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On the existence for parametric boundary value problems of a coupled system of nonlinear fractional hybrid differential equations

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Abstract: In this paper, we consider the existence and uniqueness for parametric boundary value problems of a coupled system of nonlinear fractional hybrid differential equations. By the fixed point theorem in Banach algebra, an existence theorem for parametric boundary value problems of a coupled system of nonlinear fractional hybrid differential equations is given. Further, a uniqueness result for parametric boundary value problems of a coupled system of nonlinear fractional hybrid differential equations is proved due to Banach's contraction principle. Further, we give three examples to verify the main results.

Key words: Boundary value problems, coupled fractional hybrid differential systems, Dhage fixed point theorem, existence.

1. Introduction

In this paper, we discuss the following parametric boundary value problems (in short PBVP) of a coupled system of nonlinear fractional hybrid differential equations (in short FHDS)

$$\begin{cases} {}^{C}D^{\alpha}_{0+}\left[\frac{y(\zeta)-k_1(\zeta,y(\zeta))}{f_1(\zeta,y(\zeta))}\right] = g_1(\zeta,y(\zeta),z(\zeta)), \quad 0<\zeta<1, \\ {}^{C}D^{\beta}_{0+}\left[\frac{z(\zeta)-k_2(\zeta,z(\zeta))}{f_2(\zeta,z(\zeta))}\right] = g_2(\zeta,y(\zeta),z(\zeta)), \quad 0<\zeta<1, \\ a_1\left[\frac{y(\zeta)-k_1(\zeta,y(\zeta))}{f_1(\zeta,y(\zeta))}\right]_{\zeta=0} + b_1\left[\frac{y(\zeta)-k_1(\zeta,y(\zeta))}{f_1(\zeta,y(\zeta))}\right]_{\zeta=1} = c_1, \\ a_2\left[\frac{z(\zeta)-k_2(\zeta,z(\zeta))}{f_2(\zeta,z(\zeta))}\right]_{\zeta=0} + b_2\left[\frac{z(\zeta)-k_2(\zeta,z(\zeta))}{f_2(\zeta,y(\zeta))}\right]_{\zeta=1} = c_2, \end{cases} \tag{1.1}$$

where $0<\alpha,\ \beta\le 1$, ${}^{C}D^{\alpha}_{0+}$ and ${}^{C}D^{\beta}_{0+}$ are the Caputo derivatives, $\Delta=[0,1]$, $f_i\in C(\Delta\times\mathbb{R},\mathbb{R}\setminus\{0\})$, $k_i\in C(\Delta\times\mathbb{R},\mathbb{R})$, $g_i\in \mathcal{C}(\Delta\times\mathbb{R}\times\mathbb{R},\mathbb{R})(i=1,\ 2)$, and $a_i,\ b_i,\ c_i(i=1,\ 2)$ are real constants with $a_i+b_i\neq 0(i=1,\ 2)$.

The theory and the applications of the FDE have gained many researchers' attention, see the study in [10]. Many papers on the solvability of the nonlinear FDE and FDS, see [6, 12, 14, 15, 17]. In recent years, the theory of the HDE has been a hot research topic; see [1–5, 7, 8, 11, 13, 16, 18]. Dhage [4] discussed the following first order hybrid differential equation

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\frac{x(\zeta)-k(\zeta,x(\zeta))}{f(\zeta,x(\zeta))}\right] = g(\zeta,x(\zeta)), \quad \zeta\in[\zeta_0,\zeta_0+a], \\ x(\zeta_0)=x_0\in\mathbb{R}, \end{cases}$$

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and developed the theory of the HDE and gave some original and interesting results. Dumitru Baleanu et al. [2] investigated sufficient conditions for existence and uniqueness of solutions for a coupled system of fractional order hybrid differential equations with multi-point hybrid boundary conditions due to Dhage and Banach's contraction principle. Amjad Ali [1] established necessary and sufficient conditions for existence of positive solutions for coupled systems of nonlinear hybrid differential equations subject to nonhomogeneous boundary conditions using hybrid fixed point theorem due to Dhage. Dhage et al. [5] studied a system of two nonhomogeneous boundary value problems of coupled hybrid integro-differential equations of fractional order by a hybrid fixed point theorem due to Dhage in Banach algebras. You and Sun [13] discussed a class of impulsive coupled hybrid fractional differential system due to a new hybrid fixed point theorem in Banach algebra.

To the best of our knowledge, there are no results for the PBVP (1.1) of the nonlinear coupled FHDS. From the above works, we consider the existence and uniqueness of the BVP (1.1) of the nonlinear coupled FHDS. An existence theorem and a uniqueness result for the PBVP (1.1) of the nonlinear coupled FDS are obtained. Further, three examples are given to verify the main results. To some extent, our work fills the gap on some basic theory for BVP of the nonlinear coupled FHDS.

The paper is organized as follows: Section 2 gives some theory of fractional calculus. Section 3 establishes an existence theorem for the PBVP (1.1) of the nonlinear coupled FHDS by the fixed point theorem in Banach algebra. Section 4 considers a uniqueness result for the PBVP (1.1) of the nonlinear coupled FHDS by Banach's contraction principle. Section 5 presents three examples to verify the existence theorem.

2. Preliminary

In this section, we give some basic theory from fractional calculus, see [3, 9, 13, 18].

Definition 2.1 *([9]) The Caputo fractional derivative of order* $0 < \alpha < 1$ *of a continuous function* $f : (0, +\infty) \to \mathbb{R}$ *is given by*

$$^{C}D^{\alpha}_{0+}f(\zeta) = \frac{1}{\Gamma(1-\alpha)} \int_0^{\zeta} \frac{f'(\omega)}{(\zeta-\omega)^{\alpha}} d\omega.$$

Definition 2.2 *([9]) The Riemann-Liouville fractional integral of order* $\alpha > 0$ *of an integrable function* $f : (0, +\infty) \to \mathbb{R}$ *is given by*

$$I^{\alpha}_{0+}f(\zeta) = \frac{1}{\Gamma(\alpha)} \int_0^{\zeta} (\zeta-\omega)^{\alpha-1} f(\omega) d\omega,$$

provided that the right side is pointwise defined on $(0, +\infty)$.

Lemma 2.3 *([9]) Let* $\alpha > 0$. *If we assume* $y \in C(0,1) \cap L(0,1)$, *then the fractional differential equation*

$$^{C}D^{\alpha}_{0+}y(\zeta) = 0$$

has $y(\zeta) = c_0 + c_1\zeta + c_2\zeta^2 + \cdots + c_{n-1}\zeta^{n-1}$, $c_i \in \mathbb{R}$, $i = 0, 1, 2, \cdots, n-1$, *as unique solutions, where* n *is the smallest integer greater than or equal to* α.

