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Solving nonlinear integro-differential equations using numerical method

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Abstract: The aim of this paper is to establish conditions for the existence and uniqueness of the solution of a nonlinear integro-differential equation. Moreover, it is to propose a quadrature method in order to find an approximate solution and establish the convergence of the method. We conclude by providing the algorithm and some numerical simulation to confirm our theoretical results.

Key words: Nonlinear integro-differential equation, existence and uniqueness, Gauss–Kronrod, quadrature method, backward difference

1. Introduction

Nowadays, integral equations (IE) and integro-differential equations (IDE) appear in many areas of real life problems such as: dynamics, mechanics [11, Chapter 2], physics and biology [12, 13] and many mathematicians are interested in studying this type of equations ([2, 4, 5, 7, 15, 16, 20, 21]). The classical methods for solving (IE) and (IDE) are valid for linear equations. On the other hand, for nonlinear equations, they are much more difficult. For instance, existence and uniqueness of the solution is easy to obtain for the linear equations, whereas for the nonlinear equations it is complicated and is one of the most studied problems. See for example [8, 17, 22] and the references therein. Consider the nonlinear integro-differential equation of the form

$$x'(s) = y(s) + \int_{-1}^{1} k(s,t,x(t))dt, \tag{1.1}$$

$$x(-1) = 0,$$

where $x(s)$ is an unknown function that occurs inside and outside the integral sign, $k(.,.,.)$ is the kernel and $y(s)$ is the source term. In our case, we are interested in the numerical study of Equation (1.1), because of the crucial role that the solution of Equation (1.1) plays in science and engineering [1]. More precisely, we intend to present an iterative method based on the Gauss–Kronrod quadrature ([6, 14, 19]).
The existence and uniqueness of this kind of equations will be shown later.

This paper contains 5 sections and a brief conclusion. In Section 2, conditions for existence and uniqueness are given. In Section 3, we discretize Equation (1.1) using a numerical scheme. Then, we solve Equation (1.1) and prove the vanishing of the error. In Section 4, we introduce a numerical approach based on Gauss–Kronrod

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quadrature. Then we establish the convergence of the approximate solution. In Section 5, we give the algorithm of the method and we provide some numerical examples to illustrate our theoretical results.

2. Existence and uniqueness of the solution

The purpose of this section is to obtain the existence and uniqueness of the solution of Equation (1.1) in $C([-1,1],\mathbb{R})$. We therefore apply the contraction principle and Schauder's theorem to reach our results.

2.1. Existence of solution in $C([-1,1],\mathbb{R})$

We consider Equation (1.1) and assume that the following conditions are satisfied:

(H_1) $k \in C([-1,1]\times[-1,1]\times\mathbb{R},\mathbb{R})$

(H_2) $y \in C([-1,1],\mathbb{R})$

In addition, suppose that

(H_3) there exists $M > 0$ such that

$$|k(s,t,x)| \leq M,\ \forall s,t \in [-1,1],\ x \in \mathbb{R}$$

and we have the following existence theorem

Theorem 2.1 *Suppose that the conditions $(H_1)-(H_3)$ are satisfied. Then Equation (1.1) has at least one solution $x^* \in C([-1,1],\mathbb{R})$.*

Proof We follow the same steps as [8]. Firstly, we introduce $D : C([-1,1],\mathbb{R}) \to C([-1,1],\mathbb{R})$, defined by

$$D(x)(s) := \int_{-1}^{s} x(w)dw,$$

where D is a compact operator. Let $A : C([-1,1],\mathbb{R}) \to C([-1,1],\mathbb{R})$, such that

$$A(x)(s) := \int_{-1}^{s} y(w)dw + \int_{-1}^{s}\int_{-1}^{1} k(w,t,x(t))dtdw, \quad s \in [-1,1] \tag{2.1}$$

