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Discrete impulsive Sturm–Liouville equation with hyperbolic eigenparameter

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Abstract: Let L denote the selfadjoint difference operator of second order with boundary and impulsive conditions generated in $\ell_2(\mathbb{N})$ by

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = (2\cosh z)\, y_n \ , \ n \in \mathbb{N} \setminus \{k-1, k, k+1\},$$

$$y_0 = 0 \ ,$$

$$\begin{cases} y_{k+1} = \theta_1 y_{k-1} \\ \triangle y_{k+1} = \theta_2 \bigtriangledown y_{k-1} \end{cases} , \ \theta_1, \theta_2 \in \mathbb{R},$$

where $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ are real sequences and $\triangle, \bigtriangledown$ are respectively forward and backward operators. In this paper, the spectral properties of L such as the resolvent operator, the spectrum, the eigenvalues, the scattering function and their properties are investigated. Moreover, an example about the scattering function and the existence of eigenvalues is given in the special cases, if

$$\sum_{n=1}^{\infty} n\left(|1-a_n| + |b_n|\right) < \infty.$$

Key words: Discrete equations, impulsive condition, hyperbolic eigenparameter, spectral analysis, scattering function, resolvent operator, eigenvalues

1. Introduction

Difference equations are among the important study topics of many mathematicians from the recent past to the present. It can be shown as reasons for this are that the solutions of difference equations can be obtained with the help of computers and these equations arise as a mathematical model in many daily events about biology, economics, engineering and physics, especially Newtonian mechanics. Some problems about spectral theory of difference equations with spectral singularities have been debated in [1,3,4,6,9,13,17–19,22,23].

Let us consider the discrete Sturm–Liouville problem (DSP)

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \ n \in \mathbb{N} = \{1, 2, \ldots\} \ , \tag{1.1}$$

with the boundary condition $\sum_{n=0}^{\infty} h_n y_n = 0$, where $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$, $\{h_n\}_{n\in\mathbb{N}}$ are complex sequences and λ is a spectral parameter. The spectral properties of the DSP (1.1) such as the spectrum, spectral singularities

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and principal functions have been investigated in [10]. It is concluded that the spectral singularities are poles of the resolvent that are imbedded in the continuous spectrum and are not eigenvalues. Also, the results about the spectrum of the DSP (1.1) are applied to the nonselfadjoint Jacobi matrices and discrete Schrödinger operators in the same study. The spectral analysis of a nonselfadjoint second order difference equation with principal functions has been investigated in [2]. In that study, it is shown that the Jost solution of this equation has an analytic continuation to the lower half-plane and the finiteness of the eigenvalues and the spectral singularities of the difference equation is obtained as a result of this analytic continuation. Furthermore, in the course of many physical events, impulses are observed at certain moments of the time. The solutions of the impulsive difference equations, which occur when the mathematical model of the impulsive physical phenomena such as temperature distribution and heat transfer are made in the set of integers, have extra equation jumps at certain points. Therefore, examining problems involving impulsive difference equations is very important, both theoretically and practically. The spectral analysis of some impulsive difference equation problems with scattering function have been debated by various authors in recent years [5,7,8,11,20,21,23,24] and their studies have led to the rapid development of the theory of discrete difference equations.

Let L denote the selfadjoint difference operator of second order generated in $\ell_2(\mathbb{N})$ by

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n,\ n \in \mathbb{N} \setminus \{k-1, k, k+1\}, \tag{1.2}$$

with boundary and impulsive conditions

$$y_0 = 0\ , \tag{1.3}$$

$$\begin{cases} y_{k+1} = \theta_1 y_{k-1} \\ \triangle y_{k+1} = \theta_2 \bigtriangledown y_{k-1} \end{cases},\ \theta_1, \theta_2 \in \mathbb{R}, \tag{1.4}$$

where $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ are real sequences, $a_n \neq 0$ for all $n \in \mathbb{N} \cup \{0\}$, $\theta_1\theta_2 \neq 0$, λ is a hyperbolic eigenparameter and $\triangle, \bigtriangledown$ are forward and backward operators, respectively. Moreover, we can write the difference equation (1.2) in the following Sturm–Liouville form:

$$\bigtriangledown(a_n \triangle y_n) + v_n y_n = \lambda y_n,\ n \in \mathbb{N},$$

where $v_n = a_{n-1} + a_n + b_n$.