Lemma 2.4 *([9]) Assume that* $y \in C^n[0,1]$ *with a fractional derivative of order* $\alpha > 0$ *that belongs to* $C^n[0,1]$. *Then*

$$I^{\alpha}_{0+} {}^{C}D^{\alpha}_{0+}y(\zeta) = y(\zeta) + c_0 + c_1\zeta + c_2\zeta^2 + \cdots + c_{n-1}\zeta^{n-1},$$

for some $c_i \in \mathbb{R},\ i = 0, 1, 2, \cdots, n-1,$ *where* n *is the smallest integer greater than or equal to* α*.*

Let $C(\Delta, \mathbb{R})$ be the space of all continuous functions defined on Δ. $\|\cdot\|$ denotes a supremum norm in $C(\Delta, \mathbb{R})$ by

$$\|y\| = \sup_{\zeta\in\Delta} |y(\zeta)|,$$

and a multiplication "$\cdot$" in $C(\Delta, \mathbb{R})$ by

$$(y \cdot z)(\zeta) = (yz)(\zeta) = y(\zeta)z(\zeta)$$

for $y,\ z \in C(\Delta, \mathbb{R})$. Clearly $C(\Delta, \mathbb{R})$ is a Banach algebra with respect to above norm and multiplication in it.

Lemma 2.5 *([18]) Suppose that* $a,\ b,\ c$ *are real constants with* $a + b \neq 0$*. Then for any* $z \in L(\Delta, \mathbb{R})$*, the function* y *is a solution of the PBVP*

$$ {}^{C}D^{\alpha}_{0+}\Big[\frac{y(\zeta) - k(\zeta, y(\zeta))}{f(\zeta, y(\zeta))}\Big] = z(\zeta),\ 0 < \alpha \leq 1,\ \zeta \in \Delta, \tag{2.1}$$

and

$$a\Big[\frac{y(\zeta) - k(\zeta, y(\zeta))}{f(\zeta, y(\zeta))}\Big]_{\zeta=0} + b\Big[\frac{y(\zeta) - k(\zeta, y(\zeta))}{f(\zeta, y(\zeta))}\Big]_{\zeta=1} = c, \tag{2.2}$$

if and only if y *satisfies the integral equation*

$$\begin{aligned} y(\zeta) \quad &= \quad f(\zeta, y(\zeta))\Big(\frac{1}{\Gamma(\alpha)}\int_0^\zeta (\zeta-\omega)^{\alpha-1} z(\omega) d\omega \\ &\quad + \frac{1}{a+b}\Big(c - \frac{b}{\Gamma(\alpha)}\int_0^1 (1-\omega)^{\alpha-1} z(\omega) d\omega\Big)\Big) + k(\zeta, y(\zeta)), \quad \zeta \in \Delta. \end{aligned} \tag{2.3}$$

Let $\tilde{U} = C(\Delta, \mathbb{R})$. Define multiplication and the sum on $X \times X$ as

$$(y_1, z_1) + (y_2, z_2) = (y_1 + y_2, z_1 + z_2),$$

$$p(y, z) = (py, pz), p \in \mathbb{R}.$$

Lemma 2.6 *([3]) Let* $\overline{U} = U \times U$*. Define the product in* X *by*

$$(y_1, z_1)(y_2, z_2) = (y_1y_2, z_1z_2),$$

and

$$\|(y, z)\| = 2(\|y\| + \|z\|).$$

Then $\overline{U}$ *is a Banach algebra with respect to the above norm and multiplication.*

Lemma 2.7 *([13]) Let* $\tilde{S}$ *be a nonempty, closed, convex and bounded subset of a Banach algebra* X *and* $\overline{U} = \tilde{U} \times \tilde{U}, \overline{S} = \tilde{S} \times \tilde{S}$*. Suppose* $\tilde{A}_i, \tilde{C}_i : \tilde{U} \to \tilde{U}, \tilde{B}_i : \tilde{S} \to \tilde{U}(i = 1, 2)$ *are operators satisfying* *(a) There exist* $0 < \tilde{\rho_i}, \tilde{\delta_i} < 1$*, such that, respectively,*

$$\| \tilde{A}_i y - \tilde{A}_i z \| \leq \tilde{\rho_i} \| y - z \|, \| \tilde{C}_i y - \tilde{C}_i z \| \leq \tilde{\delta_i} \| y - z \|,$$

for all $y, z \in \tilde{U}$*,* $i = 1, 2$*,* $\tilde{\rho} = \max\{\tilde{\rho_1}, \tilde{\rho_2}\}, \delta = \max\{\tilde{\delta_1}, \tilde{\delta_2}\}$*,*

(b) $\tilde{B}_i$ *is completely continuous,* $i = 1, 2$*,*

(c) $y = \tilde{A}_1 y \tilde{B}_1 z + \tilde{C}_1 y$ *for all* $z \in \tilde{S} \Rightarrow y \in \tilde{S}$*, and* $z = \tilde{A}_2 z \tilde{B}_2 y + \tilde{C}_2 z$ *for all* $y \in \tilde{S} \Rightarrow z \in \tilde{S}$*,*

(d) $4\rho \|\tilde{B}(\tilde{S})\| + \tilde{\delta} < 1, \|\tilde{B}(\tilde{S})\| = \max\{\sup\{\|\tilde{B}_1(y)\| : y \in \tilde{S}\}, \sup\{\|\tilde{B}_2(y)\| : y \in \tilde{S}\}\}$*.*

Then the operator equation $(\tilde{T_1}(y, z), \tilde{T_2}(y, z)) = (y, z)$ *has a fixed point in* $\overline{S}$*, where* $\tilde{T_1}, \tilde{T_2} : \overline{U} \to \tilde{U}$ *are defined by*

$$\tilde{T_1}(y, z) = \tilde{A}_1 y \tilde{B}_1 z + \tilde{C}_1 y, \ \tilde{T_2}(y, z) = \tilde{A}_2 z \tilde{B}_2 y + \tilde{C}_2 z, \quad (y, z) \in \overline{U}.$$

3. Existence result

In this section, we discuss the existence results for the PBVP (1.1) of the nonlinear coupled FHDS.

We present the following hypotheses.