Now, by using the Chebyshev norm and the compactness property of D, we obtain:

$$\|A(x)\|_C \leq \|Dy\|_C + \|D\| \left\| \int_{-1}^{1} k(s,t,x(t))dt \right\|_C \leq C_0\left[\|y\|_C + 2M\right], \quad \forall x \in C([-1,1],\mathbb{R})$$

where $\|D\| = C_0$. Thus, $A(X)$ is equal bounded. In addition, let $X \subset C([-1,1],\mathbb{R})$ a bounded subset. Then $A(X)$ is also a bounded subset. From the uniform continuity of k with respect to s (Heine–Cantor theorem), it follows that $A(X)$ is equal continuous. Therefore, $A(X)$ is compact subset. Let $X = \overline{conv}A(C([-1,1],\mathbb{R}))$. On the other hand, by the Ascoli–Arzela's theorem it results that A is completely continuous operator. At the end, since X is an invariant subset by A i.e. $X \in I(A)$, it follows that the conditions of the Schauder's theorem are satisfied and the proof is complete. □

Remark 2.2 *Equation (1.1) has at least one solution, if we have the compactness property of k instead of the conditions $(H_1)-(H_3)$.*

2.2. Existence and uniqueness of the solution in $C([-1,1],\mathbb{R})$

Suppose now that the following conditions are hold:

(H_4) there exists $\gamma > 0$ such that

$$|k(s,t,u) - k(s,t,v)| \leq \gamma|u-v|, \forall s,t \in [-1,1], u,v \in \mathbb{R}$$

(H_5) $2C_0\gamma < 1$

and we have the following existence and uniqueness theorem.

Theorem 2.3 *Suppose that the conditions $(H_1)-(H_2)$ and $(H_4)-(H_5)$ are satisfied. Then Equation (1.1) has a unique solution $x^* \in C([-1,1],\mathbb{R})$.*

Proof We follow the same steps as [8]. We attach to Equation (1.1) the operator $A : C([-1,1],\mathbb{R}) \to C([-1,1],\mathbb{R})$, defined by the relation (2.1). The set of the solutions of integral equation (1.1) coincides with the set of the fixed points of the operator A. Furthermore, by using (H_4) and Chebyshev norm we have

$$\|A(x_1) - A(x_2)\|_C \leq 2C_0\gamma \|x_1 - x_2\|_C .$$

Therefore, by (H_5) it results that the operator A is an ν-contraction with the coefficient $\nu = 2C_0\gamma$, which completes the proof. □

3. Solving the integro-differential equation

Firstly, we suppose that Equation (1.1) has at least one solution. In this section, we discretize Equation (1.1) by using a numerical method of integration called Gauss–Kronrod quadrature and the backward difference. Then, we study the convergence analysis. Let $\Gamma = s_1, s_2 \cdots, s_n$ be the Gauss points [9] (pp.59, 115), for $n \geq 1$.
We have the backward difference:

$$x'(s) = \frac{x(s) - x(s-h)}{h} + O(h),$$

where $h = s_i - s_{i-1}$ is the discretization step (length between the nodes) with h assume to goes to 0 as $n \to \infty, s_i = s$ and $s_{i-1} = s - h$. In addition, $O(h)$ is the truncation error.
Now if $x^*(s)$ is an analytical solution of Equation (1.1), then for Γ on $[-1,1]$, we have

$$x^*(s_i) - x^*(s_{i-1}) = hy(s_i) + h\int_{-1}^{1} k(s_i, t, x^*(t))\,dt - O\left(h^2\right), i = 1,2,\cdots,n. \tag{3.1}$$

The integral term in Equation (3.1) can be estimated by a numerical integration method. In our case, we use Gauss–Kronrod quadrature [6]. Thus, Equation (3.1) becomes

$$x_i^* - x_{i-1}^* = hy_i + h\sum_{j=1}^{n} w_j k\left(s_i, t_j, x_j^*\right) + hR(\varepsilon) - O\left(h^2\right), \quad i = 1,2,\cdots,n, -1 < \varepsilon < 1 \tag{3.2}$$

where $x_i^* = x^*(s_i), y_i = y(s_i), w_i$ are the weight functions $i = 1,2,\cdots,n$, which satisfy the following requirements

$$\sum_{i=1}^{n} w_i = 2,$$

and $R(\varepsilon)$ is the Gauss–Kronrod quadrature error [6, Chapter 5], which is expressed as follows, for $n \geq 1$

$$R(\varepsilon) = \frac{2^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} k_\varepsilon^{(2n)}\left(s, \varepsilon, x^*\right), -1 < \varepsilon < 1.$$

where $k_\varepsilon^{(2n)}$ is the $(2n)^{th}$ derivatives with respect to ε.

