Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 587 – 611
© TÜBİTAK
doi:10.3906/mat-2106-1

On a solvable system of rational difference equations of higher order

Merve KARA[1], **Yasin YAZLIK**[2,*]
[1]Department of Mathematics, Kamil Özdağ Science Faculty, Karamanoğlu Mehmetbey University, Karaman, Turkey
[2]Department of Mathematics, Faculty of Science and Art, Nevşehir Hacı Bektaş Veli University, Nevşehir, Turkey

Received: 01.06.2021 • **Accepted/Published Online:** 20.07.2021 • **Final Version:** 21.01.2022

Abstract: In this paper, we present that the following system of difference equations

$$x_n = \frac{x_{n-k} z_{n-l}}{b_n x_{n-k} + a_n z_{n-k-l}}, \; y_n = \frac{y_{n-k} x_{n-l}}{d_n y_{n-k} + c_n x_{n-k-l}}, \; z_n = \frac{z_{n-k} y_{n-l}}{f_n z_{n-k} + e_n y_{n-k-l}},$$

where $n \in \mathbb{N}_0$, $k, l \in \mathbb{N}$, the initial values x_{-i}, y_{-i}, z_{-i} are real numbers, for $i \in \overline{1, k+l}$, and sequences $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$, $(c_n)_{n\in\mathbb{N}_0}$, $(d_n)_{n\in\mathbb{N}_0}$, $(e_n)_{n\in\mathbb{N}_0}$ and $(f_n)_{n\in\mathbb{N}_0}$ are non-zero real numbers, for all $n \in \mathbb{N}_0$, which can be solved in closed form. We describe the forbidden set of the initial values using the obtained formulas and also determine the asymptotic behavior of solutions for the case $k = 3$, $l = 1$, and the sequences $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$, $(c_n)_{n\in\mathbb{N}_0}$, $(d_n)_{n\in\mathbb{N}_0}$, $(e_n)_{n\in\mathbb{N}_0}$ and $(f_n)_{n\in\mathbb{N}_0}$ are constant. Our results considerably extend and improve some recent results in the literature.

Key words: System of difference equations, closed form, forbidden set

1. Introduction and preliminaries

The main problem of theory of difference equations is to determine the behaviour of the solutions of difference equations. See, for example, the references [1, 2, 4, 10, 11, 14–23, 25, 29, 31–33, 35–37].

One of the ways to examine the asymptotic behavior of solutions of difference equations or systems of difference equations is to obtain solutions of difference equations or systems of difference equations. Obtaining solutions to difference equations started at the beginning of the 18th century by De Moivre. Firstly, he solved the following homogeneous linear difference equation

$$x_{n+2} = a x_{n+1} + b x_n \; n \in \mathbb{N}_0, \tag{1.1}$$

when $b \neq 0$ and $a^2 \neq -4b$. He found the general solution for Eq. (1.1) is given by the following formula:

$$x_n = \frac{(x_1 - \lambda_2 x_0)\lambda_1^n + (\lambda_1 x_0 - x_1)\lambda_2^n}{\lambda_1 - \lambda_2}, n \in \mathbb{N}_0, \tag{1.2}$$

where λ_1 and λ_2 are roots of the polynomial $P(\lambda) = \lambda^2 - a\lambda - b = 0$. Eq. (1.2) is called the De Moivre formula, whereas the polynomial P is called the characteristic polynomial associated to the linear equation (1.1) in [3].

*Correspondence: yyazlik@nevsehir.edu.tr
2010 *AMS Mathematics Subject Classification:* 39A10, 39A20, 39A23.

Ideas and methods of De Moivre were later improved by Euler in [12]. The study was followed by Lagrange, Laplace and many other mathematicians.

After learning the solution methods of linear difference equations, a new problem emerged. This problem is how to turn nonlinear difference equations into linear difference equations.

Firstly, the following equations

$$x_{n+1}=\frac{x_n x_{n-1}}{x_n+x_{n-2}} \quad \text{and} \quad x_{n+1}=\frac{x_n x_{n-1}}{x_{n-1}+x_{n-2}},\ n\in\mathbb{N}_0, \tag{1.3}$$

were presented, among other things, by Elmetwally et al. in [6]. The solutions of difference equations in (1.3) were found by using induction. This method didn't give much detail on how solutions were obtained. We believe that the results can be obtained by computer programs. In addition, the solutions of some of the difference equations investigated in [6] are associated with number sequences. For this reason, the last aforementioned study has been considered important by mathematicians.

Difference equations of the type of difference equations in (1.3) have been generalized in different ways by many mathematicians in [5, 7, 8, 13, 20, 24, 26, 28, 30, 34]. That is, generalizations are adding the parameters, increasing order, adding periodic coefficients and increasing dimensional, such as two-dimensional or three-dimensional systems. For example, in [30], the most general form of the difference equations in (1.3), which is a theoretical explanation of the studies using the induction, is the following difference equation

$$x_n=\alpha x_{n-k}+\frac{\delta x_{n-k}x_{n-(k+l)}}{\beta x_{n-(k+l)}+\gamma x_{n-l}},\ n\in\mathbb{N}_0, \tag{1.4}$$

where k and l are fixed natural numbers, $\alpha,\ \beta,\ \gamma,\ \delta\in\mathbb{R}$, and the initial values $x_{-i},\ i=\overline{1,k+l}$ are real numbers. Authors showed that equation (1.4) is solvable in closed form and presented formulas for the solutions by using transformation. They also studied the long-term behavior of the solutions of equation (1.4). It is possible to find special cases of this equation in the literature. See, for example, the references [5, 7–9, 13, 24, 26, 28, 30].

Another example for aforementioned generalization is that the second of the difference equations in (1.3) was generalized to the following two dimensional systems

$$x_{n+1}=\frac{x_{n-1}y_n}{\pm x_{n-1}\pm y_{n-2}},\ y_{n+1}=\frac{y_{n-1}x_n}{\pm y_{n-1}\pm x_{n-2}},\ n\in\mathbb{N}_0, \tag{1.5}$$

and was solved by using induction in [7]. In addition, Elmetwally didn't give theoretical explanation of how solutions were obtained.

Moreover, in [27], systems (1.5) were extended to the following two-dimensional system of difference equations

$$x_n=\frac{x_{n-k}y_{n-l}}{b_n x_{n-k}+a_n y_{n-k-l}},\ y_n=\frac{y_{n-k}x_{n-l}}{d_n y_{n-k}+c_n x_{n-k-l}},\ n\in\mathbb{N}_0, \tag{1.6}$$

and solved in closed form.

Our aim in this paper is to extend both the there-dimensional form of equations in (1.3) and more general systems of (1.5) and (1.6) to solve them in closed form. Another goal in this study is to prevent the solutions of difference equations and systems of difference equations from being obtained by induction and to extend the solutions of difference equations and their systems obtained by induction, such as equations in (1.3), systems (1.5), to the following three-dimensional system

$$x_n = \frac{x_{n-k}z_{n-l}}{b_n x_{n-k} + a_n z_{n-k-l}},\ y_n = \frac{y_{n-k}x_{n-l}}{d_n y_{n-k} + c_n x_{n-k-l}},\ z_n = \frac{z_{n-k}y_{n-l}}{f_n z_{n-k} + e_n y_{n-k-l}}, \tag{1.7}$$

where $n \in \mathbb{N}_0$, $k, l \in \mathbb{N}$, the initial values x_{-i}, y_{-i}, z_{-i} are real numbers, for $i \in \overline{1, k+l}$, and sequences $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$, $(c_n)_{n\in\mathbb{N}_0}$, $(d_n)_{n\in\mathbb{N}_0}$, $(e_n)_{n\in\mathbb{N}_0}$ and $(f_n)_{n\in\mathbb{N}_0}$ are non-zero real numbers, for all $n \in \mathbb{N}_0$. System (1.7) can be solved in closed form by using transformation. In addition, we determine the asymptotic behavior of solutions and the forbidden set of the initial values by using the obtained formulas. Note that system (1.7) is a natural generalization of both equations in (1.3), system (1.5), system (1.6), the general systems of system (1.5), and the general equations in (1.3).

The following definition gives us the set of all initial values, which yields undefined solutions.

Definition 1.1 *[27] (Forbidden set): Consider the following system of difference equations*

$$\begin{aligned}
x_n^{(1)} &= f_1\left(x_{n-1}^{(1)}, \ldots, x_{n-k}^{(1)}, x_{n-1}^{(2)}, \ldots, x_{n-k}^{(2)}, \ldots, x_{n-1}^{(m)}, \ldots, x_{n-k}^{(m)}\right),\\
x_n^{(2)} &= f_2\left(x_{n-1}^{(1)}, \ldots, x_{n-k}^{(1)}, x_{n-1}^{(2)}, \ldots, x_{n-k}^{(2)}, \ldots, x_{n-1}^{(m)}, \ldots, x_{n-k}^{(m)}\right),\\
&\ \ \vdots\\
x_n^{(m)} &= f_m\left(x_{n-1}^{(1)}, \ldots, x_{n-k}^{(1)}, x_{n-1}^{(2)}, \ldots, x_{n-k}^{(2)}, \ldots, x_{n-1}^{(m)}, \ldots, x_{n-k}^{(m)}\right),
\end{aligned} \tag{1.8}$$

$n \in \mathbb{N}_0$, where $m, k \in \mathbb{N}$ and $x_{-j}^{(i)} \in \mathbb{R}$, $j = \overline{1,k}$, $i = \overline{1,m}$. The string of vectors $\left(x_j^{(1)}, x_j^{(2)}, \ldots, x_j^{(m)}\right)$, $-k \le j \le n_0$, where $n_0 \ge -1$, is called an undefined solution of system (1.8) if

$$x_j^{(i)} = f_i\left(x_{j-1}^{(1)}, \ldots, x_{j-k}^{(1)}, x_{j-1}^{(2)}, \ldots, x_{j-k}^{(2)}, \ldots, x_{j-1}^{(m)}, \ldots, x_{j-k}^{(m)}\right)$$

for $i = \overline{1,m}$, $0 \le j < n_0 + 1$, and $x_{n_0+1}^{(i_0)}$ is not defined for an $i_0 \in \{1, \ldots, m\}$, that is, the quantity $f_{i_0}\left(x_{n_0}^{(1)}, \ldots, x_{n_0-k+1}^{(1)}, x_{n_0}^{(2)}, \ldots, x_{n_0-k+1}^{(2)}, \ldots, x_{n_0}^{(m)}, \ldots, x_{n_0-k+1}^{(m)}\right)$ is not defined. The set of all initial values $x_{-j}^{(i)}$, $j = \overline{1,k}, i = \overline{1,m}$, which generate undefined solutions of system of difference equation (1.8), is called domain of undefinable solutions of the system of difference equations (or called forbidden set).

2. Closed solutions of the system (1.7)

First assume that $l \le k$. If $x_{n_0} = 0$ for some $n_0 \ge -l$, and $x_n \ne 0$, $y_n \ne 0$, $z_n \ne 0$, $-l \le n \le n_0 - 1$, then from the second equation in (1.7) we have that if $y_{n_0+l} = 0$, which implies that y_{n_0+l+k} is not defined.

If $y_{n_1} = 0$ for some $n_1 \ge -l$, and $x_n \ne 0$, $y_n \ne 0$, $z_n \ne 0$, $-l \le n \le n_1 - 1$, then from the third equation in (1.7) we have that if $z_{n_1+l} = 0$, which implies that z_{n_1+l+k} is not defined.

If $z_{n_2} = 0$ for some $n_2 \ge -l$, and $x_n \ne 0$, $y_n \ne 0$, $z_n \ne 0$, $-l \le n \le n_2 - 1$, then from the first equation in (1.7) we have that if $x_{n_2+l} = 0$, which implies that x_{n_2+l+k} is not defined.

If $x_{n_3}=0$ for some $-l>n_3\geq -k$, and $x_n\neq 0$, $y_n\neq 0$, $z_n\neq 0$, $-k\leq n\leq n_3-1$, then, from the first equation in (1.7), we obtain $x_{mk+n_3}=0$, $m\in\mathbb{N}_0$, as far as these numbers are defined. Hence, x_{n_3+k} is not defined, or $x_{n_3+k}=0$ (note that $n_3+k\geq 0$), which, according to the first case, would imply that y_{n_3+2k+l} is not defined.

If $y_{n_4}=0$ for some $-l>n_4\geq -k$, and $x_n\neq 0$, $y_n\neq 0$, $z_n\neq 0$, $-k\leq n\leq n_4-1$, then, from the second equation in (1.7), we obtain $y_{mk+n_4}=0$, $m\in\mathbb{N}_0$, as far as these numbers are defined. Hence, y_{n_4+k} is not defined, or $y_{n_4+k}=0$ (note that $n_4+k\geq 0$), which, according to the second case, would imply that z_{n_4+2k+l} is not defined.

If $z_{n_5}=0$ for some $-l>n_5\geq -k$, and $x_n\neq 0$, $y_n\neq 0$, $z_n\neq 0$, $-k\leq n\leq n_5-1$, then, from the third equation in (1.7), we obtain $z_{mk+n_5}=0$, $m\in\mathbb{N}_0$, as far as these numbers are defined. Hence, z_{n_5+k} is not defined, or $z_{n_5+k}=0$ (note that $n_5+k\geq 0$), which, according to the third case, would imply that x_{n_5+2k+l} is not defined.

The case $l>k$ is treated similarly, so we omit it.