(H_1) There exist constants $L_i > 0$ $(i = 1, 2)$ and $\bar{L}_i > 0$ $(i = 1, 2)$ such that

$$|f_i(\zeta, y) - f_i(\zeta, z)| \leq L_i |y - z|$$

and

$$|k_i(\zeta, y) - k_i(\zeta, z)| \leq \bar{L}_i |y - z|$$

for all $\zeta \in \Delta$ and $y,\ z \in \mathbb{R}$.

(H_2) There exist two nonnegative functions $d_i(\zeta) \in L(0, 1)$ $(i = 1, 2)$ such that $g_i(\zeta, y, z) \leq d_i(\zeta) + m_i |y|^{\rho_i} + n_i |z|^{\theta_i}$, where $m_i, n_i \geq 0,\ 0 < \rho_i, \theta_i < 1,$ for $i = 1,\ 2$.

Now we will give the following existence theorem for the PBVP (1.1) of the nonlinear coupled FHDS.

Theorem 3.1 *Suppose that* (H_1) *and* (H_2) *hold. Then the PBVP (1.1) of the nonlinear coupled FHDS has a solution.*

Proof. By Lemma 2.5, the system (1.1) is equivalent to the following integral system

$$\begin{cases} y(\zeta) &= f_1(\zeta, y(\zeta)) \Big(\frac{1}{\Gamma(\alpha)} \int_0^\zeta (\zeta - \omega)^{\alpha - 1} g_1(\omega, y(\omega), z(\omega)) d\omega \\ & \quad + \frac{1}{a_1 + b_1} \Big(c_1 - \frac{b_1}{\Gamma(\alpha)} \int_0^1 (1 - \omega)^{\alpha - 1} g_1(\omega, y(\omega), z(\omega)) d\omega \Big) \Big) \\ & \quad + k_1(\zeta, y(\zeta)), \quad \zeta \in \Delta. \\ z(\zeta) &= f_2(\zeta, z(\zeta)) \Big(\frac{1}{\Gamma(\beta)} \int_0^\zeta (\zeta - \omega)^{\beta - 1} g_2(\omega, y(\omega), z(\omega)) d\omega \\ & \quad + \frac{1}{a_2 + b_2} \Big(c_2 - \frac{b_2}{\Gamma(\beta)} \int_0^1 (1 - \omega)^{\beta - 1} g_2(\omega, y(\omega), z(\omega)) d\omega \Big) \Big) \\ & \quad + k_2(\zeta, z(\zeta)), \quad \zeta \in \Delta. \end{cases} \tag{3.1}$$

Set $U = C(\Delta, \mathbb{R})$ and define a subset S of U by

$$S = \{y \in U | \ \|y\| \leq R\},$$

where

$$R \geq \max\Big\{5l_1, 5\bar{l}_1, \frac{5|c_1|}{|a_1+b_1|}, 5l_2, 5\bar{l}_2, \frac{5|c_2|}{|a_2+b_2|}, \left(\frac{5m_1(|b_1|+|a_1+b_1|)}{|a_1+b_1|\Gamma(\alpha+1)}\right)^{\frac{1}{1-\rho_1}}, \left(\frac{5n_1(|b_1|+|a_1+b_1|)}{|a_1+b_1|\Gamma(\alpha+1)}\right)^{\frac{1}{1-\theta_1}}, \left(\frac{5m_2(|b_2|+|a_2+b_2|)}{|a_2+b_2|\Gamma(\beta+1)}\right)^{\frac{1}{1-\rho_2}}, \left(\frac{5n_2(|b_2|+|a_2+b_2|)}{|a_2+b_2|\Gamma(\beta+1)}\right)^{\frac{1}{1-\theta_2}}, \frac{F_0M_1+K_0}{1-L_1M_1-\bar{L}_1}, \frac{F_0M_2+K_0}{1-L_2M_2-\bar{L}_2}\Big\},$$

and

$$l_1 = \max_{\zeta\in\Delta}\frac{1}{\Gamma(\alpha)}\int_0^t(\zeta-\omega)^{\alpha-1}d_1(\omega)d\omega,\ \bar{l}_1 = \frac{|b_1|}{|a_1+b_1|\Gamma(\alpha)}\int_0^1(1-\omega)^{\alpha-1}d_1(\omega)d\omega,$$

$$l_2 = \max_{\zeta\in\Delta}\frac{1}{\Gamma(\beta)}\int_0^\zeta(\zeta-\omega)^{\beta-1}d_2(\omega)d\omega,\ \bar{l}_2 = \frac{|b_2|}{|a_2+b_2|\Gamma(\beta)}\int_0^1(1-\omega)^{\beta-1}d_2(\omega)d\omega,$$

$$M_1 = \frac{K}{\Gamma(\alpha+1)}\left(1+\frac{|b_1|}{|a_1+b_1|}\right)+\frac{|c_1|}{|a_1+b_1|},$$

$$M_2 = \frac{K}{\Gamma(\beta+1)}\left(1+\frac{|b_2|}{|a_2+b_2|}\right)+\frac{|c_2|}{|a_2+b_2|},$$

$$K = \max_{\zeta\in\Delta}\{|g_1(\zeta,y(\zeta),z(\zeta))|, |g_2(\zeta,y(\zeta),z(\zeta))|\},$$

$$F_0 = \max\{\sup_{\zeta\in\Delta}|f_1(\zeta,0)|, \sup_{\zeta\in\Delta}|f_2(\zeta,0)|\},$$

$$K_0 = \max\{\sup_{\zeta\in\Delta}|k_1(\zeta,0)|, \sup_{\zeta\in\Delta}|k_2(\zeta,0)|\},\ L=\max\{L_1,L_2\},\ \bar{L}=\max\{\bar{L}_1,\bar{L}_2\},$$

$$M=\max\{M_1,M_2\},\ 4LM+4\bar{L}<1.$$

Clearly, S is a closed, convex and bounded subset of the Banach space U.

