{}^{c}D^{\alpha,\Phi}x-{}^{c}D^{\alpha,\Phi}y\right\|, \\ \leq & \left(\frac{6\|\varphi\|\left(\Phi\left(1\right)-\Phi\left(0\right)\right)^{\alpha+\beta-p}}{\Gamma\left(\alpha+\beta\right)}\left(\frac{1-p}{\alpha+\beta-p}\right)^{1-p}+\frac{6\|\varphi\|\left(\Phi\left(1\right)-\Phi\left(0\right)\right)^{\beta-p}}{\Gamma\left(\beta\right)}\left(\frac{1-p}{\beta-p}\right)^{1-p}\right. \\ & \left.+\left(\left(\Phi\left(1\right)-\Phi\left(0\right)\right)^{\alpha}+\Gamma\left(\alpha+1\right)\right)\left(\frac{2L_{g}}{\Gamma\left(\alpha+1\right)}+\frac{L_{h}}{\left(\Phi\left(1\right)-\Phi\left(0\right)\right)^{\alpha}}\right)\right)\|x-y\|_{E}.\end{aligned}$$

Hence, $\Im$ is a contraction since we have already seen that $\digamma<1$.

By the BCP principle, we confirm that $\Im$ has a unique fixed point, which is the unique solution of (1.1). □

3.4. A solution via Darbo's theorem

Now, we prove an existence result for the problem (1.1) by Kuratowski MNC and Darbo's fixed point theorem. We have:

Theorem 3.4 *Suppose that* $(\mathbf{H}_1)-(\mathbf{H}_3)$ *are valid. Then, the problem* (1.1) *has at least one solution on* $[0,1]$.

Proof Let ϱ be a positive constant. We consider the set defined by: $B_{\varrho}=\{x\in E:\|x\|_{E}\leq\varrho\}$ and let

$$\sup_{t\in[0,1]}\left|f\left(t,0,0,0\right)\right|:=N_{f}<\infty,$$

$$\sup_{t\in[0,1]}\left|g\left(t,0\right)\right|:=N_{g}<\infty,$$

and

$$\sup_{t\in[0,1]}\left|h\left(t,0\right)\right|:=N_{h}<\infty.$$

The set B_{ϱ} is a closed, bounded and convex of the Banach space E. The proof will be developed as follows:

Claim 1: We prove that $\Im(B_\varrho)$ is bounded for any bounded set B_ϱ.
For $x \in B_\varrho$, and $t \in [0,1]$, we have

$$\begin{aligned}
|\Im x(t)| \leq & \int_0^t \frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\alpha+\beta-1}\left|f\left(\tau, x(\tau), x(\lambda\tau), {}^c D^{\alpha,\Phi}x(\tau)\right)\right|}{\Gamma(\alpha+\beta)}d\tau \\
& +\int_0^t \frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\alpha-1}|g(\tau,x(\tau))|}{\Gamma(\alpha)}d\tau \\
& +\left(\frac{(\Phi(t)-\Phi(0))}{(\Phi(1)-\Phi(0))}\right)^{\alpha}(\int_0^1 |h(s,x(s))|\,ds \\
& +\int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha+\beta-1}\left|f\left(\tau, x(\tau), x(\lambda\tau), {}^c D^{\alpha,\Phi}x(\tau)\right)\right|}{\Gamma(\alpha+\beta)}d\tau \\
& +\int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha-1}|g(\tau,x(\tau))|}{\Gamma(\alpha)}d\tau).
\end{aligned}$$