On the other hand, if $x_{n_6}=0$ for some $n_6\in\mathbb{N}_0$, then, according to the first equation in (1.7) we have that $x_{n_6-k}=0$ or $z_{n_6-l}=0$. If $-k\leq n_6-k\leq -1$ or $-l\leq n_6-l\leq -1$, then we have a $j_0\in\{1,\ldots,s\}$, where $s=max\{k,l\}$, such that $x_{-j_0}=0$ or $z_{-j_0}=0$. If $n_6\geq s$, then, by using the equations in (1.7), we have that $x_{n_6-2k}=0$ or $z_{n_6-k-l}=0$ if $x_{n_6-k}=0$ or $z_{n_6-k-l}=0$ or $y_{n_6-2l}=0$ if $z_{n_6-l}=0$. If $-s\leq n_6-2k\leq -1$ or $-s\leq n_6-k-l\leq -1$ in the first case, or $-s\leq n_6-2l\leq -1$ or $-s\leq n_6-k-l\leq -1$ in the second case, then we have a $j_1\in\{1,\ldots,s\}$ such that $x_{-j_1}=0$ or $y_{-j_1}=0$ or $z_{-j_1}=0$. Repeating this procedure we find a $p\in\{1,\ldots,s\}$ such that $x_{-p}=0$ or $y_{-p}=0$ or $z_{-p}=0$. As we have proved above, such solutions are not defined.

Hence, we will consider $x_ny_nz_n\neq 0$ for all $n\in\mathbb{N}_0$. Note that the system (1.7) can be written in the form

$$\frac{z_{n-l}}{x_n}=a_n\frac{z_{n-k-l}}{x_{n-k}}+b_n,\ \frac{x_{n-l}}{y_n}=c_n\frac{x_{n-k-l}}{y_{n-k}}+d_n,\ \frac{y_{n-l}}{z_n}=e_n\frac{y_{n-k-l}}{z_{n-k}}+f_n, \tag{2.1}$$

for $n\in\mathbb{N}_0$. Let

$$u_n=\frac{z_{n-l}}{x_n},\ v_n=\frac{x_{n-l}}{y_n},\ w_n=\frac{y_{n-l}}{z_n},\ n\geq -k. \tag{2.2}$$

Then, system (2.1) can be written as

$$u_n=a_nu_{n-k}+b_n,\ v_n=c_nv_{n-k}+d_n,\ w_n=e_nw_{n-k}+f_n,\ n\in\mathbb{N}_0. \tag{2.3}$$

Hence, the sequences

$$u_{km+i}=u_m^{(i)},\ v_{km+i}=v_m^{(i)},\ w_{km+i}=w_m^{(i)},\ i=\overline{0,k-1},\quad m\geq -1. \tag{2.4}$$

are solutions of the equations

$$\begin{aligned} u_m^{(i)} &= a_{km+i}u_{m-1}^{(i)}+b_{km+i},\\ v_m^{(i)} &= c_{km+i}v_{m-1}^{(i)}+d_{km+i},\\ w_m^{(i)} &= e_{km+i}w_{m-1}^{(i)}+f_{km+i}, \end{aligned} \tag{2.5}$$

for $m \geq -1$. So, for each fixed $i \in \{0, 1, \ldots, k-1\}$ equations in (2.5)

$$u_m^{(i)} = u_{-1}^{(i)} \prod_{j=0}^{m} a_{kj+i} + \sum_{j=0}^{m} b_{kj+i} \prod_{s=j+1}^{m} a_{ks+i}, \tag{2.6}$$

$$v_m^{(i)} = v_{-1}^{(i)} \prod_{j=0}^{m} c_{kj+i} + \sum_{j=0}^{m} d_{kj+i} \prod_{s=j+1}^{m} c_{ks+i}, \tag{2.7}$$

$$w_m^{(i)} = w_{-1}^{(i)} \prod_{j=0}^{m} e_{kj+i} + \sum_{j=0}^{m} f_{kj+i} \prod_{s=j+1}^{m} e_{ks+i}, \tag{2.8}$$

for $m \geq -1$.

If $a_n = a$, $b_n = b$, $c_n = c$, $d_n = d$, $e_n = e$ and $f_n = f$, for every $n \in \mathbb{N}_0$, $i \in \{0, 1, \ldots, k-1\}$, then we get

$$u_m^{(i)} = \frac{\left(u_{-1}^{(i)}(1-a) - b\right) a^{m+1} + b}{1-a}, \quad m \geq -1, \tag{2.9}$$

if $a \neq 1$, and

$$u_m^{(i)} = u_{-1}^{(i)} + (m+1)\, b, \quad m \geq -1, \tag{2.10}$$

if $a = 1$, while

$$v_m^{(i)} = \frac{\left(v_{-1}^{(i)}(1-c) - d\right) c^{m+1} + d}{1-c}, \quad m \geq -1, \tag{2.11}$$

if $c \neq 1$, and

$$v_m^{(i)} = v_{-1}^{(i)} + (m+1)\, d, \quad m \geq -1, \tag{2.12}$$

if $c = 1$, while

$$w_m^{(i)} = \frac{\left(w_{-1}^{(i)}(1-e) - f\right) e^{m+1} + f}{1-e}, \quad m \geq -1, \tag{2.13}$$

if $e \neq 1$, and

$$w_m^{(i)} = w_{-1}^{(i)} + (m+1)\, f, \quad m \geq -1, \tag{2.14}$$

if $e = 1$.

Using (2.2), it follows that

$$x_n = \frac{z_{n-l}}{u_n} = \frac{y_{n-2l}}{u_n w_{n-l}} = \frac{x_{n-3l}}{u_n w_{n-l} v_{n-2l}}, \; n \geq 2l - k, \tag{2.15}$$

$$y_n = \frac{x_{n-l}}{v_n} = \frac{z_{n-2l}}{v_n u_{n-l}} = \frac{y_{n-3l}}{v_n u_{n-l} w_{n-2l}}, \; n \geq 2l - k, \tag{2.16}$$

$$z_n = \frac{y_{n-l}}{w_n} = \frac{x_{n-2l}}{w_n v_{n-l}} = \frac{z_{n-3l}}{w_n v_{n-l} u_{n-2l}}, \ n \geq 2l-k. \tag{2.17}$$

From (2.15)-(2.17), we have

$$x_{3lm+i} = \frac{x_{3lm+i-3l}}{u_{3lm+i} w_{3lm+i-l} v_{3lm+i-2l}} = \cdots = \frac{x_{i-3l}}{\prod_{j=0}^{m} u_{3lj+i} w_{3lj+i-l} v_{3lj+i-2l}}, \tag{2.18}$$

$$y_{3lm+i} = \frac{y_{3lm+i-3l}}{v_{3lm+i} u_{3lm+i-l} w_{3lm+i-2l}} = \cdots = \frac{y_{i-3l}}{\prod_{j=0}^{m} v_{3lj+i} u_{3lj+i-l} w_{3lj+i-2l}}, \tag{2.19}$$

$$z_{3lm+i} = \frac{z_{3lm+i-3l}}{w_{3lm+i} v_{3lm+i-l} u_{3lm+i-2l}} = \cdots = \frac{z_{i-3l}}{\prod_{j=0}^{m} w_{3lj+i} v_{3lj+i-l} u_{3lj+i-2l}}, \tag{2.20}$$

for every $m \in \mathbb{N}_0$, $i = \overline{2l-k, 5l-k-1}$. Since every non-negative integer can be written in the form $km_1 + j$, where $m_1 \in \mathbb{N}_0$ and $j \in \{0, 1, \ldots, k-1\}$, we get that

$$x_{3lkm_1+3lj+i} = \frac{x_{3lj+i-3l}}{\prod_{s=0}^{km_1} u_{3ls+3lj+i} w_{3ls+3lj+i-l} v_{3ls+3lj+i-2l}}, \tag{2.21}$$

$$y_{3lkm_1+3lj+i} = \frac{y_{3lj+i-3l}}{\prod_{s=0}^{km_1} v_{3ls+3lj+i} u_{3ls+3lj+i-l} w_{3ls+3lj+i-2l}}, \tag{2.22}$$

$$z_{3lkm_1+3lj+i} = \frac{z_{3lj+i-3l}}{\prod_{s=0}^{km_1} w_{3ls+3lj+i} v_{3ls+3lj+i-l} u_{3ls+3lj+i-2l}}, \tag{2.23}$$

for every $m_1 \in \mathbb{N}_0$, $j = \overline{0, k-1}$, $i = \overline{2l-k, 5l-k-1}$.
A simple analysis shows that formulas (2.6)-(2.8) can be efficiently applied in (2.21)-(2.23) if $3l = k$.

Theorem 2.1 *Assume that $a_n \neq 0$, $b_n \neq 0$, $c_n \neq 0$, $d_n \neq 0$, $e_n \neq 0$, $f_n \neq 0$, every $n \in \mathbb{N}_0$. Then the forbidden set of the initial values for system (1.7) is given by the set*

$$\begin{aligned}
\mathcal{F} \ = \ & \bigcup_{m \in \mathbb{N}_0} \bigcup_{i=0}^{k-1} \Big\{ (x_{-k-l}, \ldots, x_{-1}, y_{-k-l}, \ldots, y_{-1}, z_{-k-l}, \ldots, z_{-1}) \in \mathbb{R}^{3(k+l)} : \\
& \frac{z_{i-k-l}}{x_{i-k}} = -\sum_{j=0}^{m} \frac{b_{kj+i}}{a_{kj+i}} \prod_{l=0}^{j-1} \frac{1}{a_{kl+i}} \neq 0, \frac{x_{i-k-l}}{y_{i-k}} = -\sum_{j=0}^{m} \frac{d_{kj+i}}{c_{kj+i}} \prod_{l=0}^{j-1} \frac{1}{c_{kl+i}} \neq 0, \\
& \frac{y_{i-k-l}}{z_{i-k}} = -\sum_{j=0}^{m} \frac{f_{kj+i}}{e_{kj+i}} \prod_{l=0}^{j-1} \frac{1}{e_{kl+i}} \neq 0 \Big\} \bigcup \\
& \bigcup_{j=1}^{k+l} \Big\{ (x_{-k-l}, \ldots, x_{-1}, y_{-k-l}, \ldots, y_{-1}, z_{-k-l}, \ldots, z_{-1}) \in \mathbb{R}^{3(k+l)} : \\
& x_{-j} = 0, \ y_{-j} = 0, \ z_{-j} = 0 \Big\}.
\end{aligned} \tag{2.24}$$

Proof At the beginning of Section 2, we have acquired that the set

$$\bigcup_{j=1}^{k+l}\Big\{(x_{-k-l},\dots,x_{-1},y_{-k-l},\dots,y_{-1},z_{-k-l},\dots,z_{-1})\in\mathbb{R}^{3(k+l)}:$$
$$x_{-j}=0,\ y_{-j}=0,\ z_{-j}=0\Big\}$$

belongs to the forbidden set of the initial values for system (1.7). Now, we assume that $x_n\neq 0$, $y_n\neq 0$ and $z_n\neq 0$ for every $n\in\mathbb{N}_0$. Note that the system (1.7) is undefined, when the conditions $b_nx_{n-k}+a_nz_{n-k-l}=0$ or $d_ny_{n-k}+c_nx_{n-k-l}=0$ or $f_nz_{n-k}+e_ny_{n-k-l}=0$, that is, $\frac{z_{n-k-l}}{x_{n-k}}=-\frac{b_n}{a_n}$ or $\frac{x_{n-k-l}}{y_{n-k}}=-\frac{d_n}{c_n}$ or $\frac{y_{n-k-l}}{z_{n-k}}=-\frac{f_n}{e_n}$, for some $n\in\mathbb{N}_0$ or are satisfied (Here we consider that $a_n\neq 0$, $c_n\neq 0$ and $e_n\neq 0$ for every $n\in\mathbb{N}_0$). From this and the substitution $u_n=\frac{z_{n-l}}{x_n}$, $v_n=\frac{x_{n-l}}{y_n}$, $w_n=\frac{y_{n-l}}{z_n}$, we get

$$u_{k(m-1)+i}=-\frac{b_{km+i}}{a_{km+i}},\ v_{k(m-1)+i}=-\frac{d_{km+i}}{c_{km+i}},\ w_{k(m-1)+i}=-\frac{f_{km+i}}{e_{km+i}} \tag{2.25}$$

for some $m\in\mathbb{N}_0$ and $i\in\{0,1,\dots,k-1\}$. Hence, we can determine the forbidden set of the initial values for system (1.7) by using the substitution $u_n=\frac{z_{n-l}}{x_n}$, $v_n=\frac{x_{n-l}}{y_n}$, $w_n=\frac{y_{n-l}}{z_n}$. Now, we consider the functions

$$\widehat{f}_{km+i}(t):=a_{km+i}t+b_{km+i},\ g_{km+i}(t):=c_{km+i}t+d_{km+i},\ h_{km+i}(t):=e_{km+i}t+f_{km+i}, \tag{2.26}$$

for some $m\in\mathbb{N}_0$ and $i\in\{0,1,\dots,k-1\}$, which correspond to the equations in (2.3). From (2.25) and (2.26), we can write

$$u_{km+i}=\widehat{f}_{km+i}\circ\widehat{f}_{k(m-1)+i}\circ\cdots\circ\widehat{f}_i(u_{i-k}), \tag{2.27}$$

$$v_{km+i}=g_{km+i}\circ g_{k(m-1)+i}\circ\cdots\circ g_i(v_{i-k}), \tag{2.28}$$

$$w_{km+i}=h_{km+i}\circ h_{k(m-1)+i}\circ\cdots\circ h_i(w_{i-k}), \tag{2.29}$$

where $m\in\mathbb{N}_0$, and $i\in\{0,1,\dots,k-1\}$. By using (2.25) and implicit forms (2.27)-(2.29) and considering $\widehat{f}^{-1}_{km+i}(0)=-\frac{b_{km+i}}{a_{km+i}}$, $g^{-1}_{km+i}(0)=-\frac{d_{km+i}}{c_{km+i}}$, $h^{-1}_{km+i}(0)=-\frac{f_{km+i}}{e_{km+i}}$, for $m\in\mathbb{N}_0$ and $i\in\{0,1,\dots,k-1\}$, we have

$$u_{i-k}=\widehat{f}^{-1}_i\circ\cdots\circ\widehat{f}^{-1}_{km+i}(0),\ v_{i-k}=g^{-1}_i\circ\cdots\circ g^{-1}_{km+i}(0),\ w_{i-k}=h^{-1}_i\circ\cdots\circ h^{-1}_{km+i}(0), \tag{2.30}$$

where $\widehat{f}^{-1}_{km+i}(t)=\frac{t-b_{km+i}}{a_{km+i}}$, $g^{-1}_{km+i}(t)=\frac{t-d_{km+i}}{c_{km+i}}$, $h^{-1}_{km+i}(t)=\frac{t-f_{km+i}}{e_{km+i}}$, $m\in\mathbb{N}_0$, $i\in\{0,1,\dots,k-1\}$. From (2.30), we obtain

$$u_{i-k}=-\sum_{j=0}^{m}\frac{b_{kj+i}}{a_{kj+i}}\prod_{l=0}^{j-1}\frac{1}{a_{kl+i}},\ v_{i-k}=-\sum_{j=0}^{m}\frac{d_{kj+i}}{c_{kj+i}}\prod_{l=0}^{j-1}\frac{1}{c_{kl+i}},\ w_{i-k}=-\sum_{j=0}^{m}\frac{f_{kj+i}}{e_{kj+i}}\prod_{l=0}^{j-1}\frac{1}{e_{kl+i}},$$

for some $m\in\mathbb{N}_0$ and $i\in\{0,1,\dots,k-1\}$. This means that if one of the conditions in (2.30) holds, then m-th iteration or $(m+1)$-th iteration in system (1.7) can not be calculated. □