For Γ, we treat a nonlinear system obtained by neglecting the truncation errors of (3.1), as below,

$$\xi_i - \xi_{i-1} = hy_i + h\sum_{j=1}^{n} w_j k\left(s_i, t_j, \xi_j\right), \quad i = 1, 2, \cdots, n \tag{3.3}$$

and let $\xi^* = \left(\xi_1^*, \xi_2^*, \cdots, \xi_n^*\right)^T$ be the exact solution of the nonlinear system (3.3).

Now, we look for the conditions of vanishing $\|x^* - \xi^*\|_\infty$, where $x^* = \left(x_1^*, x_2^*, \cdots, x_n^*\right)^T$.

Proposition 3.1 *Assume that* $(H_1) - (H_4)$ *are hold and for* $n \geq 1$,

(i) $k \in C^{2n}\left([-1,1]^2 \times \mathbb{R}, \mathbb{R}_+^*\right)$

(ii) $y \in C\left([-1,1], \mathbb{R}_+\right)$

(iii) $\gamma < \frac{1-a}{2h}$, *and* $0 < \alpha < 1$ *where*

$$\alpha = \max_{1 \leq \mathrm{p} \leq n} \frac{\left|x_{p-1}^* - \xi_{p-1}^*\right|}{\left|x_p^* - \xi_p^*\right|}$$

Therefore,

$$\|x^* - \xi^*\|_\infty \leq \frac{1}{1 - (2\gamma h + \alpha)}\left[\frac{2^{2n+1}(n!)^4 h}{(2n+1)[(2n!)]^3}\left|k_\varepsilon^{(2n)}\left(s, \varepsilon, x^*\right)\right| - \left|O\left(h^2\right)\right|\right], \quad -1 < \varepsilon < 1 \tag{3.4}$$

Proof Let

$$|x_p^* - \xi_p^*| = \|x^* - \xi^*\|_\infty, \quad 1 \leq p \leq n.$$

By substituting (3.2) and (3.3), we get

$$x_p^* - \xi_p^* = x_{p-1}^* - \xi_{p-1}^* + h\sum_{j=1}^{n} w_j\left[k\left(s_p, t_j, x_j^*\right) - k\left(s_p, t_j, \xi_j^*\right)\right] + E(\varepsilon, h),$$

where $E(\varepsilon, h) = hR(\varepsilon) - O\left(h^2\right)$. According to (H_4) there exist $\gamma > 0$, such that

$$\left|k\left(s_p, t_j, x_j^*\right) - k\left(s_p, t_j, \xi_j^*\right)\right| \leq \gamma\left|x_j^* - \xi_p^*\right|, \quad j = 1, 2, \cdots, n.$$

and by using the above inequality, we obtain

$$\begin{aligned}\left|x_p^* - \xi_p^*\right| &\le \left|x_{p-1}^* - \xi_{p-1}^*\right| + \left|h\sum_{j=1}^{n} w_j \left[k\left(s_p, t_j, x_j^*\right) - k\left(s_p, t_j, \xi_j^*\right)\right]\right| + |E(\varepsilon, h)| \\ &\le \left|x_{p-1}^* - \xi_{p-1}^*\right| + \gamma h \sum_{j=1}^{n} w_j \left|x_j^* - \xi_j^*\right| + |E(\varepsilon, h)|\end{aligned}$$

As $\sum_{j-1}^{n} w_j = 2$. Then,

$$\left|x_p^* - \xi_p^*\right| \le \left|x_{p-1}^* - \xi_{p-1}^*\right| + 2\gamma h \left|x_p^* - \xi_p^*\right| + |E(\varepsilon, h)|$$

We divide the previous inequality by $\left|x_p^* - \xi_p^*\right|$, we get

$$\frac{\left|x_p^* - \xi_p^*\right|}{\left|x_p^* - \xi_p^*\right|} \le 2\gamma h \frac{\left|x_p^* - \xi_p^*\right|}{\left|x_p^* - \xi_p^*\right|} + \frac{\left|x_{p-1}^* - \xi_{p-1}^*\right|}{\left|x_p^* - \xi_p^*\right|} + \frac{|E(\varepsilon, h)|}{\left|x_p^* - \xi_p^*\right|}$$