Differently other studies in the literature, the specific feature of this paper which is one of the articles have applicability in a lot of branches of mathematical physics and chemistry is the discrete Sturm–Liouville equation with boundary and impulsive conditions is taken under investigation for hyperbolic eigenparameter. In this paper, we investigate various spectral properties of L; i.e. we investigate the spectrum, the scattering function and their properties if

$$\sum_{n=1}^{\infty} n\left(|1-a_n| + |b_n|\right) < \infty. \tag{1.5}$$

2. Spectral properties of scattering function and spectrum

Let us define two semistrips $T_- = \{z \in \mathbb{C} : z = \xi + i\tau,\ \xi < 0,\ \tau \in [0, 2\pi]\}$ and $T = T_- \cup T_0$, where $T_0 = \{z \in \mathbb{C} : z = i\tau,\ \tau \in [0, 2\pi]\}$. Under the condition (1.5), Eq. (1.2) has the solution

$$f_n(z) = \alpha_n e^{nz}\left(1 + \sum_{m=1}^{\infty} A_{nm} e^{mz}\right),\ n \in \{k+1, k+2, \ldots\},$$

satisfying the condition $\lim_{n\to\infty} e^{-nz} f_n(z) = 1,\ z \in T_-$ for $\lambda = 2\cosh z$ and $\alpha_n,\ A_{nm}$ are expressed in terms of $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ as

$$\begin{aligned}
\alpha_n &= \left(\prod_{i=n}^{\infty} a_i\right)^{-1} \\
A_{n,1} &= -\sum_{i=n+1}^{\infty} b_i \\
A_{n,2} &= \sum_{i=n+1}^{\infty} \left(1-a_i^2\right) + \sum_{i=n+1}^{\infty} b_i \sum_{j=i+1}^{\infty} b_j \\
A_{n,m+2} &= A_{n+1,m} + \sum_{i=n+1}^{\infty} \left(1-a_i^2\right) A_{i+1,m} - \sum_{i=n+1}^{\infty} b_i A_{i,m+1}
\end{aligned}$$

from [15, 16]. Moreover

$$|A_{nm}| \leq B \sum_{k=n+\left[\left|\frac{m}{2}\right|\right]}^{\infty} \left(|1-a_k| + |b_k|\right)$$

holds, where $B > 0$ is constant and $\left[\left|\frac{m}{2}\right|\right]$ is the integer part of $\frac{m}{2}$. Hence, $f_n(z)$ is analytic with respect to z in $\mathbb{C}_{left} = \{z \in \mathbb{C} : Rez < 0\}$, continuous on the real axis and $f_n(z) = f_n(z+2\pi i)$ for all $z \in \overline{\mathbb{C}}_{left}$.

Now, we consider the elementary $\{\varphi_n(z)\}$ and $\{\psi_n(z)\}$, $n = 0, 1, \ldots, k-1$ solutions of (1.2) for $z \in T$ subject to the initial conditions

$$\begin{array}{ll} \varphi_0(z) = 0 & , \ \varphi_1(z) = 1 \\ \psi_0(z) = a_0^{-1} & , \ \psi_1(z) = 0\ . \end{array} \tag{2.1}$$

Then, we find the wronskian of $\varphi_n(z)$ and $\psi_n(z)$ as

$$\begin{aligned}
W\left[\varphi_n(z), \psi_n(z)\right] &= a_n\left[\varphi_n(z)\psi_{n+1}(z) - \varphi_{n+1}(z)\psi_n(z)\right] \\
&= a_0\left[\varphi_0(z)\psi_1(z) - \varphi_1(z)\psi_0(z)\right] \\
&= -1
\end{aligned}$$

and also they are entire functions for $z \in \mathbb{C}$.

In here, we define the Jost solution of L by using $f_n(z)$, $\varphi_n(z)$ and $\psi_n(z)$

$$J_n(z) = \begin{cases} p(z)\varphi_n(z) + r(z)\psi_n(z) & ;\ n = 0, 1, \ldots, k-1 \\ f_n(z) & ;\ n = k+1, k+2, \ldots \end{cases} \tag{2.2}$$

for $z \in T$, where

$$\begin{aligned}
p(z) &= \frac{a_{k-2}}{\theta_1\theta_2}\left[\theta_1\psi_{k-1}(z) \bigtriangleup f_{k+1}(z) - \theta_2 f_{k+1}(z) \bigtriangledown \psi_{k-1}(z)\right] \\
r(z) &= -\frac{a_{k-2}}{\theta_1\theta_2}\left[\theta_1\varphi_{k-1}(z) \bigtriangleup f_{k+1}(z) - \theta_2 f_{k+1}(z) \bigtriangledown \varphi_{k-1}(z)\right]\ ,\ z \in \overline{\mathbb{C}}_{left}
\end{aligned}$$