Define the operators $A_i,\ C_i:\ U\to U$ and $B_i:\ S\to U$ $(i=1,2)$ by

$$A_1y(\zeta)=f_1(\zeta,y(\zeta)),\ A_2z(\zeta)=f_2(\zeta,z(\zeta)),\quad \zeta\in\Delta, \tag{3.2}$$

$$\begin{cases} B_1z(\zeta) &= \frac{1}{\Gamma(\alpha)}\int_0^\zeta(\zeta-\omega)^{\alpha-1}g_1(\omega,y(\omega),z(\omega))d\omega \\ &\quad +\frac{1}{a_1+b_1}\Big(c_1-\frac{b_1}{\Gamma(\alpha)}\int_0^1(1-\omega)^{\alpha-1}g_1(\omega,y(\omega),z(\omega))d\omega\Big),\quad \zeta\in\Delta. \\ B_2y(\zeta) &= \frac{1}{\Gamma(\beta)}\int_0^\zeta(\zeta-\omega)^{\beta-1}g_2(\omega,y(\omega),z(\omega))d\omega \\ &\quad +\frac{1}{a_2+b_2}\Big(c_2-\frac{b_2}{\Gamma(\beta)}\int_0^1(1-\omega)^{\beta-1}g_2(\omega,y(\omega),z(\omega))d\omega\Big),\quad \zeta\in\Delta. \end{cases} \tag{3.3}$$

and

$$C_1y(\zeta)=k_1(\zeta,y(\zeta)),\ C_2z(\zeta)=k_2(\zeta,z(\zeta)),\quad \zeta\in\Delta. \tag{3.4}$$

Then the system (3.1) is transformed into the system of operator equation as

$$\begin{cases} y(\zeta) & = A_1y(\zeta)B_1z(\zeta) + C_1y(\zeta), \\ z(\zeta) & = A_2z(\zeta)B_2y(\zeta) + C_2z(\zeta). \end{cases}$$

Set $\overline{U} = U \times U$, $\overline{S} = S \times S$. Define operators $\overline{A}$, $\overline{C}:\ \overline{U} \to \overline{U}$ and $\overline{B}:\ \overline{S} \to \overline{U}$ by

$$\overline{A}(y(\zeta), z(\zeta)) = (A_1y(\zeta), A_2z(\zeta)),\ \overline{B}(y(\zeta), z(\zeta)) = (B_1z(\zeta), B_2y(\zeta)),$$

$$\overline{C}(y(\zeta), z(\zeta)) = (C_1y(\zeta), C_2z(\zeta)).$$

and define the operators $T_1:\ \overline{U} \to U$, $T_2:\ \overline{U} \to U$ by

$$T_1(y, z) = A_1yB_1z + C_1y,\ T_2(y, z) = A_2zB_2y + C_2z.$$

Thus, we just prove that $(T_1(y, z), T_2(y, z)) = (y, z)$ has one solution in $\overline{S}$.

Next, we prove the operators A_i, B_i, and C_i $(i = 1, 2)$ satisfy all the conditions of Lemma 2.7.

Firstly, we prove that (a) of Lemma 2.7 is satisfied. Let $y,\ z \in U$. Then by (H_1),

$$|A_iy(\zeta) - A_iz(\zeta)| = |f_i(\zeta, y(\zeta)) - f_i(\zeta, z(\zeta))| \le L_i|y(\zeta) - z(\zeta)| \le L_i\|y - z\|,\ i = 1, 2,$$

for all $\zeta \in \Delta$. Taking supremum over ζ, then we have

$$\|A_iy - A_iz\| \le L_i\|y - z\|, \quad i = 1, 2,$$

for all $y,\ z \in U$. Similarly, it can be implied that

$$\|C_iy - C_iz\| \le \bar{L}_i\|y - z\|, \quad i = 1, 2,$$

for all $y,\ z \in U$. Thus, A_i and C_i $(i = 1, 2)$ satisfy (a) of Lemma 2.7.

Next, we prove B_i $(i = 1, 2)$ are compact and continuous operators on S into U. Firstly, we prove B_i $(i = 1, 2)$ are continuous on S. Let $\{(y_n, z_n)\}$ be a sequence in $\overline{S}$ converging to $(y, z) \in \overline{S}$. Then by the

Lebesgue dominated convergence theorem,

$$
\begin{aligned}
&\lim_{n\to\infty} B_1 z_n(\zeta)\\
&= \lim_{n\to\infty}\Big(\frac{1}{\Gamma(\alpha)}\int_0^{\zeta}(\zeta-\omega)^{\alpha-1}g_1(\omega,y_n(\omega),z_n(\omega))d\omega\\
&\quad+\frac{1}{a_1+b_1}\left(c_1-\frac{b_1}{\Gamma(\alpha)}\int_0^1(1-\omega)^{\alpha-1}g_1(\omega,y_n(\omega),z_n(\omega))d\omega\right)\Big)\\
&= \lim_{n\to\infty}\frac{1}{\Gamma(\alpha)}\int_0^{\zeta}(\zeta-\omega)^{\alpha-1}g_1(\omega,y_n(\omega),z_n(\omega))d\omega\\
&\quad+\lim_{n\to\infty}\frac{1}{a_1+b_1}\left(c_1-\frac{b_1}{\Gamma(\alpha)}\int_0^1(1-\omega)^{\alpha-1}g_1(\omega,y_n(\omega),z_n(\omega))d\omega\right)\\
&= \frac{1}{\Gamma(\alpha)}\int_0^{\zeta}(\zeta-\omega)^{\alpha-1}g_1(\omega,y(\omega),z(\omega))d\omega\\
&\quad+\frac{1}{a_1+b_1}\left(c_1-\frac{b_1}{\Gamma(\alpha)}\int_0^1(1-\omega)^{\alpha-1}g_1(\omega,y(\omega),z(\omega))d\omega\right)\\
&= B_1 z(\zeta),
\end{aligned}
$$

for all $\zeta\in\Delta$. This shows that B_1 is a continuous operator on S. Similarly, we can prove that B_2 is also a continuous operator on S.

Next we prove B_i $(i=1,2)$ are compact operators on S. It is enough to show that $B_i(S)$ $(i=1,2)$ are uniformly bounded and equicontinuous sets in U. On the one hand, let $y\in S$ be arbitrary. Then by (H_2),

$$
\begin{aligned}
|B_1 z(\zeta)| &= \Big|\frac{1}{\Gamma(\alpha)}\int_0^{\zeta}(\zeta-\omega)^{\alpha-1}g_1(\omega,y(\omega),z(\omega))d\omega\\
&\quad+\frac{1}{a_1+b_1}\left(c_1-\frac{b_1}{\Gamma(\alpha)}\int_0^1(1-\omega)^{\alpha-1}g_1(\omega,y(\omega),z(\omega))d\omega\right)\Big|\\
&\le \frac{1}{\Gamma(\alpha)}\int_0^{\zeta}(\zeta-\omega)^{\alpha-1}\left|g_1(\omega,y(\omega),z(\omega))\right|d\omega\\
&\quad+\frac{|b_1|}{|a_1+b_1|\Gamma(\alpha)}\int_0^1(1-\omega)^{\alpha-1}|g_1(\omega,y(\omega),z(\omega))|d\omega+\frac{|c_1|}{|a_1+b_1|}\\
&\le l_1+\frac{m_1R^{\rho_1}+n_1R^{\theta_1}}{\Gamma(\alpha+1)}+\bar{l}_1+\frac{|b_1|(m_1R^{\rho_1}+n_1R^{\theta_1})}{|a_1+b_1|\Gamma(\alpha+1)}+\frac{|c_1|}{|a_1+b_1|}\le R,
\end{aligned}
$$

for all $\zeta\in\Delta$. Taking supremum over ζ, $\|B_1 z\|\le R$ for all $y\in S$. Similarly, we can conclude that $\|B_2 z\|\le R$ for all $z\in S$. This shows that B_i $(i=1,2)$ are uniformly bounded on S.