Now, from (i) and (ii) we have

$$x' > 0,$$

which means that x is strictly increasing, then

$$\max_{1 \le p \le n} \frac{\left|x_{p-1}^* - \xi_{p-1}^*\right|}{\left|x_p^* - \xi_p\right|} = \alpha,$$

where $0 < \alpha < 1$. Hence,

$$\|x^* - \xi^*\|_\infty \le \frac{1}{1 - (2\gamma h + \alpha)} \left[\frac{2^{2n+1}(n!)^4 h}{(2n+1)[(2n!)]^3} \left|k_\varepsilon^{(2n)}\left(s, \varepsilon, x^*\right)\right| - O\left(h^2\right)\right], \quad -1 < \varepsilon < 1$$

which completes the proof. □

Next, a nonlinear system with a special form is given by (3.3). A numerical approach could now be found to obtain the approximate solution.

4. The numerical approach

The purpose of this section is to find an approximate solution of nonlinear systems, using iterative methods for more details we refer to [19, Chapter 4, 5] and [18, Chapter 2]. As we previously mentioned, the nonlinear system (3.3) has a special form which allows us to approximate its solution using an iterative method.

In linear systems, Gauss–Seidel's method [18, Chapter 2, Section5] is an effective method for solving such types of systems of equations. Thus, we apply a successive substitution similar to this one and define an iterative process which leads to the vector sequence $\left\{\xi^{(k)}\right\}$, where $\xi^{(k)} = \left(\xi_1^{(k)}, \xi_2^{(k)}, \cdots, \xi_n^{(k)}\right)$ and each element of $\xi^{(k)}$ satisfy the following iteration formula,

$$\xi_i^{(k+1)} - \xi_{i-1}^{(k+1)} = hy_i + h\sum_{j=1}^{n} w_j k(s_i, t_j, \xi_j^{(k)}),\ i = 1, 2, \cdots, n,\ k = 0, 1, \cdots . \tag{4.1}$$

Here, we seek for the conditions that guarantee the convergence of the sequence $\{\xi^{(k)}\}$.

Accordingly, we yield the following theorem.

Theorem 4.1 *Under the same assumption as Proposition 3.1 and for $1 \leq n < \infty$ the sequence $\{\xi^{(k)}\}$ comes from (4.1) tends to the exact solution ξ^* of (3.3), for any arbitrary $\xi^{(0)}$.*

Proof By substituting (3.3) and (4.1), we get

$$\left(\xi_i^{(k+1)} - \xi_{i-1}^{(k+1)}\right) - \left(\xi_i^* - \xi_{i-1}^*\right) = h \sum_{j=1}^{n} w_j \left[k(s_i, t_j, \xi_j^{(k)}) - k(s_i, t_j, \xi_j^*)\right],$$

with $i = 1, 2, \cdots, n,\ k = 0, 1, \cdots$. From (i) and (ii) we get the increasing property of ξ and from (H_4) we get

$$\sum_{j=1}^{n} w_j \left|k\left(s_i, t_j, \xi_j^{(k)}\right) - k\left(s_i, t_j, \xi_j^*\right)\right| \leq \gamma \sum_{j=1}^{n} w_j \left|\xi_j^{(k)} - \xi_j^*\right|$$

Thus

$$\left|\xi_i^{(k+1)} - \xi_i^*\right| \leq \left|\xi_{i-1}^{(k+1)} - \xi_{i-1}^*\right| + 2\gamma h \left\|\xi^{(k)} - \xi^*\right\|_\infty$$

$$(1-\beta)\left|\xi_i^{(k+1)} - \xi_i^*\right| \leq 2\gamma h \left\|\xi^{(k)} - \xi^*\right\|_\infty$$

$$\left|\xi_i^{(k+1)} - \xi_i^*\right| \leq \frac{2\gamma h}{1-\beta} \left\|\xi^{(k)} - \xi^*\right\|_\infty$$

where $\beta = \max_{1 \leq i \leq n} \frac{\left|\xi_{i-1}^{(k+1)} - \xi_{i-1}^*\right|}{\left|\xi_i^{(k+1)} - \xi_i^*\right|}$ and $0 < \beta < 1$. Now, let $\lambda = \frac{2\gamma h}{1-\beta}$, we have

$$\left\|\xi^{(k+1)} - \xi^*\right\|_\infty \leq \lambda \left\|\xi^{(k)} - \xi^*\right\|_\infty \leq \lambda^2 \left\|\xi^{(k-1)} - \xi^*\right\|_\infty \leq \cdots \leq \lambda^k \left\|\xi^{(0)} - \xi^*\right\|_\infty, k = 0, 1, \cdots$$