3. Case of the constant coefficients

In this section, we suppose that $a_n = a$, $b_n = b$, $c_n = c$, $d_n = d$, $e_n = e$, $f_n = f$ for every $n \in \mathbb{N}_0$. Then system (1.7) becomes

$$x_n = \frac{x_{n-k}z_{n-l}}{bx_{n-k} + az_{n-k-l}},\ y_n = \frac{y_{n-k}x_{n-l}}{dy_{n-k} + cx_{n-k-l}},\ z_n = \frac{z_{n-k}y_{n-l}}{fz_{n-k} + ey_{n-k-l}},\ n \in \mathbb{N}_0. \tag{3.1}$$

Then, we may assume that $gcd\,(k,l) = 1$. Indeed, if $|gcd\,(k,l)| = r > 1$, where $gcd\,(k,l)$ denotes the greatest common divisor of natural numbers k and l, then $k = rk_1$ and $l = rl_1$ for some $k_1, l_1 \in \mathbb{N}$ such that $gcd\,(k_1,l_1) = 1$. Since every $n \in \mathbb{N}_0$ has the form $n = mr + i$ for some $m \in \mathbb{N}_0$ and $i = \overline{0, r-1}$, from system (3.1) we get

$$\begin{aligned}
x_{mr+i} &= \frac{x_{r(m-k_1)+i}z_{r(m-l_1)+i}}{bx_{r(m-k_1)+i} + az_{r(m-k_1-l_1)+i}},\\
y_{mr+i} &= \frac{y_{r(m-k_1)+i}x_{r(m-l_1)+i}}{dy_{r(m-k_1)+i} + cx_{r(m-k_1-l_1)+i}},\\
z_{mr+i} &= \frac{z_{r(m-k_1)+i}y_{r(m-l_1)+i}}{fz_{r(m-k_1)+i} + ey_{r(m-k_1-l_1)+i}},\ n \in \mathbb{N}_0.
\end{aligned} \tag{3.2}$$

The change of variables

$$x_m^{(i)} = x_{mr+i},\ y_m^{(i)} = y_{mr+i},\ z_m^{(i)} = z_{mr+i},\ m \in \mathbb{N}_0,\ i = \overline{0, r-1},$$

in (3.2) yields that $\left(x_m^{(i)}, y_m^{(i)}, z_m^{(i)}\right)_{m \geq -(k_1+l_1)}$, $i = \overline{0, r-1}$, are r independent solutions of the system

$$x_m^{(i)} = \frac{x_{m-k_1}^{(i)}z_{m-l_1}^{(i)}}{bx_{m-k_1}^{(i)} + az_{m-k_1-l_1}^{(i)}},\ y_m^{(i)} = \frac{y_{m-k_1}^{(i)}x_{m-l_1}^{(i)}}{dy_{m-k_1}^{(i)} + cx_{m-k_1-l_1}^{(i)}},\ z_m^{(i)} = \frac{z_{m-k_1}^{(i)}y_{m-l_1}^{(i)}}{fz_{m-k_1}^{(i)} + ey_{m-k_1-l_1}^{(i)}}. \tag{3.3}$$

Note that system (3.3) can get by taking k_1 and l_1, respectively, instead of k and l in system (3.1). From now on, we assume that the greatest common divisor of k and l is equal to 1; that is, $gcd\,(k,l) = 1$. By putting the formulas (2.6)-(2.8) into (2.21)-(2.23), we obtain the well-defined solutions of system (3.1) when $gcd\,(k,l) = 1$.

3.1. Case k=3, l=1

In this subsection, we will give solutions of system (3.1) for the case $k = 3$, $l = 1$. In this case, system (3.1) becomes

$$x_n = \frac{x_{n-3}z_{n-1}}{bx_{n-3} + az_{n-4}},\ y_n = \frac{y_{n-3}x_{n-1}}{dy_{n-3} + cx_{n-4}},\ z_n = \frac{z_{n-3}y_{n-1}}{fz_{n-3} + ey_{n-4}},\ n \in \mathbb{N}_0. \tag{3.4}$$

First note that formulas (2.9)-(2.14), in this case, can be written in the following form

$$u_{3m+i_1} = \frac{\left(u_{i_1-3}\left(1-a\right) - b\right)a^{m+1} + b}{1-a},\quad m \geq -1, i_1 \in \{0,1,2\}, \tag{3.5}$$

if $a \neq 1$, and

$$u_{3m+i_1} = u_{i_1-3} + (m+1)\,b,\quad m \geq -1, i_1 \in \{0,1,2\}, \tag{3.6}$$

if $a=1$, while

$$v_{3m+i_1} = \frac{\left(v_{i_1-3}\left(1-c\right)-d\right)c^{m+1}+d}{1-c}, \quad m \geq -1, i_1 \in \{0,1,2\}, \tag{3.7}$$

if $c \neq 1$, and

$$v_{3m+i_1} = v_{i_1-3} + \left(m+1\right)d, \quad m \geq -1, i_1 \in \{0,1,2\}, \tag{3.8}$$

if $c=1$, while

$$w_{3m+i_1} = \frac{\left(w_{i_1-3}\left(1-e\right)-f\right)e^{m+1}+f}{1-e}, \quad m \geq -1, i_1 \in \{0,1,2\}, \tag{3.9}$$

if $e \neq 1$, and

$$w_{3m+i_1} = w_{i_1-3} + \left(m+1\right)f, \quad m \geq -1, i_1 \in \{0,1,2\}, \tag{3.10}$$

if $e=1$.
We obtain following equations from (2.18)–(2.20) for the case $k=3$, $l=1$,

$$x_{3m+i} = \frac{x_{i-3}}{\prod_{j=0}^{m} u_{3j+i}w_{3j+i-1}v_{3j+i-2}}, \tag{3.11}$$

$$y_{3m+i} = \frac{y_{i-3}}{\prod_{j=0}^{m} v_{3j+i}u_{3j+i-1}w_{3j+i-2}}, \tag{3.12}$$

$$z_{3m+i} = \frac{z_{i-3}}{\prod_{j=0}^{m} w_{3j+i}v_{3j+i-1}u_{3j+i-2}}, \tag{3.13}$$

for $m \in \mathbb{N}_0$, $i \in \{-1,0,1\}$.
Let

$$p_1 := \begin{cases} 0, & i \equiv 0 \ (\text{mod}3) \\ 1, & i \equiv 1 \ (\text{mod}3) \\ 2, & i \equiv 2 \ (\text{mod}3) \end{cases}, \ p_2 := \begin{cases} 0, & i-1 \equiv 0 \ (\text{mod}3) \\ 1, & i-1 \equiv 1 \ (\text{mod}3) \\ 2, & i-1 \equiv 2 \ (\text{mod}3) \end{cases}, \ p_3 := \begin{cases} 0, & i-2 \equiv 0 \ (\text{mod}3) \\ 1, & i-2 \equiv 1 \ (\text{mod}3) \\ 2, & i-2 \equiv 2 \ (\text{mod}3) \end{cases}.$$

3.1.1. Case $a \neq 1$, $c \neq 1$, $e \neq 1$

In this case, if (3.5)–(3.10) are used in (3.11)–(3.13), it can be easily seen that the solutions of system (3.4) are as follows.

$$x_{3m+i} = \frac{x_{i-3}}{\prod_{j=0}^{m} \frac{\left(u_{p_1-3}(1-a)-b\right)a^{j+\lfloor\frac{i}{3}\rfloor+1}+b}{1-a} \frac{\left(w_{p_2-3}(1-e)-f\right)e^{j+\lfloor\frac{i-1}{3}\rfloor+1}+f}{1-e} \frac{\left(v_{p_3-3}(1-c)-d\right)c^{j+\lfloor\frac{i-2}{3}\rfloor+1}+d}{1-c}}, \tag{3.14}$$

$$y_{3m+i} = \frac{y_{i-3}}{\prod_{j=0}^{m} \frac{\left(v_{p_1-3}(1-c)-d\right)c^{j+\lfloor\frac{i}{3}\rfloor+1}+d}{1-c} \frac{\left(u_{p_2-3}(1-a)-b\right)a^{j+\lfloor\frac{i-1}{3}\rfloor+1}+b}{1-a} \frac{\left(w_{p_3-3}(1-e)-f\right)e^{j+\lfloor\frac{i-2}{3}\rfloor+1}+f}{1-e}}, \tag{3.15}$$

$$z_{3m+i} = \frac{z_{i-3}}{\prod_{j=0}^{m} \frac{\left(w_{p_1-3}(1-e)-f\right)e^{j+\lfloor \frac{i}{3} \rfloor+1}+f}{1-e} \frac{\left(v_{p_2-3}(1-c)-d\right)c^{j+\lfloor \frac{i-1}{3} \rfloor+1}+d}{1-c} \frac{\left(u_{p_3-3}(1-a)-b\right)a^{j+\lfloor \frac{i-2}{3} \rfloor+1}+b}{1-a}}, \tag{3.16}$$

for every $m \in \mathbb{N}_0$, $i \in \{-1, 0, 1\}$. Now we will apply these formulas.

Theorem 3.1 *Assume that* $k = 3l = 3$, $abcdef \neq 0$, $a \neq 1$, $c \neq 1$, $e \neq 1$, *and that* $(x_n, y_n, z_n)_{n \geq -4}$ *is a well-defined solution of system* (3.4) *and* $x_{-p}, y_{-p}, z_{-p} \notin \mathcal{F}$, *for* $p = \overline{1,4}$, $k = 3$, $l = 1$. *Then the following statements hold.*

(a) *If* $|a| > 1$ *and* $u_{p_1-3} \neq \frac{b}{1-a}$ *or* $|e| > 1$ *and* $w_{p_2-3} \neq \frac{f}{1-e}$ *or* $|c| > 1$ *and* $v_{p_3-3} \neq \frac{d}{1-c}$, *then* $x_{3m+i} \to 0$, *for* $i \in \{-1, 0, 1\}$, *as* $m \to \infty$.

(b) *If* $|c| > 1$ *and* $v_{p_1-3} \neq \frac{d}{1-c}$ *or* $|a| > 1$ *and* $u_{p_2-3} \neq \frac{b}{1-a}$ *or* $|e| > 1$ *and* $w_{p_3-3} \neq \frac{f}{1-e}$, *then* $y_{3m+i} \to 0$, *for* $i \in \{-1, 0, 1\}$, *as* $m \to \infty$.

(c) *If* $|e| > 1$ *and* $w_{p_1-3} \neq \frac{f}{1-e}$ *or* $|c| > 1$ *and* $v_{p_2-3} \neq \frac{d}{1-c}$ *or* $|a| > 1$ *and* $u_{p_3-3} \neq \frac{b}{1-a}$, *then* $z_{3m+i} \to 0$, *for* $i \in \{-1, 0, 1\}$, *as* $m \to \infty$.

(d) *If* $|a| < 1$, $|e| < 1$, $|c| < 1$ *and* $|bfd| < |(1-a)(1-e)(1-c)|$, *then* $|x_{3m+i}| \to \infty$, $|y_{3m+i}| \to \infty$, $|z_{3m+i}| \to \infty$, *for* $i \in \{-1, 0, 1\}$, *as* $m \to \infty$.

(e) *If* $|a| < 1$, $|e| < 1$, $|c| < 1$ *and* $|bfd| > |(1-a)(1-e)(1-c)|$, *then* $x_{3m+i} \to 0$, $y_{3m+i} \to 0$, $z_{3m+i} \to 0$, *for* $i \in \{-1, 0, 1\}$, *as* $m \to \infty$.