On the other hand, let $\zeta_1,\ \zeta_2 \in \Delta$. Then for any $z \in S$, we get

$$
\begin{aligned}
& |B_1 z(\zeta_1) - B_1 z(\zeta_2)| \\
&= \Big| \frac{1}{\Gamma(\alpha)} \int_0^{\zeta_1} (\zeta_1 - \omega)^{\alpha-1} g_1(\omega, y(\omega), z(\omega)) d\omega \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_0^{\zeta_2} (\zeta_2 - \omega)^{\alpha-1} g_1(\omega, y(\omega), z(\omega)) d\omega \Big| \\
&\le \frac{1}{\Gamma(\alpha)} \Big| \int_0^{\zeta_1} (\zeta_1 - \omega)^{\alpha-1} g_1(\omega, y(\omega), z(\omega)) d\omega \\
&\quad - \int_0^{\zeta_1} (\zeta_2 - \omega)^{\alpha-1} g_1(\omega, y(\omega), z(\omega)) d\omega \\
&\quad + \int_0^{\zeta_1} (\zeta_2 - \omega)^{\alpha-1} g_1(\omega, y(\omega), z(\omega)) d\omega - \int_0^{\zeta_2} (\zeta_2 - \omega)^{\alpha-1} g_1(\omega, y(\omega), z(\omega)) d\omega \Big| \\
&= \frac{K}{\Gamma(\alpha+1)} \left(\zeta_2^\alpha - \zeta_1^\alpha - (\zeta_2 - \zeta_1)^\alpha \right).
\end{aligned}
$$

Since the functions ζ^α is uniformly continuous on compact Δ, from the above analysis, $B_1(S)$ is an equicontinuous set in U. Thus, B_1 is completely continuous likewise B_2.

Next, we show that (c) of Lemma 2.7 is satisfied. Let $y \in S$ and $z \in S$ be arbitrary such that $y = A_1 y B_1 z + C_1 y,\ z = A_2 z B_2 y + C_2 z$. Then, by assumption (H_1), we have

$$
\begin{aligned}
& |y(\zeta)| \\
&\le |A_1 y(\zeta)||B_1 z(\zeta)| + |C_1 y(\zeta)| \\
&= |f_1(\zeta, y(\zeta))| \Big| \frac{1}{\Gamma(\alpha)} \int_0^{\zeta} (\zeta - \omega)^{\alpha-1} g_1(\omega, y(\omega), z(\omega)) d\omega \\
&\quad + \frac{1}{a_1 + b_1} \Big(c_1 - \frac{b_1}{\Gamma(\alpha)} \int_0^1 (1 - \omega)^{\alpha-1} g_1(\omega, y(\omega), z(\omega)) d\omega \Big) \Big| + |k_1(\zeta, y(\zeta))| \\
&\le [|f_1(\zeta, y(\zeta)) - f_1(\zeta, 0)| + |f_1(\zeta, 0)|] \cdot \left(\frac{K}{\Gamma(\alpha+1)} \left(1 + \frac{|b_1|}{|a_1 + b_1|} \right) + \frac{|c_1|}{|a_1 + b_1|} \right) \\
&\quad + |k_1(\zeta, y(\zeta)) - k_1(\zeta, 0)| + |k_1(\zeta, 0)| \\
&\le [L_1 |y(\zeta)| + F_0] M_1 + \bar{L}_1 |y(\zeta)| + K_0.
\end{aligned}
$$

Thus, we get

$$
|y(\zeta)| \le \frac{F_0 M_1 + K_0}{1 - L_1 M_1 - \bar{L}_1}.
$$

Taking supremum over ζ,

$$
\|y\| \le \frac{F_0 M_1 + K_0}{1 - L_1 M_1 - \bar{L}_1} \le R.
$$

Similarly, we can obtain that

$$
\|z\| \le \frac{F_0 M_2 + K_0}{1 - L_2 M_2 - \bar{L}_2} \le R.
$$

This shows that (c) of Lemma 2.7 is satisfied.

Finally, we obtain

$$\begin{aligned}
&\|B(S)\| \\
&= \max\{\sup\{\|B_1(y)\|: \ y \in S\}, \sup\{\|B_2(y)\|: \ y \in S\}\} \\
&= \max\left\{\frac{K}{\Gamma(\alpha+1)}\left(1+\frac{|b_1|}{|a_1+b_1|}\right)+\frac{|c_1|}{|a_1+b_1|},\right. \\
&\qquad \left.\frac{K}{\Gamma(\beta+1)}\left(1+\frac{|b_2|}{|a_2+b_2|}\right)+\frac{|c_2|}{|a_2+b_2|}\right\} \\
&= \max\{M_1, M_2\} = M.
\end{aligned}$$

and so,

$$4L\|B(S)\| + \bar{L} = 4LM + \bar{L} < 1.$$

This shows that (d) of Lemma 2.7 is satisfied.

Thus, all the conditions of Lemma 2.7 are satisfied and $(T_1(y,z), T_2(y,z)) = (y,z)$ has one solution in $\overline{S}$. Therefore, the PBVP (1.1) of the nonlinear coupled FHDS has a solution.

4. Uniqueness of solution

In this section, we give the uniqueness of the solution for the PBVP (1.1) of the nonlinear coupled FHDS.

We present the following hypotheses.

(H_3) There exist constants $h_i > 0$ $(i = 1,2)$ and $\bar{h}_i > 0$ $(i = 1,2)$ such that

$$|g_i(\zeta, y, z) - g_i(\zeta, y', z')| \le h_i|y - y'| + \bar{h}_i|z - z'|$$

for all $\zeta \in \Delta$ and $y,\ y',\ z,\ z' \in \mathbb{R}$.