Since, we have $0 < 2\gamma h < 1 - \beta$ and $0 < 1 - \beta < 1$, then $0 < \lambda < 1$. Hence, $\left\|\xi^{(k+1)} - \xi^*\right\|_\infty$ goes to zero, when k goes to ∞. The proof is complete. □

5. Algorithm of the method and numerical simulations

This section is devoted to numerical tests to verify the efficiency of the method.

5.1. Algorithm of the method

The purpose of this section is to define an algorithm for the nonlinear integro-differential equation (1.1) and by taking in consideration the assumptions of Proposition 3.1.

The algorithm is as follows: [3]

The initial data:

- Choose any $\varepsilon > 0$.
- Let $\Gamma = \{-1 = s_0 = t_0, s_1 = t_1, \cdots, s_{n-1} = t_{n-1}, s_n = t_n = 1\}$ on $[-1, 1]$.

- Let $h = s_i - s_{i-1},\ i = 1, \cdots, n$, be the step size.
- The initial vector $\xi^{(0)} = (\xi_0^{(0)}, \xi_1^{(0)}, \cdots, \xi_n^{(0)})^T$ and $\xi_0^{(k)}$ is given.
- Set $k = 0$ and go to the main steps

Main steps:

(1) Compute $\xi^{(k+1)}$ using Equation (4.1) and go to the next step.

(2) Compute the error $\|\xi^{(k+1)} - \xi^{(k)}\|_\infty$ and go the next step.

(3) If $\|\xi^{(k+1)} - \xi^{(k)}\|_\infty \leq \varepsilon$, stop; otherwise, set $k = k + 1$ and go to step (1).

5.2. Numerical simulation

Let x^* be the exact solution and ξ^* be the approximate solution using Gauss–Kronrod quadrature with the following error function $\delta(s_i) = \|x^*(s_i) - \xi^*(s_i)\|_\infty$.

5.2.1. Example 1

Consider the nonlinear integro-differential equations

$$x'(s) = 2s + \int_{-1}^{1} 4ts \exp(x(t))dt$$
$$x(-1) = 0,$$

with the exact solution

$$x(s) = s^2 - 1.$$

Where, the initial vector $\xi^{(0)} = 0$, $\xi_0^{(k)} = (1, \cdots, 1)^T$, $n = 199$ and $\epsilon = 4.4409e - 8$.
Figure 1 shows the comparison between the exact and the approximate solution and Figure 2 (Figure 3) shows the error function $\delta(s)$ for $n = 199$ ($n = 900$, respectively).

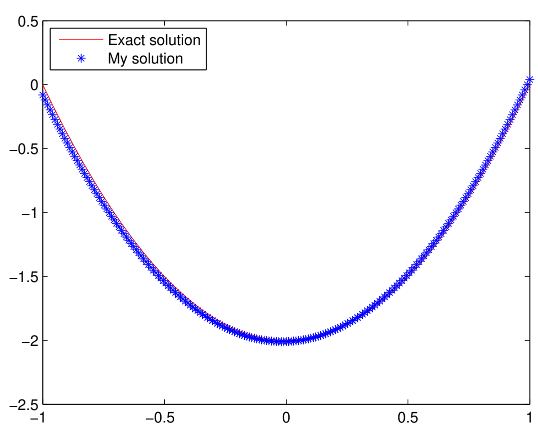

Figure 1. The exact and approximate solution of Example 5.2.1, with $n = 199$.

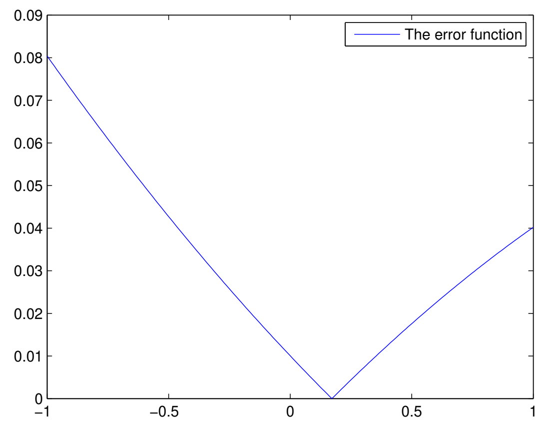

Figure 2. The error function for Example 5.2.1, with $n = 199$.