(f) *If* $|a| < 1$, $|e| < 1$, $|c| < 1$ *and* $bfd = (1-a)(1-e)(1-c)$, *then the sequences* x_{3m+i}, y_{3m+i}, z_{3m+i}, *for* $i \in \{-1, 0, 1\}$, *are convergent.*

(g) *If* $|a| < 1$, $|e| < 1$, $|c| < 1$ *and* $bfd = -(1-a)(1-e)(1-c)$, *then the sequences* x_{6m+i}, x_{6m+3+i}, y_{6m+i}, y_{6m+3+i}, z_{6m+i}, z_{6m+3+i}, *for* $i \in \{-1, 0, 1\}$, *are convergent.*

Proof (a)-(c) Suppose that

$$\alpha_m^{(1)} := \frac{(u_{p_1-3}(1-a)-b)\,a^{m+\lfloor \frac{i}{3} \rfloor+1}+b}{1-a} \frac{(w_{p_2-3}(1-e)-f)\,e^{m+\lfloor \frac{i-1}{3} \rfloor+1}+f}{1-e} \frac{(v_{p_3-3}(1-c)-d)\,c^{m+\lfloor \frac{i-2}{3} \rfloor+1}+d}{1-c},$$

$$\beta_m^{(1)} := \frac{(v_{p_1-3}(1-c)-d)\,c^{m+\lfloor \frac{i}{3} \rfloor+1}+d}{1-c} \frac{(u_{p_2-3}(1-a)-b)\,a^{m+\lfloor \frac{i-1}{3} \rfloor+1}+b}{1-a} \frac{(w_{p_3-3}(1-e)-f)\,e^{m+\lfloor \frac{i-2}{3} \rfloor+1}+f}{1-e},$$

$$\gamma_m^{(1)} := \frac{(w_{p_1-3}(1-e)-f)\,e^{m+\lfloor \frac{i}{3} \rfloor+1}+f}{1-e} \frac{(v_{p_2-3}(1-c)-d)\,c^{m+\lfloor \frac{i-1}{3} \rfloor+1}+d}{1-c} \frac{(u_{p_3-3}(1-a)-b)\,a^{m+\lfloor \frac{i-2}{3} \rfloor+1}+b}{1-a}.$$

We have that

$$\lim_{m\to\infty} |\alpha_m^{(1)}| = \lim_{m\to\infty} |\beta_m^{(1)}| = \lim_{m\to\infty} |\gamma_m^{(1)}| = +\infty,$$

the results easily follow by using formulas (3.14)–(3.16).

(d)-(e) In this case, we have that

$$\lim_{m\to\infty}|\alpha_m^{(1)}| = \lim_{m\to\infty}|\beta_m^{(1)}| = \lim_{m\to\infty}|\gamma_m^{(1)}| = \frac{|bfd|}{|\,(1-a)\,(1-e)\,(1-c)\,|},$$

from which along with (3.14)–(3.16) these results easily follow.

(f) After some calculation, we have that

$$\begin{aligned}\alpha_m^{(1)} &= \frac{(u_{p_1-3}\,(1-a)-b)\,a^{m+\lfloor\frac{i}{3}\rfloor+1}+b}{1-a}\,\frac{(w_{p_2-3}\,(1-e)-f)\,e^{m+\lfloor\frac{i-1}{3}\rfloor+1}+f}{1-e}\,\frac{(v_{p_3-3}\,(1-c)-d)\,c^{m+\lfloor\frac{i-2}{3}\rfloor+1}+d}{1-c}\\ &= 1+\frac{u_{p_1-3}\,(1-a)-b}{b}a^{m+\lfloor\frac{i}{3}\rfloor+1}+\frac{w_{p_2-3}\,(1-e)-f}{f}e^{m+\lfloor\frac{i-1}{3}\rfloor+1}+\frac{v_{p_3-3}\,(1-c)-d}{d}c^{m+\lfloor\frac{i-2}{3}\rfloor+1}+\mathcal{O}\left((aec)^m\right),\end{aligned}$$

$$\begin{aligned}\beta_m^{(1)} &= \frac{(v_{p_1-3}\,(1-c)-d)\,c^{m+\lfloor\frac{i}{3}\rfloor+1}+d}{1-c}\,\frac{(u_{p_2-3}\,(1-a)-b)\,a^{m+\lfloor\frac{i-1}{3}\rfloor+1}+b}{1-a}\,\frac{(w_{p_3-3}\,(1-e)-f)\,e^{m+\lfloor\frac{i-2}{3}\rfloor+1}+f}{1-e}\\ &= 1+\frac{v_{p_1-3}\,(1-c)-d}{d}c^{m+\lfloor\frac{i}{3}\rfloor+1}+\frac{u_{p_2-3}\,(1-a)-b}{b}a^{m+\lfloor\frac{i-1}{3}\rfloor+1}+\frac{w_{p_3-3}\,(1-e)-f}{f}e^{m+\lfloor\frac{i-2}{3}\rfloor+1}+\mathcal{O}\left((cae)^m\right),\end{aligned}$$

and

$$\begin{aligned}\gamma_m^{(1)} &= \frac{(w_{p_1-3}\,(1-e)-f)\,e^{m+\lfloor\frac{i}{3}\rfloor+1}+f}{1-e}\,\frac{(v_{p_2-3}\,(1-c)-d)\,c^{m+\lfloor\frac{i-1}{3}\rfloor+1}+d}{1-c}\,\frac{(u_{p_3-3}\,(1-a)-b)\,a^{m+\lfloor\frac{i-2}{3}\rfloor+1}+b}{1-a}\\ &= 1+\frac{w_{p_1-3}\,(1-e)-f}{f}e^{m+\lfloor\frac{i}{3}\rfloor+1}+\frac{v_{p_2-3}\,(1-c)-d}{d}c^{m+\lfloor\frac{i-1}{3}\rfloor+1}+\frac{u_{p_3-3}\,(1-a)-b}{b}a^{m+\lfloor\frac{i-2}{3}\rfloor+1}+\mathcal{O}\left((eca)^m\right),\end{aligned}$$

from which the convergence of the sequences $\left(\prod_{s=0}^m\alpha_s^{(1)}\right)_{m\in\mathbb{N}_0}$, $\left(\prod_{s=0}^m\beta_s^{(1)}\right)_{m\in\mathbb{N}_0}$ and $\left(\prod_{s=0}^m\gamma_s^{(1)}\right)_{m\in\mathbb{N}_0}$, and, consequently, the convergence of the sequences x_{3m+i}, y_{3m+i} and z_{3m+i}, for $i\in\{-1,0,1\}$ from formulas (3.14)–(3.16) easily follows.

(g) Similar to (f), we have that

$$\begin{aligned}\alpha_m^{(1)} &= \frac{(u_{p_1-3}\,(1-a)-b)\,a^{m+\lfloor\frac{i}{3}\rfloor+1}+b}{1-a}\,\frac{(w_{p_2-3}\,(1-e)-f)\,e^{m+\lfloor\frac{i-1}{3}\rfloor+1}+f}{1-e}\,\frac{(v_{p_3-3}\,(1-c)-d)\,c^{m+\lfloor\frac{i-2}{3}\rfloor+1}+d}{1-c}\\ &= -\left(1+\frac{u_{p_1-3}\,(1-a)-b}{b}a^{m+\lfloor\frac{i}{3}\rfloor+1}+\frac{w_{p_2-3}\,(1-e)-f}{f}e^{m+\lfloor\frac{i-1}{3}\rfloor+1}+\frac{v_{p_3-3}\,(1-c)-d}{d}c^{m+\lfloor\frac{i-2}{3}\rfloor+1}+\mathcal{O}\left((aec)^m\right)\right),\end{aligned}$$

$$\begin{aligned}\beta_m^{(1)} &= \frac{(v_{p_1-3}\,(1-c)-d)\,c^{m+\lfloor\frac{i}{3}\rfloor+1}+d}{1-c}\,\frac{(u_{p_2-3}\,(1-a)-b)\,a^{m+\lfloor\frac{i-1}{3}\rfloor+1}+b}{1-a}\,\frac{(w_{p_3-3}\,(1-e)-f)\,e^{m+\lfloor\frac{i-2}{3}\rfloor+1}+f}{1-e}\\ &= -\left(1+\frac{v_{p_1-3}\,(1-c)-d}{d}c^{m+\lfloor\frac{i}{3}\rfloor+1}+\frac{u_{p_2-3}\,(1-a)-b}{b}a^{m+\lfloor\frac{i-1}{3}\rfloor+1}+\frac{w_{p_3-3}\,(1-e)-f}{f}e^{m+\lfloor\frac{i-2}{3}\rfloor+1}+\mathcal{O}\left((cae)^m\right)\right),\end{aligned}$$

and

$$\begin{aligned}\gamma_m^{(1)} &= \frac{(w_{p_1-3}\,(1-e)-f)\,e^{m+\lfloor\frac{i}{3}\rfloor+1}+f}{1-e}\,\frac{(v_{p_2-3}\,(1-c)-d)\,c^{m+\lfloor\frac{i-1}{3}\rfloor+1}+d}{1-c}\,\frac{(u_{p_3-3}\,(1-a)-b)\,a^{m+\lfloor\frac{i-2}{3}\rfloor+1}+b}{1-a}\\ &= -\left(1+\frac{w_{p_1-3}\,(1-e)-f}{f}e^{m+\lfloor\frac{i}{3}\rfloor+1}+\frac{v_{p_2-3}\,(1-c)-d}{d}c^{m+\lfloor\frac{i-1}{3}\rfloor+1}+\frac{u_{p_3-3}\,(1-a)-b}{b}a^{m+\lfloor\frac{i-2}{3}\rfloor+1}+\mathcal{O}\left((eca)^m\right)\right),\end{aligned}$$

from which the convergence of the sequences $\left(\prod_{s=0}^{3m+i}\alpha_s^{(1)}\right)_{m\in\mathbb{N}_0}$, $\left(\prod_{s=0}^{3m+i}\beta_s^{(1)}\right)_{m\in\mathbb{N}_0}$ and $\left(\prod_{s=0}^{3m+i}\gamma_s^{(1)}\right)_{m\in\mathbb{N}_0}$, $i\in\{-1,0,1\}$ and consequently the convergence of the sequences x_{6m+i}, x_{6m+3+i}, y_{6m+i}, y_{6m+3+i}, z_{6m+i}, z_{6m+3+i} and formulas (3.14)-(3.16) easily follows.

□

Let

$$M_1:=\frac{b^2w_{p_2-3}v_{p_3-3}\left(f-w_{p_2-3}\right)\left(d-v_{p_3-3}\right)}{(1-a)^2},\quad M_2:=\frac{b^2v_{p_1-3}w_{p_3-3}\left(f-w_{p_3-3}\right)\left(d-v_{p_1-3}\right)}{(1-a)^2},$$

$$M_3:=\frac{b^2w_{p_1-3}v_{p_2-3}\left(f-w_{p_1-3}\right)\left(d-v_{p_2-3}\right)}{(1-a)^2},\quad M_4:=\frac{d^2u_{p_1-3}w_{p_2-3}\left(b-u_{p_1-3}\right)\left(f-w_{p_2-3}\right)}{(1-c)^2},$$

$$M_5:=\frac{d^2u_{p_2-3}w_{p_3-3}\left(b-u_{p_2-3}\right)\left(f-w_{p_3-3}\right)}{(1-c)^2},\quad M_6:=\frac{d^2u_{p_3-3}w_{p_1-3}\left(b-u_{p_3-3}\right)\left(f-w_{p_1-3}\right)}{(1-c)^2},$$

$$M_7:=\frac{f^2u_{p_1-3}v_{p_3-3}\left(b-u_{p_1-3}\right)\left(d-v_{p_3-3}\right)}{(1-e)^2},\quad M_8:=\frac{f^2v_{p_1-3}u_{p_2-3}\left(b-u_{p_2-3}\right)\left(d-v_{p_1-3}\right)}{(1-e)^2},$$

$$M_9:=\frac{f^2u_{p_3-3}v_{p_2-3}\left(b-u_{p_3-3}\right)\left(d-v_{p_2-3}\right)}{(1-e)^2},$$

$$M_{10}:=u_{p_1-3}w_{p_2-3}v_{p_3-3}\left(b-u_{p_1-3}\right)\left(f-w_{p_2-3}\right)\left(d-v_{p_3-3}\right),$$

$$M_{11}:=v_{p_1-3}u_{p_2-3}w_{p_3-3}\left(d-v_{p_1-3}\right)\left(b-u_{p_2-3}\right)\left(f-w_{p_3-3}\right),$$

$$M_{12}:=w_{p_1-3}v_{p_2-3}u_{p_3-3}\left(f-w_{p_1-3}\right)\left(d-v_{p_2-3}\right)\left(b-u_{p_3-3}\right).$$

Theorem 3.2 *Suppose that* $k=3l=3$, $abdf\neq 0$, $|a|<1$, $c=-1$, $e=-1$, *and that* $(x_n,y_n,z_n)_{n\geq -4}$ *is a well-defined solution of system* (3.4) *and* $x_{-p},y_{-p},z_{-p}\notin\mathcal{F}$, *for* $p=\overline{1,4}$, $k=3$, $l=1$. *Then the following statements hold.*

(a) *If* $|M_1|>1$, *then* $x_{3m+i}\to 0$, *for* $i\in\{-1,0,1\}$, *as* $m\to\infty$.

(b) *If* $|M_1|<1$, *then* $|x_{3m+i}|\to\infty$, *for* $i\in\{-1,0,1\}$, *as* $m\to\infty$.

(c) *If* $M_1=1$, *then for* $i\in\{-1,0,1\}$, *the sequences* x_{6m+i}, x_{6m+3+i} *are convergent.*

(d) *If* $M_1=-1$, *then the sequences* $x_{12m+3j+i}$, *for* $i\in\{-1,0,1\}$, $j=\overline{0,3}$, *are convergent.*

(e) *If* $|M_2|>1$, *then* $y_{3m+i}\to 0$, *for* $i\in\{-1,0,1\}$, *as* $m\to\infty$.