Thus,

$$\begin{aligned}
|\Im x(t)| \leq & \int_0^t \frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \\
& \times\left|f\left(\tau, x(\tau), x(\lambda\tau), {}^c D^{\alpha,\Phi}x(\tau)\right)-f(\tau,0,0,0)\right|+|f(\tau,0,0,0)|\,d\tau \\
& +\int_0^t \frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\alpha-1}|g(\tau,x(\tau))-g(\tau,0)|+|g(\tau,0)|}{\Gamma(\alpha)}d\tau \\
& +\left(\frac{(\Phi(t)-\Phi(0))}{(\Phi(1)-\Phi(0))}\right)^{\alpha}(\int_0^1 |h(s,x(s))-h(s,0)|+|h(s,0)|\,ds \\
& +\int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \\
& \times\left|f\left(\tau, x(\tau), x(\lambda\tau), {}^c D^{\alpha,\Phi}x(\tau)\right)-f(\tau,0,0,0)\right|+|f(\tau,0,0,0)|\,d\tau \\
& +\int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha-1}|g(\tau,x(\tau))-g(\tau,0)|+|g(\tau,0)|}{\Gamma(\alpha)}d\tau).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\Im x\| \leq & \frac{2(\Phi(1)-\Phi(0))^{\alpha+\beta}\left(2L_f\|x\|+M_f\left\|{}^c D^{\alpha,\Phi}x\right\|+N_f\right)}{\Gamma(\alpha+\beta+1)} \\
& +\frac{2(\Phi(1)-\Phi(0))^{\alpha}(L_g\|x\|+N_g)}{\Gamma(\alpha+1)}+L_h\|x\|+N_h.
\end{aligned} \tag{3.13}$$

On other hand, in view of $(\mathbf{H}_1)-(\mathbf{H}_3)$, we can state that

$$
\begin{aligned}
\left|{}^cD^{\alpha,\Phi}\Im x(t)\right| \leq\ & \int_0^t \frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\beta-1}}{\Gamma(\beta)} \\
& \times\left|f\left(\tau,x(\tau),x(\lambda\tau),{}^cD^{\alpha,\Phi}x(\tau)\right)-f(\tau,0,0,0)\right|+|f(\tau,0,0,0)|\,d\tau \\
& +|g(\tau,x(\tau))-g(\tau,0)|+|g(\tau,0)| \\
& +\left(\frac{\Gamma(\alpha+1)}{(\Phi(1)-\Phi(0))^{\alpha}}\right)\Big(\int_0^1 |h(s,x(s))-h(s,0)|+|h(s,0)|\,ds \qquad (3.14)\\
& +\int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \\
& \times\left|f\left(\tau,x(\tau),x(\lambda\tau),{}^cD^{\alpha,\Phi}x(\tau)\right)-f(\tau,0,0,0)\right|+|f(\tau,0,0,0)|\,d\tau \\
& +\int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha-1}|g(\tau,x(\tau))-g(\tau,0)|+|g(\tau,0)|}{\Gamma(\alpha)}d\tau\Big).
\end{aligned}
$$

Then, we obtain

$$
\begin{aligned}
\left\|{}^cD^{\alpha,\Phi}\Im x\right\| \leq\ & \frac{(\Phi(1)-\Phi(0))^{\beta}\left(2L_f\|x\|+M_f\left\|{}^cD^{\alpha,\Phi}x\right\|+N_f\right)}{\Gamma(\beta+1)}+L_g\|x\|+N_g+\left(\frac{\Gamma(\alpha+1)}{(\Phi(1)-\Phi(0))^{\alpha}}\right) \\
& \Big(L_h\|x\|+N_h+\frac{(\Phi(1)-\Phi(0))^{\alpha+\beta}\left(2L_f\|x\|+M_f\left\|{}^cD^{\alpha,\Phi}x\right\|+N_f\right)}{\Gamma(\alpha+\beta+1)} \qquad (3.15)\\
& +\frac{(\Phi(1)-\Phi(0))^{\alpha}(L_g\|x\|+N_g)}{\Gamma(\alpha+1)}\Big).
\end{aligned}
$$

Thus, (3.13) and (3.15) imply that,

$$
\begin{aligned}
\|\Im x\|_E \leq\ & \frac{2(\Phi(1)-\Phi(0))^{\alpha+\beta}\left(2L_f\|x\|+M_f\left\|{}^cD^{\alpha,\Phi}x\right\|+N_f\right)}{\Gamma(\alpha+\beta+1)} \\
& +\frac{2(\Phi(1)-\Phi(0))^{\alpha}(L_g\|x\|+N_g)}{\Gamma(\alpha+1)}+L_h\|x\|+N_h. \\
& \frac{(\Phi(1)-\Phi(0))^{\beta}\left(2L_f\|x\|+M_f\left\|{}^cD^{\alpha,\Phi}x\right\|+N_f\right)}{\Gamma(\beta+1)}+L_g\|x\| \\
& +N_g+\left(\frac{\Gamma(\alpha+1)}{(\Phi(1)-\Phi(0))^{\alpha}}\right)\Big(L_h\|x\|+N_h \\
& +\frac{(\Phi(1)-\Phi(0))^{\alpha+\beta}\left(2L_f\|x\|+M_f\left\|{}^cD^{\alpha,\Phi}x\right\|+N_f\right)}{\Gamma(\alpha+\beta+1)} \\
& +\frac{(\Phi(1)-\Phi(0))^{\alpha}(L_g\|x\|+N_g)}{\Gamma(\alpha+1)}\Big).
\end{aligned}
$$