can be obtained from the conditions (1.4). Furthermore,

$$\begin{aligned}
W\left[f_n(z), f_n(-z)\right] &= \lim_{n\to\infty}\left\{a_n\left[f_n(z)f_{n+1}(-z)-f_{n+1}(z)f_n(-z)\right]\right\} \\
&= \lim_{n\to\infty}\left\{a_n\left[e^{-z}f_n(z)e^{-nz}f_{n+1}(-z)e^{(n+1)z}\right]\right\} \\
&\quad -\lim_{n\to\infty}\left\{a_n\left[e^{z}f_{n+1}(z)e^{-(n+1)z}f_n(-z)e^{nz}\right]\right\} \\
&= e^{-z}-e^{z} \\
&= -2\sinh z \qquad (2.3)
\end{aligned}$$

for $z\in T_0\setminus\{i\pi\}$. If we take the another F solution of L as

$$F_n(z)=\begin{cases}\varphi_n(z) & ;\ n=0,1,\ldots,k-1 \\ q(z)f_n(z)+t(z)f_n(-z) & ;\ n=k+1,k+2,\ldots\end{cases} \qquad (2.4)$$

for $z\in T_0\setminus\{i\pi\}$, where

$$\begin{aligned}
q(z) &= -\frac{a_{k+1}}{2\sinh z}\left[\theta_1\varphi_{k-1}(z)\bigtriangleup f_{k+1}(-z)-\theta_2 f_{k+1}(-z)\bigtriangledown\varphi_{k-1}(z)\right] \\
t(z) &= \frac{a_{k+1}}{2\sinh z}\left[\theta_1\varphi_{k-1}(z)\bigtriangleup f_{k+1}(z)-\theta_2 f_{k+1}(z)\bigtriangledown\varphi_{k-1}(z)\right]\ ,
\end{aligned}$$

then we get $t(z)=q(-z)=\overline{q(z)}$ due to $\varphi_n(-z)=\varphi_n(z)$ and the following result:

Lemma 2.1

$$W\left[J_n(z),F_n(z)\right]=\begin{cases} r(z) & ;\ n=0,1,\ldots,k-1 \\ \frac{a_{k+1}}{a_{k-2}}\theta_1\theta_2 r(z) & ;\ n=k+1,k+2,\ldots\end{cases}$$

for $z\in T_0\setminus\{i\pi\}$.

Proof From (2.1)–(2.4), we find

$$\begin{aligned}
W\left[J_n(z),F_n(z)\right] &= a_0\left[J_0(z)F_1(z)-J_1(z)F_0(z)\right] \\
&= a_0\left[p(z)\varphi_0(z)+r(z)\psi_0(z)\right]\varphi_1(z) \\
&\quad -a_0\left[p(z)\varphi_1(z)+r(z)\psi_1(z)\right]\varphi_0(z) \\
&= r(z)
\end{aligned}$$

for $n=0,1,\ldots,k-1$ and

$$\begin{aligned}
W\left[J_n(z),F_n(z)\right] &= a_{k+1}\left[J_{k+1}(z)F_{k+2}(z)-J_{k+2}(z)F_{k+1}(z)\right] \\
&= a_{k+1}t(z)\left[f_{k+1}(z)f_{k+2}(-z)-f_{k+2}(z)f_{k+1}(-z)\right] \\
&= t(z)W\left[f_n(z),f_n(-z)\right] \\
&= \frac{a_{k+1}}{a_{k-2}}\theta_1\theta_2 r(z)
\end{aligned}$$

for $n=k+1,k+2,\ldots$ since $t(z)=-\frac{a_{k+1}}{a_{k-2}}\frac{\theta_1\theta_2 r(z)}{2\sinh z}$. □