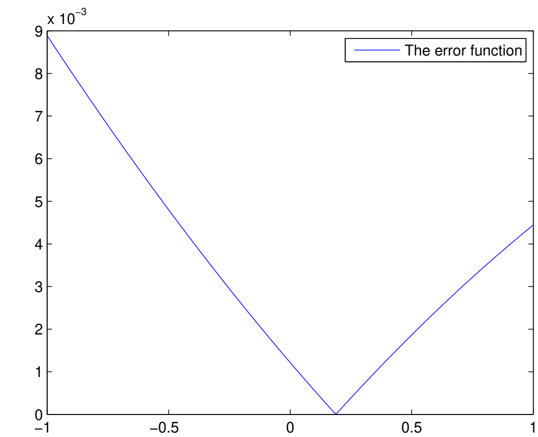

Figure 3. The error function for Example 5.2.1, with $n = 900$.

5.2.2. Example 2

Consider the nonlinear Fredholm integral equation of the second kind

$$x'(s) = -\pi sin(\pi s) + \int_{-1}^{1} sin(t)sx(t)^2 dt$$
$$x(-1) = 0,$$

with the exact

$$x(s) = cos(\pi s) + 1.$$

Where the initial vector $\xi^{(0)} = 0$, $\xi_0^{(k)} = 0$, $n = 250$ and $\epsilon = 4.4409e - 8$.
Figure 4 shows the comparison between the exact and the approximate solution and Figure 5 (Figure 6) shows the error function $\delta(s)$ for $n = 250$ ($n = 1200$, respectively).

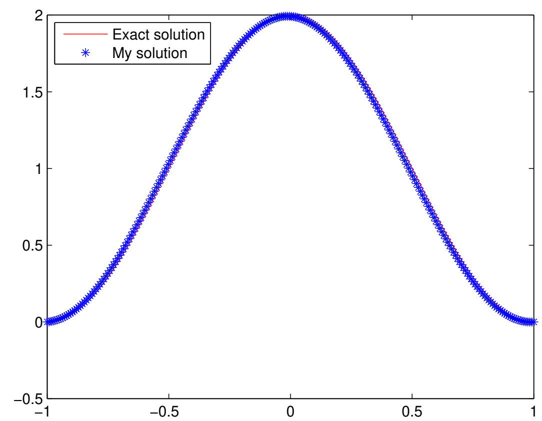

Figure 4. The exact and approximate solution of Example 5.2.2, with $n = 250$.

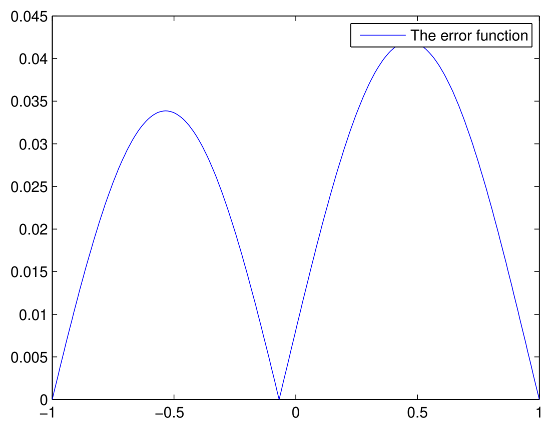

Figure 5. The error function for Example 5.2.2, with $n = 250$.

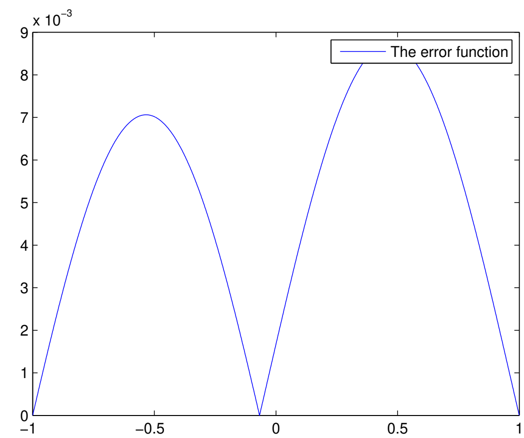

Figure 6. The error function for Example 5.2.2, with $n = 1200$.