Next define the following notations

$$K' = \max_{\zeta\in\Delta}\{|f_1(\zeta, y(\zeta))|, |f_2(\zeta, z(\zeta))|\},\ \mu_1 = \frac{K'}{\Gamma(\alpha+1)}\left(1+\frac{|b_1|}{|a_1+b_1|}\right),\ \mu_2 = \frac{K'}{\Gamma(\beta+1)}\left(1+\frac{|b_2|}{|a_2+b_2|}\right).$$

Now we will present the following uniqueness of solutions for the PBVP (1.1) of the nonlinear coupled FHDS.

Theorem 4.1 *Suppose that (H_1) and (H_3) hold. If*

$$2\left(\mu_1(h_1+\bar{h}_1) + \mu_2(h_2+\bar{h}_2) + 2M + 2\bar{L}\right) < 1.$$

Then the PBVP (1.1) of the nonlinear coupled FHDS has a unique solution.

Proof. Define $U = C(\Delta, \mathbb{R})$ and choose a subset S' of U by

$$S' = \{y \in U|\ \|y\| \le R'\},$$

where

$$R' \geq \frac{4MF_0 + 4K_0}{1 - 2(LM + \bar{L})},$$

and $F_0,\ K_0,\ L,\ \bar{L},\ M$ are defined in Theorem 3.1. Thus, we just prove that $(T_1(y,z), T_2(y,z)) = (y,z)$ has one solution in $\overline{S'} = S' \times S'$.

For $(y,z) \in \overline{S'}$ and $\zeta \in \Delta,$ we have

$$\begin{aligned}
&|T_1(y(\zeta), z(\zeta))| \\
&\leq |A_1 y(\zeta)||B_1 z(\zeta)| + |C_1 y(\zeta)| \\
&= |f_1(\zeta, y(\zeta))| \Big| \frac{1}{\Gamma(\alpha)} \int_0^\zeta (\zeta - \omega)^{\alpha-1} g_1(\omega, y(\omega), z(\omega)) d\omega \\
&\quad + \frac{1}{a_1 + b_1} \Big(c_1 - \frac{b_1}{\Gamma(\alpha)} \int_0^1 (1-\omega)^{\alpha-1} g_1(\omega, y(\omega), z(\omega)) d\omega \Big) \Big| + |k_1(\zeta, y(\zeta))| \\
&\leq [|f_1(\zeta, y(\zeta)) - f_1(\zeta, 0)| + |f_1(\zeta, 0)|] \cdot \left(\frac{K}{\Gamma(\alpha+1)} \left(1 + \frac{|b_1|}{|a_1 + b_1|} \right) + \frac{|c_1|}{|a_1 + b_1|} \right) \\
&\quad + |k_1(\zeta, y(\zeta)) - k_1(\zeta, 0)| + |k_1(\zeta, 0)| \\
&\leq [L_1 |y(\zeta)| + F_0] M_1 + \bar{L}_1 |y(\zeta)| + K_0.
\end{aligned}$$

Taking supremum over $\zeta,$ we have

$$\|T_1(y,z)\| \leq [L_1 \|y\| + F_0] M_1 + \bar{L}_1 \|y\| + K_0.$$

Similarly, we can obtain that

$$\|T_2(y,z)\| \leq [L_2 \|z\| + F_0] M_2 + \bar{L}_2 \|z\| + K_0.$$

Thus, we get

$$\begin{aligned}
\|T(y,z)\| &= 2(\|T_1(y,z)\| + \|T_2(y,z)\|) \\
&\leq 2(L_1 M_1 + \bar{L}_1)\|y\| + 2(L_2 M_2 + \bar{L}_2)\|z\| + 2(M_1 + M_2) F_0 + 4K_0 \\
&\leq 2(LM + \bar{L}) R' + 4MF_0 + 4K_0 \leq R'.
\end{aligned}$$

For $(y_1, z_1)(\zeta), (y_2, z_2)(\zeta) \in U \times U$ and any $\zeta \in \Delta$, we have