Moreover, if $\widetilde{f}_n(z)$ is unbounded solution of Eq. (1.2) for $n = k+1, k+2, \ldots$ with $\lim_{n\to\infty} e^{nz}\widetilde{f}_n(z) = 1$, $z \in \overline{\mathbb{C}}_{left}$, then

$$W\left[f_n(z), \widetilde{f}_n(z)\right] = -2\sinh z$$

for $T \setminus \{i\pi\}$, and we can define the another G solution of L as

$$G_n(z) = \begin{cases} \varphi_n(z) & ;\ n = 0, 1, \ldots, k-1 \\ k(z)f_n(z) + l(z)\widetilde{f}_n(z) & ;\ n = k+1, k+2, \ldots \end{cases} \tag{2.5}$$

for $z \in T$, where

$$\begin{aligned} k(z) &= -\frac{a_{k+1}}{2\sinh z}\left[\theta_1\varphi_{k-1}(z) \bigtriangleup \widetilde{f}_{k+1}(z) - \theta_2\widetilde{f}_{k+2}(z) \bigtriangledown \varphi_{k-1}(z)\right] \\ l(z) &= \frac{a_{k+1}}{2\sinh z}\left[\theta_1\varphi_{k-1}(z) \bigtriangleup f_{k+1}(z) - \theta_2 f_{k+1}(z) \bigtriangledown \varphi_{k-1}(z)\right] . \end{aligned}$$

Also, the function $G_n(z)$ is unbounded solution of L and

$$l(z) = t(z) = \overline{q(z)} = -\frac{a_{k+1}}{a_{k-2}}\frac{\theta_1\theta_2}{2\sinh z}r(z) \tag{2.6}$$

for $z \in T_0 \setminus \{i\pi\}$.

If Equations (2.2) and (2.5) are taken into account,

$$W\left[J_n(z), G_n(z)\right] = \begin{cases} r(z) & ;\ n = 0, 1, \ldots, k-1 \\ \frac{a_{k+1}}{a_{k-2}}\theta_1\theta_2 r(z) & ;\ n = k+1, k+2, \ldots \end{cases}$$

is obtained for $z \in T$.

Theorem 2.2 *$r(z) \neq 0$ for all z in $T_0 \setminus \{i\pi\}$.*

Proof Suppose that $r(z_0) = 0$ for at least $z_0 \in T_0 \setminus \{i\pi\}$. From (2.6), $t(z_0) = q(z_0) = 0$. Then, the solution $F_n(z_0) = 0,\ n \in \mathbb{N} \cup \{0\}$ is trivial by using the impulsive conditions (1.4) which is a contradiction. $\square$

Definition 2.3 *The scattering function of L is defined by*

$$S(z) = \frac{\overline{J_0(z)}}{J_0(z)}$$

with respect to the Jost solution of L.

Because of the fact that $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ are real sequences and $a_n \neq 0$ for all $n \in \mathbb{N} \cup \{0\}$, it can be written that $\overline{J_n(z)} = J_n(-z)$ for $z \in T_0 \setminus \{i\pi\}$. Then, the scattering function

$$S(z) = \frac{\overline{r(z)}}{r(z)} = \frac{\overline{a_0 J_0(z)}}{a_0 J_0(z)} = \frac{J_0(-z)}{J_0(z)} = \frac{r(-z)}{r(z)} \tag{2.7}$$

and so

$$S(z)=\frac{\theta_1\varphi_{k-1}(-z)\bigtriangleup f_{k+1}(-z)-\theta_2 f_{k+1}(-z)\bigtriangledown\varphi_{k-1}(-z)}{\theta_1\varphi_{k-1}(z)\bigtriangleup f_{k+1}(z)-\theta_2 f_{k+1}(z)\bigtriangledown\varphi_{k-1}(z)}$$

with $\lim_{z\to 0} S(z)=S(0)=1$. In addition,

$$S(-z)=\frac{J_0(z)}{J_0(-z)}=S^{-1}(z)=\overline{S(z)}$$

for $z\in T_0\setminus\{i\pi\}$ by using the equalities (2.7).

Theorem 2.4 *The resolvent operator of L is*

$$R_\lambda g_n=\sum_{m=1}^{\infty}R_{nm}g_m \quad , \quad \{g_m\}\in\ell_2(\mathbb{N})$$

for all $z\in T\setminus\{i\pi\}$ and $r(z)\neq 0$, where

$$R_{nm}(z)=\begin{cases}-\frac{G_m J_n}{W[J_m,G_m]} & ; \quad m\le n\\ -\frac{G_n J_m}{W[J_m,G_m]} & ; \quad m>n\end{cases}$$

is the Green function of L for $m,\ n\neq k$.