This shows that

$$\begin{aligned}
\|\Im x\|_E &\le \Omega \|x\|_E + \frac{N_f (\Phi(1)-\Phi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + 2N_g \left(\frac{(\Phi(1)-\Phi(0))^{\alpha}}{\Gamma(\alpha+1)} + 1 \right) \\
&+ N_h \left(\frac{\Gamma(\alpha+1)}{(\Phi(1)-\Phi(0))^{\alpha}} + 1 \right), \\
&\le \Omega \varrho + \frac{N_f (\Phi(1)-\Phi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + 2N_g \left(\frac{(\Phi(1)-\Phi(0))^{\alpha}}{\Gamma(\alpha+1)} + 1 \right) \\
&+ N_h \left(\frac{\Gamma(\alpha+1)}{(\Phi(1)-\Phi(0))^{\alpha}} + 1 \right), \\
&\le \varrho,
\end{aligned}$$

where

$$\varrho \ge \frac{\frac{N_f (\Phi(1)-\Phi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + 2N_g \left(\frac{(\Phi(1)-\Phi(0))^{\alpha}}{\Gamma(\alpha+1)} + 1 \right) + N_h \left(\frac{\Gamma(\alpha+1)}{(\Phi(1)-\Phi(0))^{\alpha}} + 1 \right)}{1-\Omega}.$$

Consequently, $\Im(B\varrho)$ is bounded.

Claim 2: The application $\Im$ is continuous.

To do this, let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in $B\varrho$, such that $x_n \to x$ when $n \to \infty$. Then, for all $t \in [0,1]$, we have

$$\begin{aligned}
|\Im x_n(t) - \Im x(t)| &\le \int_0^t \frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \\
&\times \left| f\left(\tau, x_n(\tau), x_n(\lambda\tau), {}^c D^{\alpha,\Phi} x_n(\tau)\right) - f\left(\tau, x(\tau), x(\lambda\tau), {}^c D^{\alpha,\Phi} x(\tau)\right) \right| d\tau \\
&+ \int_0^t \frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\alpha-1} |g(\tau, x_n(\tau)) - g(\tau, x(\tau))|}{\Gamma(\alpha)} d\tau \\
&+ \left(\frac{(\Phi(t)-\Phi(0))}{(\Phi(1)-\Phi(0))} \right)^{\alpha} \\
&\times \Big(\int_0^1 |h(s, x_n(s)) - h(s, x(s))| \, ds \\
&+ \int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \\
&\times \left| f\left(\tau, x_n(\tau), x_n(\lambda\tau), {}^c D^{\alpha,\Phi} x_n(\tau)\right) - f\left(\tau, x(\tau), x(\lambda\tau), {}^c D^{\alpha,\Phi} x(\tau)\right) \right| d\tau \\
&+ \int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha-1}}{\Gamma(\alpha)} |g(\tau, x_n(\tau)) - g(\tau, x(\tau))| \, d\tau \Big),
\end{aligned}$$

Then

$$\begin{aligned}\left\|\Im x_n-\Im x\right\| &\leq \left(\frac{4L_f\left(\Phi(1)-\Phi(0)\right)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{2L_g\left(\Phi(1)-\Phi(0)\right)^{\alpha}}{\Gamma(\alpha+1)}+L_h\right)\left\|x_n-x\right\| \\ &+\left(\frac{2M_f\left(\Phi(1)-\Phi(0)\right)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right)\left\|{}^cD^{\alpha,\Phi}x_n-{}^cD^{\alpha,\Phi}x\right\|.\end{aligned} \tag{3.16}$$