$$
\begin{aligned}
&|T_1(y_2, z_2)(\zeta) - T_1(y_1, z_1)(\zeta)| \\
\leq &|A_1 y_2(\zeta) B_1 z_2(\zeta) + C_1 y_2(\zeta) - (A_1 y_1(\zeta) B_1 z_1(\zeta) + C_1 y_1(\zeta))| \\
= &\Big| f_1(\zeta, y_2(\zeta)) \Big(\frac{1}{\Gamma(\alpha)} \int_0^\zeta (\zeta - \omega)^{\alpha-1} g_1(\omega, y_2(\omega), z_2(\omega)) d\omega \\
&+ \frac{1}{a_1 + b_1} \Big(c_1 - \frac{b_1}{\Gamma(\alpha)} \int_0^1 (1 - \omega)^{\alpha-1} g_1(\omega, y_2(\omega), z_2(\omega)) d\omega \Big) \Big) + k_1(\zeta, y_2(\zeta)) \\
&- f_1(\zeta, y_1(\zeta)) \Big(\frac{1}{\Gamma(\alpha)} \int_0^\zeta (\zeta - \omega)^{\alpha-1} g_1(\omega, y_1(\omega), z_1(\omega)) d\omega \\
&+ \frac{1}{a_1 + b_1} \Big(c_1 - \frac{b_1}{\Gamma(\alpha)} \int_0^1 (1 - \omega)^{\alpha-1} g_1(\omega, y_1(\omega), z_1(\omega)) d\omega \Big) \Big) - k(\zeta, y_1(\zeta)) \Big| \\
= &\Big| f_1(\zeta, y_2(\zeta)) \Big(\frac{1}{\Gamma(\alpha)} \int_0^\zeta (\zeta - \omega)^{\alpha-1} g_1(\omega, y_2(\omega), z_2(\omega)) d\omega \\
&+ \frac{1}{a_1 + b_1} \Big(c_1 - \frac{b_1}{\Gamma(\alpha)} \int_0^1 (1 - \omega)^{\alpha-1} g_1(\omega, y_2(\omega), z_2(\omega)) d\omega \Big) \Big) + k(\zeta, y_2(\zeta)) \\
&- f_1(\zeta, y_2(\zeta)) \Big(\frac{1}{\Gamma(\alpha)} \int_0^\zeta (\zeta - \omega)^{\alpha-1} g_1(\omega, y_1(\omega), z_1(\omega)) d\omega \\
&+ \frac{1}{a_1 + b_1} \Big(c_1 - \frac{b_1}{\Gamma(\alpha)} \int_0^1 (1 - \omega)^{\alpha-1} g_1(\omega, y_1(\omega), z_1(\omega)) d\omega \Big) \Big) \\
&+ f_1(\zeta, y_2(\zeta)) \Big(\frac{1}{\Gamma(\alpha)} \int_0^\zeta (\zeta - \omega)^{\alpha-1} g_1(\omega, y_1(\omega), z_1(\omega)) d\omega \\
&+ \frac{1}{a_1 + b_1} \Big(c_1 - \frac{b_1}{\Gamma(\alpha)} \int_0^1 (1 - \omega)^{\alpha-1} g_1(\omega, y_1(\omega), z_1(\omega)) d\omega \Big) \Big) \\
&- f_1(\zeta, y_1(\zeta)) \Big(\frac{1}{\Gamma(\alpha)} \int_0^\zeta (\zeta - \omega)^{\alpha-1} g_1(\omega, y_1(\omega), z_1(\omega)) d\omega \\
&+ \frac{1}{a_1 + b_1} \Big(c_1 - \frac{b_1}{\Gamma(\alpha)} \int_0^1 (1 - \omega)^{\alpha-1} g_1(\omega, y_1(\omega), z_1(\omega)) d\omega \Big) \Big) - k_1(\zeta, y_1(\zeta)) \Big| \\
\leq &|f_1(\zeta, y_2(\zeta))| \left| \frac{1}{\Gamma(\alpha)} \int_0^\zeta (\zeta - \omega)^{\alpha-1} \left(g_1(\omega, y_2(\omega), z_2(\omega)) - g_1(\omega, y_1(\omega), z_1(\omega)) \right) d\omega \right| \\
&+ \frac{|f_1(\zeta, y_2(\zeta))|\,|c_1|}{|a_1 + b_1|} \left| \int_0^1 (1 - \omega)^{\alpha-1} \left(g_1(\omega, y_2(\omega), z_2(\omega)) - g_1(\omega, y_1(\omega), z_1(\omega)) \right) d\omega \right| \\
&+ |f_1(\zeta, y_2(\zeta)) - f_1(\zeta, y_1(\zeta))| \Bigg(\left| \frac{1}{\Gamma(\alpha)} \int_0^\zeta (\zeta - \omega)^{\alpha-1} g_1(\omega, y_1(\omega), z_1(\omega)) d\omega \right| \\
&+ \frac{|c_1|}{|a_1 + b_1|} + \frac{|b_1|}{|a_1 + b_1|} \left| \frac{1}{\Gamma(\alpha)} \int_0^1 (1 - \omega)^{\alpha-1} g_1(\omega, y_1(\omega), z_1(\omega)) d\omega \right| \Bigg) + |k_1(\zeta, y_2(\zeta)) - k_1(\zeta, y_1(\zeta))| \,\Big|
\end{aligned}
$$

$$\le \frac{K'}{\Gamma(\alpha+1)}(h_1\|y_2-y_1\|+\bar{h}_1\|z_2-z_1\|)+\frac{K'}{\Gamma(\alpha+1)}\cdot\frac{|c_1|}{|a_1+b_1|}(h_1\|y_2-y_1\|+\bar{h}_1\|z_2-z_1\|)$$
$$+\|y_2-y_1\|\left(\frac{K}{\Gamma(\alpha+1)}+\frac{|c_1|}{|a_1+b_1|}+\frac{|b_1|}{|a_1+b_1|}\cdot\frac{K}{\Gamma(\alpha+1)}\right)+\bar{L}\|y_2-y_1\|$$
$$\le \mu_1(h_1\|y_2-y_1\|+\bar{h}_1\|z_2-z_1\|)+(M+\bar{L})\|y_2-y_1\|.$$

Taking supremum over ζ, we get

$$\begin{aligned}
&\|T_1(y_2,z_2)-T_1(y_1,z_1)\|\\
&\le \left(\mu_1 h_1+M+\bar{L}\right)\|y_2-y_1\|+\mu_1\bar{h}_1\|z_2-z_1\|\\
&\le \left(\mu_1 h_1+M+\bar{L}+\mu_1\bar{h}_1\right)\|y_2-y_1\|+\left(\mu_1 h_1+M+\bar{L}+\mu_1\bar{h}_1\right)\|z_2-z_1\|\\
&= \left(\mu_1(h_1+\bar{h}_1)+M+\bar{L}\right)(\|y_2-y_1\|+\|z_2-z_1\|).
\end{aligned}$$

Similarly, we can obtain that

$$\|T_2(y_2,z_2)-T_2(y_1,z_1)\|\le \left(\mu_2(h_2+\bar{h}_2)+M+\bar{L}\right)(\|y_2-y_1\|+\|z_2-z_1\|).$$

Thus, we get

$$\begin{aligned}
&\|T(y_2,z_2)-T(y_1,z_1)\|\\
&\le 2\left(\mu_1(h_1+\bar{h}_1)+\mu_2(h_2+\bar{h}_2)+2M+2\bar{L}\right)(\|y_2-y_1\|+\|z_2-z_1\|)\\
&\le (\|y_2-y_1\|+\|z_2-z_1\|),
\end{aligned}$$

which implies that T is a contraction. By Banach's contraction principle, the operator T has a unique fixed point which is the unique solution of the PBVP (1.1) of the nonlinear coupled FHDS.

5. Examples

In this section, we will present three examples to illustrate the main results.

Example 5.1 *Consider the following PBVP*

$$\begin{cases}
{}^{C}D_{0^+}^{\frac{1}{2}}\left[\frac{y(\zeta)-\frac{1}{32}\sin y(\zeta)}{\frac{1}{4}\sqrt{y^2(\zeta)+1}}\right]=\frac{\zeta}{2}+(\zeta-\frac{1}{4})^4(y(\zeta)^{\rho_1}+z(\zeta)^{\theta_1}), \quad 0<\zeta<1,\\
{}^{C}D_{0^+}^{\frac{1}{3}}\left[\frac{z(\zeta)-\frac{1}{24}\arctan z(\zeta)}{1+\frac{1}{3}\cos z(\zeta)}\right]=\zeta^2+(\zeta-\frac{1}{4})^4(y(\zeta)^{\rho_2}+z(\zeta)^{\theta_2}), \quad 0<\zeta<1,\\
\left[\frac{y(\zeta)-\frac{1}{8}\sin y(\zeta)}{\sqrt{u^2(\zeta)+1}}\right]_{\zeta=0}+\left[\frac{y(\zeta)-\frac{1}{8}\sin y(\zeta)}{\sqrt{u^2(\zeta)+1}}\right]_{\zeta=1}=\frac{1}{16},\\
\left[\frac{z(\zeta)-\frac{1}{24}\arctan z(\zeta)}{1+\frac{1}{3}\cos z(\zeta)}\right]_{\zeta=0}+\left[\frac{z(\zeta)-\frac{1}{24}\arctan z(\zeta)}{1+\frac{1}{3}\cos z(\zeta)}\right]_{\zeta=1}=\frac{1}{20},
\end{cases} \tag{5.1}$$

where $0<\rho_i,\theta_i<1$ $(i=1,\ 2)$.