5.2.3. Example 3

Consider the linear Fredholm integral equation of the second kind

$$x'(s) = 50s^{49} - 58s^{57} + \frac{16}{3009}s^{11} + 9s^8 - \frac{257}{273} + \int_{-1}^{1}(s^{11} - t^{11})x(t))dt$$

$$x(-1) = 0,$$

with the initial vector $\xi^{(0)} = 0$, $\xi_0^{(k)} = 0$, $n = 65$ and $\epsilon = 4.4409e - 16$.
Figure 7 shows the comparison between the exact and the approximate solution and Figure 8 shows the error function $\delta(s)$.

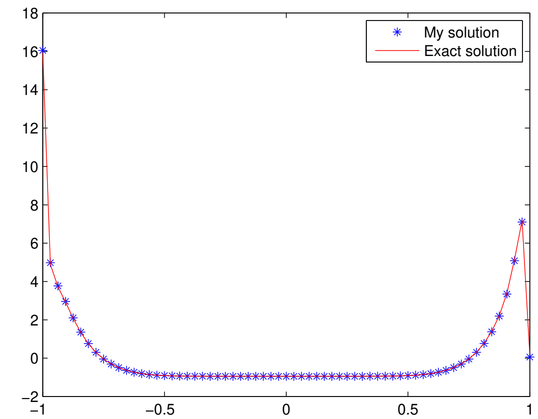

Figure 7. The exact and approximate solution of Example 5.2.3, with $n = 65$.

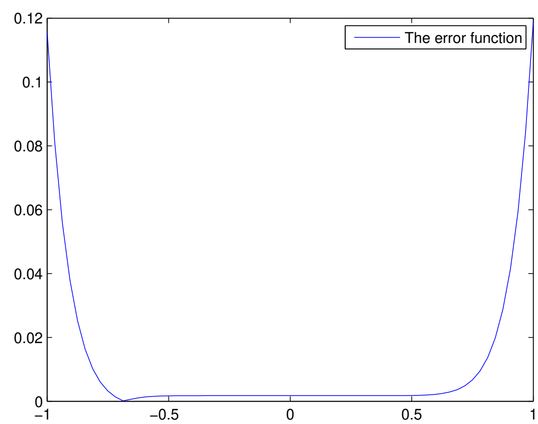

Figure 8. The error function for Example 5.2.3, with $n = 65$.

Remark 5.1 *The main reason for Example 5.2.3 is that the proposed method can be applied to both linear and nonlinear equations and the figures show the similarity between exact and approximate solutions.*

5.2.4. Example 4

Suppose the following Fredholm integro-differential equation.

$$x'(s) = se^s + e^s - s + \frac{1}{2}\int_0^1 tx(t)dt$$
$$x(0) = 0$$

with exact solution $x(s) = se^s$. The errors of the variation of parameters method (VPM) in [10] and Gauss–Kronrod method (GKM) are given in Table .

Table . Error analysis of VPM and GKM for Example 5.2.4.

s	GKM	VPM
0	0.002	0.000000000
0.1	0.002213	0.0036943633
0.2	0.002455	0.0145247634
0.3	0.002728	0.0320381598
0.4	0.003038	0.0556708598
0.5	0.003387	0.0847318684
0.6	0.003783	0.1183839981
0.7	0.004229	0.155622457
0.8	0.004733	0.195250603
0.9	0.005302	0.235852468
1	0.005937	0.275761712

Remark 5.2 *The aim of Example 5.2.4 is to compare the method (GKM) proposed above with (VPM) proposed by Ul Haq et al. [10] and Table shows the efficiency of this new method.*

6. Conclusion

In this paper, we have investigated the existence and uniqueness of the solution of a nonlinear integro-differential equation. We have presented a numerical approach named Gauss–Kronrod method with backward difference to obtain an approximate solution and we have proved the vanishing of the error. Furthermore, we have obtained precise results and the numerical simulations in Examples 5.2.1–5.2.4 show the accuracy of the proposed approach. Based on Table , the approximate solution of the proposed method is a better method compared to the VPM.

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