Similarly, we have

$$\begin{aligned}\left\|{}^cD^{\alpha,\Phi}\Im x_n-{}^cD^{\alpha,\Phi}\Im x\right\| &\leq \left(\frac{4L_f\left(\Phi(1)-\Phi(0)\right)^{\beta}}{\Gamma(\beta+1)}+2L_g+\left(\frac{\Gamma(\alpha+1)}{\left(\Phi(1)-\Phi(0)\right)^{\alpha}}\right)L_h\right)\left\|x_x-x\right\| \\ &+\frac{2M_f\left(\Phi(1)-\Phi(0)\right)^{\beta}}{\Gamma(\beta+1)}\left\|{}^cD^{\alpha,\Phi}x_n-{}^cD^{\alpha,\Phi}x\right\|,\end{aligned} \tag{3.17}$$

(3.16) and (3.17) allows us to state that

$$\left\|\Im x_n-\Im x\right\|_E\leq\Omega\left\|x_x-x\right\|_E.$$

Therefore,

$$\left\|\Im x_n-\Im x\right\|_E\to 0, n\to\infty.$$

Hence, $\Im$ is a continuous operator over $B\varrho$.

Claim 3: We shall prove that $\Im(B\varrho)$ is equicontinuous.
Let $t_1,t_2\in[0,1];t_1<t_2,\ x,y,z\in B\varrho$, where

$$\sup_{t\in[0,1]}\left|f(t,x,y,z)\right|=\vartheta_f,$$

$$\sup_{t\in[0,1]}\left|g(t,x)\right|=\vartheta_g,$$

and

$$\sup_{t\in[0,1]}\left|h(t,x)\right|=\vartheta_h.$$

Then, we have

$$
\begin{aligned}
&\left|\Im x\left(t_1\right)-\Im x\left(t_2\right)\right| \\
\leq\ & \int_0^{t_1} \frac{\Phi^{\prime}(\tau)}{\Gamma(\alpha+\beta)} \\
&\times\left[\left(\Phi\left(t_1\right)-\Phi(\tau)\right)^{\alpha+\beta-1}-\left(\Phi\left(t_2\right)-\Phi(\tau)\right)^{\alpha+\beta-1}\right]\left|f\left(\tau, x(\tau), x(\lambda \tau),{ }^c D^{\alpha, \Phi} x(\tau)\right)\right| d \tau \\
&+\int_{t_1}^{t_2} \frac{\Phi^{\prime}(\tau)\left(\Phi\left(t_2\right)-\Phi(\tau)\right)^{\alpha+\beta-1}\left|f\left(\tau, x(\tau), x(\lambda \tau),{ }^c D^{\alpha, \Phi} x(\tau)\right)\right|}{\Gamma(\alpha+\beta)} d \tau \\
&+\int_0^{t_1} \frac{\Phi^{\prime}(\tau)\left[\left(\Phi\left(t_1\right)-\Phi(\tau)\right)^{\alpha-1}-\left(\Phi\left(t_2\right)-\Phi(\tau)\right)^{\alpha-1}\right]|g(\tau, x(\tau))|}{\Gamma(\alpha)} d \tau \\
&+\int_{t_1}^{t_2} \frac{\Phi^{\prime}(\tau)\left(\Phi\left(t_2\right)-\Phi(\tau)\right)^{\alpha-1}|g(\tau, x(\tau))|}{\Gamma(\alpha)} d \tau \\
&+\frac{\left(\Phi\left(t_1\right)-\Phi(0)\right)^\alpha-\left(\Phi\left(t_2\right)-\Phi(0)\right)^\alpha}{(\Phi(1)-\Phi(0))^\alpha} \\
&\times\Big(\int_0^1|h(s, x(s))| d s+\int_0^1 \frac{\Phi^{\prime}(\tau)(\Phi(1)-\Phi(\tau))^{\alpha+\beta-1}\left|f\left(\tau, x(\tau), x(\lambda \tau),{ }^c D^{\alpha, \Phi} x(\tau)\right)\right|}{\Gamma(\alpha+\beta)} d \tau \\
&+\int_0^1 \frac{\Phi^{\prime}(\tau)(\Phi(1)-\Phi(\tau))^{\alpha-1}|g(\tau, x(\tau))|}{\Gamma(\alpha)} d \tau\Big).
\end{aligned}
$$