(f) *If* $|M_2|<1$, *then* $|y_{3m+i}|\to\infty$, *for* $i\in\{-1,0,1\}$, *as* $m\to\infty$.

(g) *If* $M_2 = 1$, *then for* $i \in \{-1, 0, 1\}$, *the sequences* y_{6m+i}, y_{6m+3+i} *are convergent.*

(h) *If* $M_2 = -1$, *then the sequences* $y_{12m+3j+i}$, *for* $i \in \{-1, 0, 1\}$, $j = \overline{0, 3}$, *are convergent.*

(i) *If* $|M_3| > 1$, *then* $z_{3m+i} \to 0$, *for* $i \in \{-1, 0, 1\}$, *as* $m \to \infty$.

(j) *If* $|M_3| < 1$, *then* $|z_{3m+i}| \to \infty$, *for* $i \in \{-1, 0, 1\}$, *as* $m \to \infty$.

(k) *If* $M_3 = 1$, *then for* $i \in \{-1, 0, 1\}$, *the sequences* z_{6m+i}, z_{6m+3+i} *are convergent.*

(l) *If* $M_3 = -1$, *then the sequences* $z_{12m+3j+i}$, *for* $i \in \{-1, 0, 1\}$, $j = \overline{0, 3}$, *are convergent.*

Proof (a), (b) In this case, we have

$$\alpha_m^{(1)} = \frac{(u_{p_1-3}(1-a) - b)\, a^{m+\lfloor \frac{i}{3} \rfloor+1} + b}{1-a} \frac{(2w_{p_2-3} - f)(-1)^{m+\lfloor \frac{i-1}{3} \rfloor+1} + f}{2} \frac{(2v_{p_3-3} - d)(-1)^{m+\lfloor \frac{i-2}{3} \rfloor+1} + d}{2},$$

from which we easily get

$$\alpha_{2m}^{(1)} \alpha_{2m+1}^{(1)} = M_1 + \mathcal{O}\left(a^{2m}\right), \tag{3.17}$$

from which along with (3.14) the results easily follow.

(c) In this case, we get

$$\prod_{s=0}^{2m-1} \alpha_s^{(1)} = \prod_{s=0}^{m-1} \left(1 + \mathcal{O}\left(a^{2s}\right)\right)$$

and

$$\prod_{s=0}^{2m} \alpha_s^{(1)} = \frac{b w_{p_2-3} v_{p_3-3}}{(1-a)} \left(1 + \mathcal{O}\left(a^{2m}\right)\right) \prod_{s=0}^{m-1} \left(1 + \mathcal{O}\left(a^{2s}\right)\right), \tag{3.18}$$

or

$$\prod_{s=0}^{2m} \alpha_s^{(1)} = \frac{b\,(f - w_{p_2-3})\, v_{p_3-3}}{(1-a)} \left(1 + \mathcal{O}\left(a^{2m}\right)\right) \prod_{s=0}^{m-1} \left(1 + \mathcal{O}\left(a^{2s}\right)\right), \tag{3.19}$$

from which it follows that the sequences $\left(\prod_{s=0}^{2m-1} \alpha_s^{(1)}\right)_{m\in\mathbb{N}_0}$ and $\left(\prod_{s=0}^{2m} \alpha_s^{(1)}\right)_{m\in\mathbb{N}_0}$ converge, so by (3.14), x_{6m+i} and x_{6m+3+i} are convergent, as claimed.

(d) In this case, we get

$$\prod_{s=0}^{2m-1} \alpha_s^{(1)} = (-1)^m \prod_{s=0}^{m-1} \left(1 + \mathcal{O}\left(a^{2s}\right)\right).$$

From this and (3.18),(3.19), we have that the sequences $\left(\prod_{s=0}^{4m+j} \alpha_s^{(1)}\right)_{m\in\mathbb{N}_0}$, $j = \overline{0, 3}$, are convergent and by (3.14), $x_{12m+3j+i}$, $i \in \{-1, 0, 1\}$, $j = \overline{0, 3}$, are convergent too.

(e), (f) In this case, we have

$$\beta_m^{(1)} = \frac{(2v_{p_1-3} - d)(-1)^{m+\lfloor \frac{i}{3} \rfloor+1} + d}{2} \frac{(u_{p_2-3}(1-a) - b) a^{m+\lfloor \frac{i-1}{3} \rfloor+1} + b}{1-a} \frac{(2w_{p_3-3} - f)(-1)^{m+\lfloor \frac{i-2}{3} \rfloor+1} + f}{2},$$

from which we easily get

$$\beta_{2m}^{(1)}\beta_{2m+1}^{(1)} = M_2 + \mathcal{O}\left(a^{2m}\right), \tag{3.20}$$

from which along with (3.15), the results easily follow.

(g) In this case, we get

$$\prod_{s=0}^{2m-1} \beta_s^{(1)} = \prod_{s=0}^{m-1} \left(1 + \mathcal{O}\left(a^{2s}\right)\right)$$

and

$$\prod_{s=0}^{2m} \beta_s^{(1)} = \frac{bv_{p_1-3}w_{p_3-3}}{(1-a)} \left(1 + \mathcal{O}\left(a^{2m}\right)\right) \prod_{s=0}^{m-1} \left(1 + \mathcal{O}\left(a^{2s}\right)\right), \tag{3.21}$$

or

$$\prod_{s=0}^{2m} \beta_s^{(1)} = \frac{b(d - v_{p_1-3})w_{p_3-3}}{(1-a)} \left(1 + \mathcal{O}\left(a^{2m}\right)\right) \prod_{s=0}^{m-1} \left(1 + \mathcal{O}\left(a^{2s}\right)\right), \tag{3.22}$$

from which it follows that the sequences $\left(\prod_{s=0}^{2m-1} \beta_s^{(1)}\right)_{m\in\mathbb{N}_0}$ and $\left(\prod_{s=0}^{2m} \beta_s^{(1)}\right)_{m\in\mathbb{N}_0}$ converge, so by (3.15), y_{6m+i} and y_{6m+3+i} are convergent, as claimed.

(h) In this case, we get

$$\prod_{s=0}^{2m-1} \beta_s^{(1)} = (-1)^m \prod_{s=0}^{m-1} \left(1 + \mathcal{O}\left(a^{2s}\right)\right).$$

From this and (3.21),(3.22), we have that the sequences $\left(\prod_{s=0}^{4m+j} \beta_s^{(1)}\right)_{m\in\mathbb{N}_0}$, $j = \overline{0,3}$, are convergent and by (3.15), $y_{12m+3j+i}$, $i \in \{-1, 0, 1\}$, $j = \overline{0,3}$, are convergent too.

(i), (j) In this case, we have

$$\gamma_m^{(1)} = \frac{(2w_{p_1-3} - f)(-1)^{m+\lfloor \frac{i}{3} \rfloor+1} + f}{2} \frac{(2v_{p_2-3} - d)(-1)^{m+\lfloor \frac{i-1}{3} \rfloor+1} + d}{2} \frac{(u_{p_3-3}(1-a) - b) a^{m+\lfloor \frac{i-2}{3} \rfloor+1} + b}{1-a},$$

from which we easily get

$$\gamma_{2m}^{(1)}\gamma_{2m+1}^{(1)} = M_3 + \mathcal{O}\left(a^{2m}\right), \tag{3.23}$$

from which along with (3.16), the results easily follow.

(k) In this case, we get

$$\prod_{s=0}^{2m-1} \gamma_s^{(1)} = \prod_{s=0}^{m-1} \left(1 + \mathcal{O}\left(a^{2s}\right)\right)$$

and

$$\prod_{s=0}^{2m} \gamma_s^{(1)} = \frac{b w_{p_1-3} v_{p_2-3}}{(1-a)} \left(1 + \mathcal{O}\left(a^{2m}\right)\right) \prod_{s=0}^{m-1} \left(1 + \mathcal{O}\left(a^{2s}\right)\right), \tag{3.24}$$

or

$$\prod_{s=0}^{2m} \gamma_s^{(1)} = \frac{b\left(f - w_{p_1-3}\right) v_{p_2-3}}{(1-a)} \left(1 + \mathcal{O}\left(a^{2m}\right)\right) \prod_{s=0}^{m-1} \left(1 + \mathcal{O}\left(a^{2s}\right)\right), \tag{3.25}$$

or

$$\prod_{s=0}^{2m} \gamma_s^{(1)} = \frac{b\left(f - w_{p_1-3}\right) \left(d v_{p_2-3}\right)}{(1-a)} \left(1 + \mathcal{O}\left(a^{2m}\right)\right) \prod_{s=0}^{m-1} \left(1 + \mathcal{O}\left(a^{2s}\right)\right), \tag{3.26}$$

from which it follows that the sequences $\left(\prod_{s=0}^{2m-1} \gamma_s^{(1)}\right)_{m\in\mathbb{N}_0}$ and $\left(\prod_{s=0}^{2m} \gamma_s^{(1)}\right)_{m\in\mathbb{N}_0}$ converge, so by (3.16), z_{6m+i} and z_{6m+3+i} are convergent, as claimed.

(l) In this case, we get

$$\prod_{s=0}^{2m-1} \gamma_s^{(1)} = (-1)^m \prod_{s=0}^{m-1} \left(1 + \mathcal{O}\left(a^{2s}\right)\right).$$

From this and (3.24),(3.25) we have that the sequences $\left(\prod_{s=0}^{4m+j} \gamma_s^{(1)}\right)_{m\in\mathbb{N}_0}$, $j = \overline{0,3}$, are convergent and by (3.16), $z_{12m+3j+i}$, $i \in \{-1,0,1\}$, $j = \overline{0,3}$, are convergent too. □

Theorem 3.3 *Assume that $k = 3l = 3$, $bcdf \neq 0$, $|c| < 1$, $a = -1$, $e = -1$ and that $(x_n, y_n, z_n)_{n\geq -4}$ is a well-defined solution of system (3.4) and $x_{-p}, y_{-p}, z_{-p} \notin \mathcal{F}$, for $p = \overline{1,4}$, $k = 3$, $l = 1$. Then the following statements hold.*

(a) *If $|M_4| > 1$, then $x_{3m+i} \to 0$, for $i \in \{-1,0,1\}$, as $m \to \infty$.*

(b) *If $|M_4| < 1$, then $|x_{3m+i}| \to \infty$, for $i \in \{-1,0,1\}$, as $m \to \infty$.*

(c) *If $M_4 = 1$, then for $i \in \{-1,0,1\}$, the sequences x_{6m+i}, x_{6m+3+i} are convergent.*

(d) *If $M_4 = -1$, then the sequences $x_{12m+3j+i}$, for $i \in \{-1,0,1\}$, $j = \overline{0,3}$, are convergent.*

(e) *If $|M_5| > 1$, then $y_{3m+i} \to 0$, for $i \in \{-1,0,1\}$, as $m \to \infty$.*

(f) *If $|M_5| < 1$, then $|y_{3m+i}| \to \infty$, for $i \in \{-1,0,1\}$, as $m \to \infty$.*

(g) *If $M_5 = 1$, then for $i \in \{-1,0,1\}$, the sequences y_{6m+i}, y_{6m+3+i} are convergent.*

(h) *If* $M_5 = -1$, *then the sequences* $y_{12m+3j+i}$, *for* $i \in \{-1, 0, 1\}$, $j = \overline{0,3}$, *are convergent.*

(i) *If* $|M_6| > 1$, *then* $z_{3m+i} \to 0$, *for* $i \in \{-1, 0, 1\}$, *as* $m \to \infty$.

(j) *If* $|M_6| < 1$, *then* $|z_{3m+i}| \to \infty$, *for* $i \in \{-1, 0, 1\}$, *as* $m \to \infty$.

(k) *If* $M_6 = 1$, *then for* $i \in \{-1, 0, 1\}$, *the sequences* z_{6m+i}, z_{6m+3+i} *are convergent.*

(l) *If* $M_6 = -1$, *then the sequences* $z_{12m+3j+i}$, *for* $i \in \{-1, 0, 1\}$, $j = \overline{0,3}$ *are convergent.*

The proof of Theorem 3.3 is similar to the proof of Theorem 3.2 and utilizes the following three relations:

$$\alpha_m^{(1)} = \frac{(2u_{p_1-3} - b)(-1)^{m+\lfloor \frac{i}{3} \rfloor+1} + b}{2} \frac{(2w_{p_2-3} - f)(-1)^{m+\lfloor \frac{i-1}{3} \rfloor+1} + f}{2} \frac{(v_{p_3-3}(1-c) - d)\, c^{m+\lfloor \frac{i-2}{3} \rfloor+1} + d}{1-c},$$

$$\beta_m^{(1)} = \frac{(v_{p_1-3}(1-c) - d)\, c^{m+\lfloor \frac{i}{3} \rfloor+1} + d}{1-c} \frac{(2u_{p_2-3} - b)(-1)^{m+\lfloor \frac{i-1}{3} \rfloor+1} + b}{2} \frac{(2w_{p_3-3} - f)(-1)^{m+\lfloor \frac{i-2}{3} \rfloor+1} + f}{2},$$

$$\gamma_m^{(1)} = \frac{(2w_{p_1-3} - f)(-1)^{m+\lfloor \frac{i}{3} \rfloor+1} + f}{2} \frac{(v_{p_2-3}(1-c) - d)\, c^{m+\lfloor \frac{i-1}{3} \rfloor+1} + d}{1-c} \frac{(2u_{p_3-3} - b)(-1)^{m+\lfloor \frac{i-2}{3} \rfloor+1} + b}{2},$$

so, it is omitted.