Choose $d_1(\zeta)=\frac{\zeta}{2}$, $d_2(\zeta)=\zeta^2$ and $m_i=n_i=\frac{81}{256}$ $(i=1,\ 2)$. Then hypotheses (H_1) and (H_2) hold. Therefore, by Theorem 3.1, the PBVP (5.1) has a solution.

Example 5.2 *Consider the following PBVP*

$$\begin{cases} {}^{C}D_{0^+}^{\frac{1}{2}}\left[\frac{y(\zeta)-\frac{\zeta}{50e^{\zeta}}|y(\zeta)|-\frac{\sin y(\zeta)}{3\zeta^4}}{1+\frac{1}{10}\cos\zeta|y(\zeta)|}\right]=(\zeta-\frac{1}{2})^4(y(\zeta)^{\rho_1}+z(\zeta)^{\theta_1}), \quad 0<\zeta<1,\\ {}^{C}D_{0^+}^{\frac{1}{3}}\left[\frac{z(\zeta)-\frac{\arctan\zeta}{40}|z(\zeta)|-\frac{\zeta^4}{7e^{\zeta}}}{\frac{1}{16}(z(\zeta)+\sqrt{z^2(\zeta)+1})}\right]=(\zeta-\frac{1}{2})^4(y(\zeta)^{\rho_2}+z(\zeta)^{\theta_2}), \quad 0<\zeta<1,\\ \left[\frac{y(\zeta)-\frac{\zeta}{50e^{\zeta}}|y(\zeta)|-\frac{\sin y}{3\zeta^4}}{1+\frac{1}{10}\cos\zeta|y(\zeta)|}\right]_{\zeta=0}+\left[\frac{y(\zeta)-\frac{\zeta}{50e^{\zeta}}|y(\zeta)|-\frac{\sin y}{3\zeta^4}}{1+\frac{1}{10}\cos\zeta|y(\zeta)|}\right]_{\zeta=1}=0,\\ \left[\frac{z(\zeta)-\frac{\arctan\zeta}{40}|z(\zeta)|-\frac{\zeta^4}{7e^{\zeta}}}{\frac{1}{16}(z(\zeta)+\sqrt{z^2(\zeta)+1})}\right]_{\zeta=0}+\left[\frac{z(\zeta)-\frac{\arctan\zeta}{40}|z(\zeta)|-\frac{\zeta^4}{7e^{\zeta}}}{\frac{1}{16}(z(\zeta)+\sqrt{z^2(\zeta)+1})}\right]_{\zeta=1}=0, \end{cases} \tag{5.2}$$

where $0<\rho_i,\theta_i<1$ $(i=1,\ 2)$.

Choose $d_1(\zeta)=d_2(\zeta)=0$ and $m_i=n_i=\frac{1}{256}$ $(i=1,\ 2)$. Then hypotheses (H_1) and (H_2) hold. Therefore, by Theorem 3.1, the PBVP (5.2) has a solution.

Example 5.3 *Consider the following PBVP*

$$\begin{cases} {}^{C}D_{0^+}^{\frac{1}{2}}\left[\frac{y(\zeta)-\frac{1}{100}\cos y(\zeta)}{1+\frac{1}{100}\sin y(\zeta)}\right]=\frac{1}{31}+\frac{1}{200}\frac{|y(\zeta)|}{1+|y(\zeta)|}+\frac{1}{100}\frac{|z(\zeta)|}{1+|z(\zeta)|}, \quad 0<\zeta<1,\\ {}^{C}D_{0^+}^{\frac{1}{2}}\left[\frac{y(\zeta)-\frac{1}{100}\sin y(\zeta)}{1+\frac{1}{100}\cos y(\zeta)}\right]=\frac{1}{31}+\frac{1}{100}\frac{|y(\zeta)|}{1+|y(\zeta)|}+\frac{1}{200}\frac{|z(\zeta)|}{1+|z(\zeta)|}, \quad 0<\zeta<1,\\ \left[\frac{y(\zeta)-\frac{1}{100}\cos y(\zeta)}{1+\frac{1}{100}\sin y(\zeta)}\right]_{\zeta=0}+\left[\frac{y(\zeta)-\frac{1}{100}\cos y(\zeta)}{1+\frac{1}{100}\sin y(\zeta)}\right]_{\zeta=1}=0,\\ \left[\frac{y(\zeta)-\frac{1}{100}\sin y(\zeta)}{1+\frac{1}{100}\cos y(\zeta)}\right]_{\zeta=0}+\left[\frac{y(\zeta)-\frac{1}{100}\sin y(\zeta)}{1+\frac{1}{100}\cos y(\zeta)}\right]_{\zeta=1}=0. \end{cases} \tag{5.3}$$

It is easy to check that $\alpha=\beta=0.5$, $a_1=b_1=a_2=b_2=1$, $c_1=c_2=0$, $L=\bar{L}=0.01$, $h_1=\bar{h}_2=0.005$, $\bar{h}_1=h_2=0.01$, $K=0.0473$, $K'=1.01$, $M=0.08$, $\mu_1=\mu_2=1.709$. Then, we have

$$2\left(\mu_1(h_1+\bar{h}_1)+\mu_2(h_2+\bar{h}_2)+2M+2\bar{L}\right)=0.46254<1.$$

Thus, hypotheses (H_1) and (H_3) hold. Therefore, by Theorem 4.1, the PBVP (5.3) has a unique solution.

6. Conclusion

In this paper, we have studied the solvability for the PBVP (1.1) of the nonlinear coupled FHDS. We have presented an existence theorem and a uniqueness result for the PBVP (1.1) of the nonlinear coupled FHDS due to the fixed point theorem in Banach algebra and Banach's contraction principle. The main results have been well illustrated with the help of three examples.

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