Therefore, we can write

$$
\begin{aligned}
&\left|\Im x\left(t_1\right)-\Im x\left(t_2\right)\right| \leq \frac{\vartheta_f}{\Gamma(\alpha+\beta)} \int_0^{t_1} \Phi^{\prime}(\tau)\left[\left(\Phi\left(t_1\right)-\Phi(\tau)\right)^{\alpha+\beta-1}-\left(\Phi\left(t_2\right)-\Phi(\tau)\right)^{\alpha+\beta-1}\right] d \tau \\
&+\frac{\vartheta_f}{\Gamma(\alpha+\beta)} \int_{t_1}^{t_2} \Phi^{\prime}(\tau)\left(\Phi\left(t_2\right)-\Phi(\tau)\right)^{\alpha+\beta-1} d \tau \\
&+\frac{\vartheta g}{\Gamma(\alpha)} \int_0^{t_1} \Phi^{\prime}(\tau)\left[\left(\Phi\left(t_1\right)-\Phi(\tau)\right)^{\alpha-1}-\left(\Phi\left(t_2\right)-\Phi(\tau)\right)^{\alpha-1}\right] d \tau \\
&+\frac{\vartheta_g}{\Gamma(\alpha)} \int_{t_1}^{t_2} \Phi^{\prime}(\tau)\left(\Phi\left(t_2\right)-\Phi(\tau)\right)^{\alpha-1} d \tau \\
&+\left(\frac{\left(\Phi\left(t_1\right)-\Phi(0)\right)^\alpha-\left(\Phi\left(t_2\right)-\Phi(0)\right)^\alpha}{(\Phi(1)-\Phi(0))^\alpha}\right)\left(\vartheta_h+\frac{\vartheta_f(\Phi(1)-\Phi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{\vartheta_g(\Phi(1)-\Phi(0))^\alpha}{\Gamma(\alpha+1)}\right), \\
\leq\ & \frac{2 \vartheta_f}{\Gamma(\alpha+\beta+1)}\left(\Phi\left(t_2\right)-\Phi\left(t_1\right)\right)^{\alpha+\beta}+\frac{2 \vartheta_g}{\Gamma(\alpha+1)}\left(\Phi\left(t_2\right)-\Phi(0)\right)^\alpha \\
&+\left(\frac{\left(\Phi\left(t_1\right)-\Phi(0)\right)^\alpha-\left(\Phi\left(t_2\right)-\Phi(0)\right)^\alpha}{(\Phi(1)-\Phi(0))^\alpha}\right)\left(\vartheta_h+\frac{\vartheta_f(\Phi(1)-\Phi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{\vartheta_g(\Phi(1)-\Phi(0))^\alpha}{\Gamma(\alpha+1)}\right) .
\end{aligned}
$$

Hence,

$$|\Im x(t_1) - \Im x(t_2)| \to 0 \text{ as } t_1 \to t_2.$$

Also, we can state that

$$\begin{aligned} \left|{}^c D^{\alpha,\Phi}\Im x(t_1) - {}^c D^{\alpha,\Phi}\Im x(t_2)\right| &\leq \frac{\vartheta_f}{\Gamma(\beta)}\int_0^{t_1}\Phi'(\tau)\left[(\Phi(t_1)-\Phi(\tau))^{\beta-1} - (\Phi(t_2)-\Phi(\tau))^{\beta-1}\right]d\tau \\ &+\frac{\vartheta_f}{\Gamma(\beta)}\int_{t_1}^{t_2}\Phi'(\tau)(\Phi(t_2)-\Phi(\tau))^{\beta-1}d\tau + [g(t_1,x(t_1)) - g(t_2,x(t_2))]. \end{aligned}$$

By taking t_1 tends to t_2, then, the right-hand side of the last inequality tends to 0.

Consequently, $\Im(B\varrho)$ is equicontinuous.

Claim 4: We show that $\Im$ is a condensing operator.

Let $W \subset B\varrho$ and $t \in [0,1]$. So, we have:

$$\mu_E(\Im W(t)) = \mu_E(\Im x(t), x \in W),$$

where μ_E be the measure of noncompactness introduced on Definition 6.