Theorem 3.4 *Assume that* $k = 3l = 3$, $bdef \neq 0$, $|e| < 1$, $a = -1$, $c = -1$ *and that* $(x_n, y_n, z_n)_{n \geq -4}$ *is a well-defined solution of system (3.4) and* $x_{-p}, y_{-p}, z_{-p} \notin \mathcal{F}$, *for* $p = \overline{1,4}$, $k = 3$, $l = 1$. *Then the following statements hold.*

(a) *If* $|M_7| > 1$, *then* $x_{3m+i} \to 0$, *for* $i \in \{-1, 0, 1\}$, *as* $m \to \infty$.

(b) *If* $|M_7| < 1$, *then* $|x_{3m+i}| \to \infty$, *for* $i \in \{-1, 0, 1\}$, *as* $m \to \infty$.

(c) *If* $M_7 = 1$, *then for* $i \in \{-1, 0, 1\}$, *the sequences* x_{6m+i}, x_{6m+3+i} *are convergent.*

(d) *If* $M_7 = -1$, *then the sequences* $x_{12m+3j+i}$, *for* $i \in \{-1, 0, 1\}$, $j = \overline{0,3}$, *are convergent.*

(e) *If* $|M_8| > 1$, *then* $y_{3m+i} \to 0$, *for* $i \in \{-1, 0, 1\}$, *as* $m \to \infty$.

(f) *If* $|M_8| < 1$, *then* $|y_{3m+i}| \to \infty$, *for* $i \in \{-1, 0, 1\}$, *as* $m \to \infty$.

(g) *If* $M_8 = 1$, *then for* $i \in \{-1, 0, 1\}$, *the sequences* y_{6m+i}, y_{6m+3+i} *are convergent.*

(h) *If* $M_8 = -1$, *then the sequences* $y_{12m+3j+i}$, *for* $i \in \{-1, 0, 1\}$, $j = \overline{0,3}$, *are convergent.*

(i) *If* $|M_9| > 1$, *then* $z_{3m+i} \to 0$, *for* $i \in \{-1, 0, 1\}$, *as* $m \to \infty$.

(j) *If* $|M_9| < 1$, *then* $|z_{3m+i}| \to \infty$, *for* $i \in \{-1, 0, 1\}$, *as* $m \to \infty$.

(k) *If* $M_9 = 1$, *then for* $i \in \{-1, 0, 1\}$, *the sequences* z_{6m+i}, z_{6m+3+i} *are convergent.*

(l) *If* $M_9 = -1$, *then the sequences* $z_{12m+3j+i}$, *for* $i \in \{-1, 0, 1\}$, $j = \overline{0, 3}$ *are convergent.*

The proof of Theorem 3.4 is similar to the proof of Theorem 3.2 and employs the following three relations:

$$\alpha_m^{(1)} = \frac{(2u_{p_1-3} - b)(-1)^{m+\lfloor \frac{i}{3} \rfloor+1} + b}{2} \frac{(w_{p_2-3}(1-e) - f)\, e^{m+\lfloor \frac{i-1}{3} \rfloor+1} + f}{1-e} \frac{(2v_{p_3-3} - d)(-1)^{m+\lfloor \frac{i-2}{3} \rfloor+1} + d}{2},$$

$$\beta_m^{(1)} = \frac{(2v_{p_1-3} - d)(-1)^{m+\lfloor \frac{i}{3} \rfloor+1} + d}{2} \frac{(2u_{p_2-3} - b)(-1)^{m+\lfloor \frac{i-1}{3} \rfloor+1} + b}{2} \frac{(w_{p_3-3}(1-e) - f)\, e^{m+\lfloor \frac{i-2}{3} \rfloor+1} + f}{1-e},$$

$$\gamma_m^{(1)} = \frac{(w_{p_1-3}(1-e) - f)\, e^{m+\lfloor \frac{i}{3} \rfloor+1} + f}{1-e} \frac{(2v_{p_2-3} - d)(-1)^{m+\lfloor \frac{i-1}{3} \rfloor+1} + d}{2} \frac{(2u_{p_3-3} - b)(-1)^{m+\lfloor \frac{i-2}{3} \rfloor+1} + b}{2},$$

so, it is omitted.

Theorem 3.5 *Assume that* $k = 3l = 3$, $bdf \neq 0$, $a = c = e = -1$, *and that* $(x_n, y_n, z_n)_{n \geq -4}$ *is a well-defined solution of system* (3.4) *and* $x_{-p}, y_{-p}, z_{-p} \notin \mathcal{F}$, *for* $p = \overline{1, 4}$, $k = 3$, $l = 1$. *Then the following statements hold.*

(a) *If* $|M_{10}| < 1$, *then* $|x_{3m+i}| \to \infty$, *for* $i \in \{-1, 0, 1\}$, *as* $m \to \infty$.

(b) *If* $|M_{10}| > 1$, *then* $x_{3m+i} \to 0$, *for* $i \in \{-1, 0, 1\}$, *as* $m \to \infty$.

(c) *If* $M_{10} = 1$, *then* x_m *is six-periodic.*

(d) *If* $M_{10} = -1$, *then* x_m *is twelve-periodic.*

(e) *If* $|M_{11}| < 1$, *then* $|y_{3m+i}| \to \infty$, *for* $i \in \{-1, 0, 1\}$, *as* $m \to \infty$.

(f) *If* $|M_{11}| > 1$, *then* $y_{3m+i} \to 0$, *for* $i \in \{-1, 0, 1\}$, *as* $m \to \infty$.

(g) *If* $M_{11} = 1$, *then* y_m *is six-periodic.*

(h) *If* $M_{11} = -1$, *then* y_m *is twelve-periodic.*

(i) *If* $|M_{12}| < 1$, *then* $|z_{3m+i}| \to \infty$, *for* $i \in \{-1, 0, 1\}$, *as* $m \to \infty$.

(j) *If* $|M_{12}| > 1$, *then* $z_{3m+i} \to 0$, *for* $i \in \{-1, 0, 1\}$, *as* $m \to \infty$.

(k) *If* $M_{12} = 1$, *then* z_m *is six-periodic.*

(l) *If* $M_{12} = -1$, *then* z_m *is twelve-periodic.*

Proof (a)-(l) Note that, in this case, we get

$$\alpha_m^{(1)} = \frac{(2u_{p_1-3} - b)(-1)^{m+\lfloor \frac{i}{3} \rfloor+1} + b}{2} \frac{(2w_{p_2-3} - f)(-1)^{m+\lfloor \frac{i-1}{3} \rfloor+1} + f}{2} \frac{(2v_{p_3-3} - d)(-1)^{m+\lfloor \frac{i-2}{3} \rfloor+1} + d}{2},$$

$$\beta_m^{(1)} = \frac{(2v_{p_1-3}-d)(-1)^{m+\lfloor\frac{i}{3}\rfloor+1}+d}{2}\frac{(2u_{p_2-3}-b)(-1)^{m+\lfloor\frac{i-1}{3}\rfloor+1}+b}{2}\frac{(2w_{p_3-3}-f)(-1)^{m+\lfloor\frac{i-2}{3}\rfloor+1}+f}{2},$$

$$\gamma_m^{(1)} = \frac{(2w_{p_1-3}-f)(-1)^{m+\lfloor\frac{i}{3}\rfloor+1}+f}{2}\frac{(2v_{p_2-3}-d)(-1)^{m+\lfloor\frac{i-1}{3}\rfloor+1}+d}{2}\frac{(2u_{p_3-3}-b)(-1)^{m+\lfloor\frac{i-2}{3}\rfloor+1}+b}{2}.$$

Hence

$$\alpha_{2m}^{(1)}\alpha_{2m+1}^{(1)} = M_{10}, \quad \beta_{2m}^{(1)}\beta_{2m+1}^{(1)} = M_{11}, \quad \gamma_{2m}^{(1)}\gamma_{2m+1}^{(1)} = M_{12}, \tag{3.27}$$

from which all the statements easily follow.

□

3.1.2. Case $a \neq 1$, $c = 1$, $e \neq 1$

In this case, if (3.5)–(3.10) are utilized in (3.11)–(3.13), it can be easily seen that the solutions of system (3.4) are as follows.

$$x_{3m+i} = \frac{x_{i-3}}{\prod_{j=0}^{m}\frac{(u_{p_1-3}(1-a)-b)a^{j+\lfloor\frac{i}{3}\rfloor+1}+b}{1-a}\frac{(w_{p_2-3}(1-e)-f)e^{j+\lfloor\frac{i-1}{3}\rfloor+1}+f}{1-e}\left(v_{p_3-3}+\left(j+\lfloor\frac{i-2}{3}\rfloor+1\right)d\right)}, \tag{3.28}$$

$$y_{3m+i} = \frac{y_{i-3}}{\prod_{j=0}^{m}\left(v_{p_1-3}+\left(j+\lfloor\frac{i}{3}\rfloor+1\right)d\right)\frac{(u_{p_2-3}(1-a)-b)a^{j+\lfloor\frac{i-1}{3}\rfloor+1}+b}{1-a}\frac{(w_{p_3-3}(1-e)-f)e^{j+\lfloor\frac{i-2}{3}\rfloor+1}+f}{1-e}}, \tag{3.29}$$

$$z_{3m+i} = \frac{z_{i-3}}{\prod_{j=0}^{m}\frac{(w_{p_1-3}(1-e)-f)e^{j+\lfloor\frac{i}{3}\rfloor+1}+f}{1-e}\left(v_{p_2-3}+\left(j+\lfloor\frac{i-1}{3}\rfloor+1\right)d\right)\frac{(u_{p_3-3}(1-a)-b)a^{j+\lfloor\frac{i-2}{3}\rfloor+1}+b}{1-a}}, \tag{3.30}$$

for every $m \in \mathbb{N}_0$, $i \in \{-1,0,1\}$.

3.1.3. Case $a \neq 1$, $c \neq 1$, $e = 1$

In this case, if (3.5)-(3.10) are employed in (3.11)–(3.13), it can be easily seen that the solutions of system (3.4) are as follows.

$$x_{3m+i} = \frac{x_{i-3}}{\prod_{j=0}^{m}\frac{(u_{p_1-3}(1-a)-b)a^{j+\lfloor\frac{i}{3}\rfloor+1}+b}{1-a}\left(w_{p_2-3}+\left(j+\lfloor\frac{i-1}{3}\rfloor+1\right)f\right)\frac{(v_{p_3-3}(1-c)-d)c^{j+\lfloor\frac{i-2}{3}\rfloor+1}+d}{1-c}}, \tag{3.31}$$

$$y_{3m+i} = \frac{y_{i-3}}{\prod_{j=0}^{m}\frac{(v_{p_1-3}(1-c)-d)c^{j+\lfloor\frac{i}{3}\rfloor+1}+d}{1-c}\frac{(u_{p_2-3}(1-a)-b)a^{j+\lfloor\frac{i-1}{3}\rfloor+1}+b}{1-a}\left(w_{p_3-3}+\left(j+\lfloor\frac{i-2}{3}\rfloor+1\right)f\right)}, \tag{3.32}$$

$$z_{3m+i} = \frac{z_{i-3}}{\prod_{j=0}^{m}\left(w_{p_1-3}+\left(j+\lfloor\frac{i}{3}\rfloor+1\right)f\right)\frac{(v_{p_2-3}(1-c)-d)c^{j+\lfloor\frac{i-1}{3}\rfloor+1}+d}{1-c}\frac{(u_{p_3-3}(1-a)-b)a^{j+\lfloor\frac{i-2}{3}\rfloor+1}+b}{1-a}}, \tag{3.33}$$

for every $m \in \mathbb{N}_0$, $i \in \{-1,0,1\}$.

3.1.4. Case $a=1,\ c\neq 1,\ e\neq 1$

In this case, if (3.5)-(3.10) are used in (3.11)-(3.13), it can be easily seen that the solutions of system (3.4) are as follows.

$$x_{3m+i}=\frac{x_{i-3}}{\prod_{j=0}^{m}\left(u_{p_1-3}+\left(j+\lfloor\frac{i}{3}\rfloor+1\right)b\right)\frac{\left(w_{p_2-3}(1-e)-f\right)e^{j+\lfloor\frac{i-1}{3}\rfloor+1}+f}{1-e}\frac{\left(v_{p_3-3}(1-c)-d\right)c^{j+\lfloor\frac{i-2}{3}\rfloor+1}+d}{1-c}},\tag{3.34}$$

$$y_{3m+i}=\frac{y_{i-3}}{\prod_{j=0}^{m}\frac{\left(v_{p_1-3}(1-c)-d\right)c^{j+\lfloor\frac{i}{3}\rfloor+1}+d}{1-c}\left(u_{p_2-3}+\left(j+\lfloor\frac{i-1}{3}\rfloor+1\right)b\right)\frac{\left(w_{p_3-3}(1-e)-f\right)e^{j+\lfloor\frac{i-2}{3}\rfloor+1}+f}{1-e}},\tag{3.35}$$

$$z_{3m+i}=\frac{z_{i-3}}{\prod_{j=0}^{m}\frac{\left(w_{p_1-3}(1-e)-f\right)e^{j+\lfloor\frac{i}{3}\rfloor+1}+f}{1-e}\frac{\left(v_{p_2-3}(1-c)-d\right)c^{j+\lfloor\frac{i-1}{3}\rfloor+1}+d}{1-c}\left(u_{p_3-3}+\left(j+\lfloor\frac{i-2}{3}\rfloor+1\right)b\right)},\tag{3.36}$$

for every $m\in\mathbb{N}_0,\ i\in\{-1,0,1\}$.

3.1.5. Case $a=1,\ c\neq 1,\ e=1$

In this case, if (3.5)-(3.10) are utilized in (3.11)-(3.13), it can be easily seen that the solutions of system (3.4) are as follows.

$$x_{3m+i}=\frac{x_{i-3}}{\prod_{j=0}^{m}\left(u_{p_1-3}+\left(j+\lfloor\frac{i}{3}\rfloor+1\right)b\right)\left(w_{p_2-3}+\left(j+\lfloor\frac{i-1}{3}\rfloor+1\right)f\right)\frac{\left(v_{p_3-3}(1-c)-d\right)c^{j+\lfloor\frac{i-2}{3}\rfloor+1}+d}{1-c}},\tag{3.37}$$

$$y_{3m+i}=\frac{y_{i-3}}{\prod_{j=0}^{m}\frac{\left(v_{p_1-3}(1-c)-d\right)c^{j+\lfloor\frac{i}{3}\rfloor+1}+d}{1-c}\left(u_{p_2-3}+\left(j+\lfloor\frac{i-1}{3}\rfloor+1\right)b\right)\left(w_{p_3-3}+\left(j+\lfloor\frac{i-2}{3}\rfloor+1\right)f\right)},\tag{3.38}$$

$$z_{3m+i}=\frac{z_{i-3}}{\prod_{j=0}^{m}\left(w_{p_1-3}+\left(j+\lfloor\frac{i}{3}\rfloor+1\right)f\right)\frac{\left(v_{p_2-3}(1-c)-d\right)c^{j+\lfloor\frac{i-1}{3}\rfloor+1}+d}{1-c}\left(u_{p_3-3}+\left(j+\lfloor\frac{i-2}{3}\rfloor+1\right)b\right)},\tag{3.39}$$

for every $m\in\mathbb{N}_0,\ i\in\{-1,0,1\}$.

3.1.6. Case $a=1,\ c=1,\ e\neq 1$

In this case, if (3.5)–(3.10) are employed in (3.11)–(3.13), it can be easily seen that the solutions of system (3.4) are as follows.

$$x_{3m+i}=\frac{x_{i-3}}{\prod_{j=0}^{m}\left(u_{p_1-3}+\left(j+\lfloor\frac{i}{3}\rfloor+1\right)b\right)\frac{\left(w_{p_2-3}(1-e)-f\right)e^{j+\lfloor\frac{i-1}{3}\rfloor+1}+f}{1-e}\left(v_{p_3-3}+\left(j+\lfloor\frac{i-2}{3}\rfloor+1\right)d\right)},\tag{3.40}$$

$$y_{3m+i}=\frac{y_{i-3}}{\prod_{j=0}^{m}\left(v_{p_1-3}+\left(j+\lfloor\frac{i}{3}\rfloor+1\right)d\right)\left(u_{p_2-3}+\left(j+\lfloor\frac{i-1}{3}\rfloor+1\right)b\right)\frac{\left(w_{p_3-3}(1-e)-f\right)e^{j+\lfloor\frac{i-2}{3}\rfloor+1}+f}{1-e}},\tag{3.41}$$

$$z_{3m+i}=\frac{z_{i-3}}{\prod_{j=0}^{m}\frac{\left(w_{p_1-3}(1-e)-f\right)e^{j+\lfloor\frac{i}{3}\rfloor+1}+f}{1-e}\left(v_{p_2-3}+\left(j+\lfloor\frac{i-1}{3}\rfloor+1\right)d\right)\left(u_{p_3-3}+\left(j+\lfloor\frac{i-2}{3}\rfloor+1\right)b\right)}, \tag{3.42}$$

for every $m\in\mathbb{N}_0$, $i\in\{-1,0,1\}$.

3.1.7. Case $a\neq 1$, $c=1$, $e=1$

In this case, if (3.5)-(3.10) are used in (3.11)-(3.13), it can be easily seen that the solutions of system (3.4) are as follows.

$$x_{3m+i}=\frac{x_{i-3}}{\prod_{j=0}^{m}\frac{\left(u_{p_1-3}(1-a)-b\right)a^{j+\lfloor\frac{i}{3}\rfloor+1}+b}{1-a}\left(w_{p_2-3}+\left(j+\lfloor\frac{i-1}{3}\rfloor+1\right)f\right)\left(v_{p_3-3}+\left(j+\lfloor\frac{i-2}{3}\rfloor+1\right)d\right)}, \tag{3.43}$$

$$y_{3m+i}=\frac{y_{i-3}}{\prod_{j=0}^{m}\left(v_{p_1-3}+\left(j+\lfloor\frac{i}{3}\rfloor+1\right)d\right)\frac{\left(u_{p_2-3}(1-a)-b\right)a^{j+\lfloor\frac{i-1}{3}\rfloor+1}+b}{1-a}\left(w_{p_3-3}+\left(j+\lfloor\frac{i-2}{3}\rfloor+1\right)f\right)}, \tag{3.44}$$

$$z_{3m+i}=\frac{z_{i-3}}{\prod_{j=0}^{m}\left(w_{p_1-3}+\left(j+\lfloor\frac{i}{3}\rfloor+1\right)f\right)\left(v_{p_2-3}+\left(j+\lfloor\frac{i-1}{3}\rfloor+1\right)d\right)\frac{\left(u_{p_3-3}(1-a)-b\right)a^{j+\lfloor\frac{i-2}{3}\rfloor+1}+b}{1-a}}, \tag{3.45}$$

for every $m\in\mathbb{N}_0$, $i\in\{-1,0,1\}$.

3.1.8. Case $a=1$, $c=1$, $e=1$

In this case, if (3.5)-(3.10) are employed in (3.11)-(3.13), it can be easily seen that the solutions of system (3.4) are as follows.

$$x_{3m+i}=\frac{x_{i-3}}{\prod_{j=0}^{m}\left(u_{p_1-3}+\left(j+\lfloor\frac{i}{3}\rfloor+1\right)b\right)\left(w_{p_2-3}+\left(j+\lfloor\frac{i-1}{3}\rfloor+1\right)f\right)\left(v_{p_3-3}+\left(j+\lfloor\frac{i-2}{3}\rfloor+1\right)d\right)}, \tag{3.46}$$

$$y_{3m+i}=\frac{y_{i-3}}{\prod_{j=0}^{m}\left(v_{p_1-3}+\left(j+\lfloor\frac{i}{3}\rfloor+1\right)d\right)\left(u_{p_2-3}+\left(j+\lfloor\frac{i-1}{3}\rfloor+1\right)b\right)\left(w_{p_3-3}+\left(j+\lfloor\frac{i-2}{3}\rfloor+1\right)f\right)}, \tag{3.47}$$

$$z_{3m+i}=\frac{z_{i-3}}{\prod_{j=0}^{m}\left(w_{p_1-3}+\left(j+\lfloor\frac{i}{3}\rfloor+1\right)f\right)\left(v_{p_2-3}+\left(j+\lfloor\frac{i-1}{3}\rfloor+1\right)d\right)\left(u_{p_3-3}+\left(j+\lfloor\frac{i-2}{3}\rfloor+1\right)b\right)}, \tag{3.48}$$

for every $m\in\mathbb{N}_0$, $i\in\{-1,0,1\}$. Now we will apply these formulas.

Theorem 3.6 *Assume that $k=3l=3$, $abcdef\neq 0$, at least one of a, c, e is equal 1 and that $(x_n,y_n,z_n)_{n\geq -4}$ is a well-defined solution of system (3.4) and $x_{-p},y_{-p},z_{-p}\notin\mathcal{F}$, for $p=\overline{1,4}$, $k=3$, $l=1$. Then $x_n\to 0$, $y_n\to 0$ and $z_n\to 0$ as $n\to\infty$.*

Proof Let

$$\alpha_m^{(2)} := \frac{(u_{p_1-3}(1-a)-b)\,a^{m+\lfloor\frac{i}{3}\rfloor+1}+b}{1-a}\,\frac{(w_{p_2-3}(1-e)-f)\,e^{m+\lfloor\frac{i-1}{3}\rfloor+1}+f}{1-e}\left(v_{p_3-3}+\left(m+\lfloor\frac{i-2}{3}\rfloor+1\right)d\right),$$

$$\beta_m^{(2)} := \left(v_{p_1-3}+\left(m+\lfloor\frac{i}{3}\rfloor+1\right)d\right)\frac{(u_{p_2-3}(1-a)-b)\,a^{m+\lfloor\frac{i-1}{3}\rfloor+1}+b}{1-a}\,\frac{(w_{p_3-3}(1-e)-f)\,e^{m+\lfloor\frac{i-2}{3}\rfloor+1}+f}{1-e},$$

$$\gamma_m^{(2)} := \frac{(w_{p_1-3}(1-e)-f)\,e^{m+\lfloor\frac{i}{3}\rfloor+1}+f}{1-e}\left(v_{p_2-3}+\left(m+\lfloor\frac{i-1}{3}\rfloor+1\right)d\right)\frac{(u_{p_3-3}(1-a)-b)\,a^{m+\lfloor\frac{i-2}{3}\rfloor+1}+b}{1-a},$$

$$\alpha_m^{(3)} := \frac{(u_{p_1-3}(1-a)-b)\,a^{m+\lfloor\frac{i}{3}\rfloor+1}+b}{1-a}\left(w_{p_2-3}+\left(m+\lfloor\frac{i-1}{3}\rfloor+1\right)f\right)\frac{(v_{p_3-3}(1-c)-d)\,c^{m+\lfloor\frac{i-2}{3}\rfloor+1}+d}{1-c},$$

$$\beta_m^{(3)} := \frac{(v_{p_1-3}(1-c)-d)\,c^{m+\lfloor\frac{i}{3}\rfloor+1}+d}{1-c}\,\frac{(u_{p_2-3}(1-a)-b)\,a^{m+\lfloor\frac{i-1}{3}\rfloor+1}+b}{1-a}\left(w_{p_3-3}+\left(m+\lfloor\frac{i-2}{3}\rfloor+1\right)f\right),$$

$$\gamma_m^{(3)} := \left(w_{p_1-3}+\left(m+\lfloor\frac{i}{3}\rfloor+1\right)f\right)\frac{(v_{p_2-3}(1-c)-d)\,c^{m+\lfloor\frac{i-1}{3}\rfloor+1}+d}{1-c}\,\frac{(u_{p_3-3}(1-a)-b)\,a^{m+\lfloor\frac{i-2}{3}\rfloor+1}+b}{1-a},$$

$$\alpha_m^{(4)} := \left(u_{p_1-3}+\left(m+\lfloor\frac{i}{3}\rfloor+1\right)b\right)\frac{(w_{p_2-3}(1-e)-f)\,e^{m+\lfloor\frac{i-1}{3}\rfloor+1}+f}{1-e}\,\frac{(v_{p_3-3}(1-c)-d)\,c^{m+\lfloor\frac{i-2}{3}\rfloor+1}+d}{1-c},$$

$$\beta_m^{(4)} := \frac{(v_{p_1-3}(1-c)-d)\,c^{m+\lfloor\frac{i}{3}\rfloor+1}+d}{1-c}\left(u_{p_2-3}+\left(m+\lfloor\frac{i-1}{3}\rfloor+1\right)b\right)\frac{(w_{p_3-3}(1-e)-f)\,e^{m+\lfloor\frac{i-2}{3}\rfloor+1}+f}{1-e},$$

$$\gamma_m^{(4)} := \frac{(w_{p_1-3}(1-e)-f)\,e^{m+\lfloor\frac{i}{3}\rfloor+1}+f}{1-e}\,\frac{(v_{p_2-3}(1-c)-d)\,c^{m+\lfloor\frac{i-1}{3}\rfloor+1}+d}{1-c}\left(u_{p_3-3}+\left(m+\lfloor\frac{i-2}{3}\rfloor+1\right)b\right),$$

$$\alpha_m^{(5)} := \left(u_{p_1-3}+\left(m+\lfloor\frac{i}{3}\rfloor+1\right)b\right)\left(w_{p_2-3}+\left(m+\lfloor\frac{i-1}{3}\rfloor+1\right)f\right)\frac{(v_{p_3-3}(1-c)-d)\,c^{m+\lfloor\frac{i-2}{3}\rfloor+1}+d}{1-c},$$

$$\beta_m^{(5)} := \frac{(v_{p_1-3}(1-c)-d)\,c^{m+\lfloor\frac{i}{3}\rfloor+1}+d}{1-c}\left(u_{p_2-3}+\left(m+\lfloor\frac{i-1}{3}\rfloor+1\right)b\right)\left(w_{p_3-3}+\left(m+\lfloor\frac{i-2}{3}\rfloor+1\right)f\right),$$

$$\gamma_m^{(5)} := \left(w_{p_1-3} + \left(m + \lfloor\frac{i}{3}\rfloor + 1\right) f\right) \frac{\left(v_{p_2-3}\left(1-c\right)-d\right) c^{m+\lfloor\frac{i-1}{3}\rfloor+1} + d}{1-c} \left(u_{p_3-3} + \left(m + \lfloor\frac{i-2}{3}\rfloor + 1\right) b\right),$$

$$\alpha_m^{(6)} := \left(u_{p_1-3} + \left(m + \lfloor\frac{i}{3}\rfloor + 1\right) b\right) \frac{\left(w_{p_2-3}\left(1-e\right)-f\right) e^{m+\lfloor\frac{i-1}{3}\rfloor+1} + f}{1-e} \left(v_{p_3-3} + \left(m + \lfloor\frac{i-2}{3}\rfloor + 1\right) d\right),$$