Obviously,

$$\begin{aligned} \mu_E(\Im W(t)) &\leq \int_0^t \frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left\{\mu_E\left(f\left(\tau,x(\tau),x(\lambda\tau),{}^c D^{\alpha,\Phi}x(\tau)\right)\right), x \in W\right\}d\tau \\ &+\int_0^t \frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\{\mu_E(g(\tau,x(\tau))), x \in W\}d\tau + \\ &\left(\frac{(\Phi(t)-\Phi(0))}{(\Phi(1)-\Phi(0))}\right)^{\alpha}(\int_0^1\{\mu_E(h(\tau,x(\tau))), x \in W\}ds \\ &+\int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left\{\mu_E\left(f\left(\tau,x(\tau),x(\lambda\tau),{}^c D^{\alpha,\Phi}x(\tau)\right)\right), x \in W\right\}d\tau \\ &+\int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\{\mu_E(g(\tau,x(\tau))), x \in W\}d\tau)), \\ &\leq \Omega\{\mu_E(x(\tau)), x \in W, \tau \in [0,1]\}, \\ &\leq \Omega\{\mu_E(W(\tau)), \tau \in [0,1]\}. \end{aligned}$$

Finally, we can state that

$$\mu_E(\Im W) \leq \Omega\mu_E(W).$$

Therefore, the operator $\Im$ is a contraction.

By Darbo's fixed point theorem, the operator has a fixed point, which is a solution of (1.1). □

4. Illustrative examples

Example 4.1 *Let consider the following pantograph fractional problem:*

$$\left\{ \begin{array}{c} {}^{c}D^{\frac{3}{5},\Phi}\left({}^{c}D^{\frac{4}{5},\Phi}x(t)+g(t,x(t))\right)=f\left(t,x(t),x(\lambda t),{}^{c}D^{\frac{4}{5},\Phi}x(t)\right), t\in J, \\ x(0)=0, \\ x(1)=\int_0^1 h(t,x(s))\,ds. \end{array} \right. \tag{4.1}$$

Let us define $\Phi(t) := 2t^2+2t+2$.
In particular, Φ *is an increasing function on* $[0,1]$ *and* $\Phi'(t)$ *is continuous over* $[0,1]$.
By taking

$$f\left(t,x(t),x(\lambda t),{}^{c}D^{\frac{4}{5},\Phi}x(t)\right)=\frac{1}{55\exp(t^2+1)\left[1+\frac{\cos t}{(t^2+1)^2}+x(t)+x\left(\frac{2t}{5}\right)+{}^{c}D^{\frac{4}{5},\Phi}x(t)\right]},$$

$$g(t,x(t))=\frac{t^3-3}{100}x(t),$$

and

$$h(t,x(t))=\frac{1}{200}x(t),$$

we constat that

$$L_f=\frac{1}{101\exp(1)}, M_f=\frac{1}{101\exp(1)}, L_g=\frac{4}{100}, L_h=\frac{1}{200},$$

$$\begin{aligned} \Omega &= (4L_f+2M_f)\left(\frac{(\Phi(1)-\Phi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{(\Phi(1)-\Phi(0))^{\beta}}{\Gamma(\beta+1)}\right) \\ &\quad +(2L_g)\left(\frac{(\Phi(1)-\Phi(0))^{\alpha}+\Gamma(\alpha+1)}{\Gamma(\alpha+1)}\right)+L_h\left(\frac{\Gamma(\alpha+1)+(\Phi(1)-\Phi(0))^{\alpha}}{(\Phi(1)-\Phi(0))^{\alpha}}\right) \\ &= \left(4\times\frac{1}{101\exp(1)}+2\times\frac{1}{101\exp(1)}\right)\left(\frac{(4)^{\frac{7}{5}}}{\Gamma\left(\frac{12}{5}\right)}+\frac{(4)^{\frac{4}{5}}}{\Gamma\left(\frac{4}{5}+1\right)}\right) \\ &\quad +\left(\frac{8}{100}\right)\left(\frac{(4)^{\frac{3}{5}}+\Gamma\left(\frac{3}{5}+1\right)}{\Gamma\left(\frac{3}{5}+1\right)}\right)+\frac{1}{200}\left(\frac{(4)^{\frac{3}{5}}+\Gamma\left(\frac{3}{5}+1\right)}{(4)^{\frac{3}{5}}}\right) \\ &= 0.486\,30<1. \end{aligned} \tag{4.2}$$

Hence, by Theorem 2, *we can state that this example has a unique solution on* $[0,1]$.