$$\beta_m^{(6)} := \left(v_{p_1-3} + \left(m + \lfloor\frac{i}{3}\rfloor + 1\right) d\right) \left(u_{p_2-3} + \left(m + \lfloor\frac{i-1}{3}\rfloor + 1\right) b\right) \frac{\left(w_{p_3-3}\left(1-e\right)-f\right) e^{m+\lfloor\frac{i-2}{3}\rfloor+1} + f}{1-e},$$

$$\gamma_m^{(6)} := \frac{\left(w_{p_1-3}\left(1-e\right)-f\right) e^{m+\lfloor\frac{i}{3}\rfloor+1} + f}{1-e} \left(v_{p_2-3} + \left(m + \lfloor\frac{i-1}{3}\rfloor + 1\right) d\right) \left(u_{p_3-3} + \left(m + \lfloor\frac{i-2}{3}\rfloor + 1\right) b\right),$$

$$\alpha_m^{(7)} := \frac{\left(u_{p_1-3}\left(1-a\right)-b\right) a^{m+\lfloor\frac{i}{3}\rfloor+1} + b}{1-a} \left(w_{p_2-3} + \left(m + \lfloor\frac{i-1}{3}\rfloor + 1\right) f\right) \left(v_{p_3-3} + \left(m + \lfloor\frac{i-2}{3}\rfloor + 1\right) d\right),$$

$$\beta_m^{(7)} := \left(v_{p_1-3} + \left(m + \lfloor\frac{i}{3}\rfloor + 1\right) d\right) \frac{\left(u_{p_2-3}\left(1-a\right)-b\right) a^{m+\lfloor\frac{i-1}{3}\rfloor+1} + b}{1-a} \left(w_{p_3-3} + \left(m + \lfloor\frac{i-2}{3}\rfloor + 1\right) f\right),$$

$$\gamma_m^{(7)} := \left(w_{p_1-3} + \left(m + \lfloor\frac{i}{3}\rfloor + 1\right) f\right) \left(v_{p_2-3} + \left(m + \lfloor\frac{i-1}{3}\rfloor + 1\right) d\right) \frac{\left(u_{p_3-3}\left(1-a\right)-b\right) a^{m+\lfloor\frac{i-2}{3}\rfloor+1} + b}{1-a},$$

$$\alpha_m^{(8)} := \left(u_{p_1-3} + \left(m + \lfloor\frac{i}{3}\rfloor + 1\right) b\right) \left(w_{p_2-3} + \left(m + \lfloor\frac{i-1}{3}\rfloor + 1\right) f\right) \left(v_{p_3-3} + \left(m + \lfloor\frac{i-2}{3}\rfloor + 1\right) d\right),$$

$$\beta_m^{(8)} := \left(v_{p_1-3} + \left(m + \lfloor\frac{i}{3}\rfloor + 1\right) d\right) \left(u_{p_2-3} + \left(m + \lfloor\frac{i-1}{3}\rfloor + 1\right) b\right) \left(w_{p_3-3} + \left(m + \lfloor\frac{i-2}{3}\rfloor + 1\right) f\right),$$

$$\gamma_m^{(8)} := \left(w_{p_1-3} + \left(m + \lfloor\frac{i}{3}\rfloor + 1\right) f\right) \left(v_{p_2-3} + \left(m + \lfloor\frac{i-1}{3}\rfloor + 1\right) d\right) \left(u_{p_3-3} + \left(m + \lfloor\frac{i-2}{3}\rfloor + 1\right) b\right).$$

Since

$$\lim_{m\to\infty} |\alpha_m^{(r)}| = \lim_{m\to\infty} |\beta_m^{(r)}| = \lim_{m\to\infty} |\gamma_m^{(r)}| = \infty, \quad r = \overline{2,8},$$

from (3.28)-(3.48) the statement easily follows. □

4. Conclusion

In this paper, we have considered the following three-dimensional system of difference equations

$$x_n = \frac{x_{n-k}z_{n-l}}{b_n x_{n-k} + a_n z_{n-k-l}},\ y_n = \frac{y_{n-k}x_{n-l}}{d_n y_{n-k} + c_n x_{n-k-l}},\ z_n = \frac{z_{n-k}y_{n-l}}{f_n z_{n-k} + e_n y_{n-k-l}},$$

where $n \in \mathbb{N}_0$, $k, l \in \mathbb{N}$, the sequences $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$, $(c_n)_{n\in\mathbb{N}_0}$, $(d_n)_{n\in\mathbb{N}_0}$, $(e_n)_{n\in\mathbb{N}_0}$ and $(f_n)_{n\in\mathbb{N}_0}$ are non-zero real numbers, for all $n \in \mathbb{N}_0$, and the initial values x_{-i}, y_{-i}, z_{-i} are real numbers for $i \in \overline{1, k+l}$.

Firstly, we have obtained the closed form of well defined solutions of the aforementioned system using suitable transformation reducing the equations of our system to linear type. Also, we describe the forbidden set of the initial values using the obtained formulas. In addition, in the case where the coefficients are constant and $k = 3$, $l = 1$ in the system, we have obtained the solutions for some possible cases of a, c and e. Finally, we have examined the asymptotic behavior of the solutions of this system for 8-case.

We will give the following important open problem for system of difference equations theory to researchers.

Open problem: The system (1.7) can extend to the following $p-$dimensional system of difference equations.

$$x_n^{(1)} = \frac{x_{n-k}^{(1)}x_{n-l}^{(3)}}{b_n^{(1)}x_{n-k}^{(1)} + a_n^{(1)}x_{n-k-l}^{(3)}},\ x_n^{(2)} = \frac{x_{n-k}^{(2)}x_{n-l}^{(4)}}{b_n^{(2)}x_{n-k}^{(2)} + a_n^{(2)}x_{n-k-l}^{(4)}}, \ldots, x_n^{(p)} = \frac{x_{n-k}^{(p)}x_{n-l}^{(2)}}{b_n^{(p)}x_{n-k}^{(p)} + a_n^{(p)}x_{n-k-l}^{(2)}},\ n \in \mathbb{N}_0, \quad (4.1)$$

where $k, l \in \mathbb{N}$, the sequences $\left(a_n^{(j)}\right)_{n\in\mathbb{N}_0}$ and $\left(b_n^{(j)}\right)_{n\in\mathbb{N}_0}$ are non-zero real numbers, for $j \in \overline{1,p}$, and the initial values $x_{-i}^{(j)}$, are real numbers for $i \in \overline{1, k+l}$, $j \in \overline{1,p}$. Can system (4.1) be solved?

References

[1] Abo-Zeid R, Kamal H. Global behavior of two rational third order difference equations. Universal Journal of Mathematics and Applications 2019; 2 (4): 212-217. doi:10.32323/ujma.626465

[2] Abo-Zeid R. Behavior of solutions of a second order rational difference equation. Mathematica Moravica 2019; 23 (1): 11-25. doi: 10.5937/MatMor1901011A

[3] De Moivre A (1756). The doctrine of chances. In: Landmark Writings in Western Mathematics, London

[4] Dekkar I, Touafek N, Yazlik Y. Global stability of a third-order nonlinear system of difference equations with period-two coefficients. Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales Serie A Matematicas 2017; 111 (2): 325-347. doi:10.1007/s13398-016-0297-z

[5] Elabbasy EM, El-Metwally HA, Elsayed EM. Global behavior of the solutions of some difference equations. Advances in Difference Equations 2011; 2011 (1): 1-16.

[6] Elmetwally ME, Elsayed EM. Dynamics of a rational difference equation. Chinese Annals of Mathematics, Series B 2009; 30B (2): 187-198. doi:10.1007/s11401-007-0456-9

[7] Elmetwally H. Solutions form for some rational systems of difference equations. Discrete Dynamics in Nature and Society 2013; Article ID 903593: 10 pages. doi:10.1155/2013/903593

[8] Elsayed EM. Qualitative behavior of a rational recursive sequence. Indagationes Mathematicae 2008; 19 (2): 189-201.

[9] Elsayed EM. Qualitative properties for a fourth order rational difference equation. Acta Applicandae Mathematicae 2010; 110 (2): 589-604. doi:10.1007/s10440-009-9463-z

[10] Elsayed EM.Solution for systems of difference equations of rational form of order two. Computational and Applied Mathematics 2014; 33 (3): 751-765.

[11] Elsayed EM. Expression and behavior of the solutions of some rational recursive sequences. Mathematical Methods in the Applied Sciences 2016; 18 (39): 5682-5694. doi:10.1002/mma.3953

[12] Euler L. Introductio in analysin infinitorum. Apud Marcum-Michaelem Bousquet & Socios 1748; 2: 1-321. (in Latin).

[13] Folly-Gbetoula M, Manda K, Gadjagboui BBI. The invariance, formulas for solutions and periodicity of some recurrence equations. International Journal of Contemporary Mathematical Sciences 2019; 14 (4): 201-210. doi:10.12988/ijcms.2019.9820

[14] Halim Y, Touafek N, Yazlik Y. Dynamic behavior of a second-order nonlinear rational difference equation. Turkish Journal of Mathematics 2015; 39 (6): 1004-1018. doi:10.3906/mat-1503-80

[15] Halim Y, Bayram M. On the solutions of a higher-order difference equation in terms of generalized Fibonacci sequence. Mathematical Methods in the Applied Science 2016; 39 (11): 2974-2982. doi:10.1002/mma.3745

[16] Halim Y, Rabago JFT. On the solutions of a second-order difference equation in terms of generalized Padovan sequences. Mathematica Slovaca 2018; 68 (3): 625-638.

[17] Kara M, Yazlik Y. Solvability of a system of nonlinear difference equations of higher order. Turkish Journal of Mathematics 2019; 43 (3): 1533-1565. doi:10.3906/mat-1902-24

[18] Kara M, Yazlik Y. On the system of difference equations $x_n = \frac{x_{n-2}y_{n-3}}{y_{n-1}\left(a_n+b_nx_{n-2}y_{n-3}\right)}$, $y_n = \frac{y_{n-2}x_{n-3}}{x_{n-1}\left(\alpha_n+\beta_ny_{n-2}x_{n-3}\right)}$. Journal of Mathematical Extension 2020; 14 (1): 41-59.

[19] Kara M, Touafek N, Yazlik Y. Well-defined solutions of a three-dimensional system of difference equations. Gazi University Journal of Science 2020; 33 (3): 767-778. doi:10.35378/gujs.641441

[20] Kara M, Yazlik Y, Tollu DT. Solvability of a system of higher order nonlinear difference equations. Hacettepe Journal of Mathematics & Statistics 2020; 49 (5): 1566-1593. doi:10.15672/hujms.474649

[21] Kara M, Yazlik Y. On a solvable three-dimensional system of difference equations. Filomat 2020; 34 (4): 1167-1186. doi:10.2298/FIL2004167K

[22] Khelifa A, Halim Y. General solutions to systems of difference equations and some of their representations. Journal of Applied Mathematics and Computing 2021; 106: 1-15. doi:10.1007/s12190-020-01476-8

[23] Öcalan Ö. Oscillation of nonlinear difference equations with several coefficients. Communications in Mathematical Analysis 2008; 4 (1): 35-44.

[24] Papaschinopoulos G, Stefanidou G. Asymptotic behavior of the solutions of a class of rational difference equations. International Journal of Difference Equations 2010; 5 (2): 233-249.

[25] Rabago JFT, Bacani JB. On a nonlinear difference equations. Dynamics of Continuous, Discrete & Impulsive Systems. Series A. Mathematical Analysis 2017; 24: 375-394.

[26] Stević S, Alghamdi MA, Shahzad N, Maturi DA. On a class of solvable difference equations. Abstract and Applied Analysis 2013; 2013: 1-7. doi:10.1155/2013/157943

[27] Stević S, Diblík J, Iričanin B, Šmarda Z. On a solvable system of rational difference equations. Journal of Difference Equations and Applications 2014; 20 (5-6): 811-825. doi:10.1080/10236198.2013.817573

[28] Stević S. General solutions to four classes of nonlinear difference equations and some of their representations. Electronic Journal of Qualitative Theory of Differential Equations 2019; 75: 1-19. doi:10.14232/ejqtde.2019.1.75

[29] Tasdemir E, Soykan Y. Stability of negative equilibrium of a non-linear difference equation. Journal of Mathematical Sciences Advances and Applications 2018; 49 (1): 51-57. doi:10.18642/jmsaa_7100121927

[30] Tollu DT, Yazlik Y, Taskara N. On a solvable nonlinear difference equation of higher order. Turkish Journal of Mathematics 2018; 42 (4): 1765-1778. doi:10.3906/mat-1705-33

[31] Tollu DT, Yazlik Y, Taskara N. Behavior of positive solutions of a difference equation. Journal of Applied Mathematics & Informatics 2017; 35: 217-230. doi:10.14317/jami.2017.217

[32] Tollu DT. Periodic solutions of a system of nonlinear difference equations with periodic coefficients. Journal of Mathematics 2020; Article ID 6636105: 7 pages. doi:10.1155/2020/6636105

[33] Touafek N. On a second order rational difference equation. Hacettepe Journal of Mathematics and Statistics 2012; 41 (6): 867-874.

[34] Touafek N, Elsayed EM. On a second order rational systems of difference equations. Hokkaido Mathematical Journal 2015; 44 (1): 29-45.

[35] Yalcinkaya I, Cinar C. Global asymptotic stability of a system of two nonlinear difference equations. Fasciculi Mathematici 2010; 43: 171-180.

[36] Yalcinkaya I, Tollu DT. Global behavior of a second order system of difference equations. Advanced Studies in Contemporary Mathematics 2016; 26 (4): 653-667.

[37] Yazlik Y, Kara M. On a solvable system of difference equations of higher-order with period two coefficients. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics 2019; 68 (2): 1675-1693. doi: 10.31801/cfsuasmas.548262