Example 4.2 *Let consider the following problem:*

$$\left\{ \begin{array}{c} {}^{c}D^{\frac{2}{3},\Phi}\left({}^{c}D^{\frac{3}{4},\Phi}x(t)+g(t,x(t))\right)=f\left(t,x(t),x(\lambda t),{}^{c}D^{\frac{3}{4},\Phi}x(t)\right), t\in J, \\ x(0)=0, \\ x(1)=\int_0^1 h(t,x(s))\,ds. \end{array} \right. \tag{4.3}$$

Let us also define $\Phi(t) := t^3 + t$.
Therefe, Φ *is an increasing function over* $[0,1]$ *and* $\Phi'(t) := 3t^2 + 1 \neq 0$ *is continuous for all* $t \in [0,1]$. *Moreover, the function* $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ *defined by*

$$f\left(t, x(t), x\left(\frac{3}{5}t\right), {}^{c}D^{\frac{3}{4},\Phi}x(t)\right) = \frac{4 + x(t) + x\left(\frac{3t}{5}\right) + {}^{c}D^{\frac{3}{4},\Phi}x(t)}{\left(98\exp(t) + 2\cos(t^2) + \sin 2t\right)\left(1 + x(t) + x\left(\frac{3t}{5}\right) + {}^{c}D^{\frac{3}{4},\Phi}x(t)\right)},$$

is continuous. In addition, let

$$g(t, x(t)) = \frac{1}{20^2} + \frac{t^2}{4}\left(\frac{1}{10^2}\sin x(t)\right),$$

and

$$h(t, x(t)) = \frac{\cos \pi t}{6(2t+9)} + \frac{\sin x(t)}{36(4t+7)}.$$

Consequently, $(\mathbf{H}_1) - (\mathbf{H}_3)$ *are satisfied with*

$$L_f = \frac{1}{100}, M_f = \frac{1}{100}, L_g = \frac{1}{200}, L_h = 2.2487 \times 10^{-2}$$

and

$$\begin{aligned} \Omega &= (4L_f + 2M_f)\left(\frac{(\Phi(1) - \Phi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{(\Phi(1) - \Phi(0))^{\beta}}{\Gamma(\beta+1)}\right) \\ &\quad + 2L_g\left(\frac{(\Phi(1) - \Phi(0))^{\alpha} + \Gamma(\alpha+1)}{\Gamma(\alpha+1)}\right) + L_h\left(\frac{\Gamma(\alpha+1) + (\Phi(1) - \Phi(0))^{\alpha}}{(\Phi(1) - \Phi(0))^{\alpha}}\right) \\ &= \left(\frac{4}{100} + \frac{2}{100}\right)\left(\frac{(2)^{\frac{2}{3}+\frac{3}{4}}}{\Gamma\left(\frac{2}{3} + \frac{3}{4} + 1\right)} + \frac{(2)^{\frac{2}{3}}}{\Gamma\left(\frac{2}{3} + 1\right)}\right) + \frac{1}{100}\left(\frac{(2)^{\frac{3}{4}} + \Gamma\left(\frac{3}{4} + 1\right)}{\Gamma\left(\frac{3}{4} + 1\right)}\right) \\ &\quad + 2.2487 \times 10^{-2}\left(\frac{\Gamma\left(\frac{3}{4} + 1\right) + (2)^{\frac{3}{4}}}{(2)^{\frac{3}{4}}}\right) \\ &= 0.29613 < 1. \end{aligned} \tag{4.4}$$

Also, we have

$$N_f = \frac{1}{25}, N_g = \frac{1}{400}, N_h = \frac{1}{54},$$

and

$$\vartheta_f = \frac{1}{100}, \vartheta_g = \frac{1}{200}, \vartheta_h = \frac{1}{54} + \frac{1}{252}.$$

It follows by Theorem 4 that the example 6 has at least one solution on $[0,1]$.

Conclusion

In this paper, we have studied a new problem of pantograph type via $\Phi-$Caputo approach. The theorems proved in this paper are new and concern results that are widespread in the literature and this study can be regarded as a contribution to the improvement of the analytic aspect of fractional calculus. An interesting extension of our problem would be to investigate the possibility of existence for positive solutions and their stability analysis in Ulam-Hyers sense for the same problem.

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