Proof Since $J_n(z)$ and $G_n(z)$ are fundamental solutions of L,

$$y_n(z)=\eta_n J_n(z)+\zeta_n G_n(z)$$

is the general solution of

$$\bigtriangledown(a_n\bigtriangleup y_n)+v_n y_n-\lambda y_n=g_n,$$

where η_n, ζ_n are nonzero coefficients and $v_n=a_{n-1}+a_n+b_n$. Then,

$$\begin{aligned}\eta_n &= -\sum_{m=1}^{n}\frac{G_m g_m}{W[J_m,G_m]} \quad , \quad m\neq k\\ \zeta_n &= -\sum_{m=n+1}^{\infty}\frac{J_m g_m}{W[J_m,G_m]} \quad , \quad m\neq k\end{aligned}$$

are obtained using the method of variation of parameters, and so they represent the Green function and resolvent operator of L. □

Theorem 2.5 *Under the condition (1.5), $\sigma_c(L)=[-2,2]$ where $\sigma_c(L)$ is the continuous spectrum of L.*

Proof Let L_1 and L_2 denote the operators generated in $\ell_2(\mathbb{N})$ by the difference expressions

$$(l_1 y)_n=y_{n-1}+y_{n+1},\ n\in\mathbb{N}\setminus\{k-1,k+1\}$$

and

$$(l_2 y)_n = (a_{n-1} - 1)\, y_{n-1} + (a_n - 1)\, y_{n+1} + b_n y_n,\ n \in \mathbb{N} \setminus \{k-1, k, k+1\},$$

respectively. It is apparent that $L = L_1 + L_2,\ L_1 = L_1^*$ and from the condition (1.5) L_2 is a compact operator in $\ell_2(\mathbb{N})$. Then, we obtain

$$\sigma_c(L) = \sigma_c(L_1) = \sigma(L_1) = [-2, 2]$$

by using the Weyl theorem of a compact perturbation ([14]). □

Theorem 2.6

$$\begin{aligned} \sigma_d(L) &= \{\lambda \in \mathbb{C} : \lambda = 2\cosh z,\ z \in T_-,\ r(z) = 0\}, \\ \sigma_{ss}(L) &= \varnothing, \end{aligned}$$

where $\sigma_d(L)$ and $\sigma_{ss}(L)$ are the sets of eigenvalues and spectral singularities of L, respectively.

Proof The Jost solution $J_n(z) \in \ell_2(\mathbb{N})$ because the first part of it consists of a finite number of elements and $f_n(z)$ is also in $\ell_2(\mathbb{N})$. Moreover, due to the condition (1.3),

$$0 = J_0(z) = p(z)\varphi_0(z) + r(z)\psi_0(z) = \frac{r(z)}{a_0}$$

hence $r(z) = 0$. Therefore, from the definition of spectral singularities and eigenvalues in [12] and Theorem 2.2.,

$$\sigma_d(L) = \{\lambda \in \mathbb{C} : \lambda = 2\cosh z,\ z \in T_-,\ r(z) = 0\}$$

and

$$\begin{aligned} \sigma_{ss}(L) &= \{\lambda \in \mathbb{C} : \lambda = 2\cosh z,\ z \in T_0 \setminus \{i\pi\},\ r(z) = 0\} \\ &= \varnothing. \end{aligned}$$

□

Therefore, in the light of the last theorem, the quantitative properties of zeros of $r(z)$ in T_- are required to investigate the quantitative properties of the eigenvalues of L.

3. Special cases

Let us choose the operator M in $\ell_2(\mathbb{N})$ generated by the equation

$$y_{n-1} + y_{n+1} = \lambda y_n,\ n \in \mathbb{N} \setminus \{4, 5, 6\}\ , \tag{3.1}$$

with boundary and impulsive conditions

$$\begin{cases} y_0 = 0\ , \\ y_6 = \theta_1 y_4 \\ \triangle y_6 = \theta_2 \bigtriangledown y_4 \end{cases},\ \theta_1, \theta_2 \in \mathbb{R}, \tag{3.2}$$

where $\theta_1\theta_2 \neq 0$ and $\lambda = 2\cosh z$ is a hyperbolic eigenparameter. Also, $\{\varphi_n(z)\}$ and $\{\psi_n(z)\}$, $n = 0,1,2,3,4$ are the elementary solutions of (3.1) for $z \in T$ subject to the initial conditions (2.1) and clearly $f_n(z) = e^{nz}$. In additon, we get from (3.1) that

$$\begin{array}{ll} \varphi_3(\lambda) = \lambda^2 - 1 & , \ \varphi_4(\lambda) = \lambda^3 - 2\lambda \\ \psi_3(\lambda) = -\lambda & , \ \psi_4(\lambda) = 1 - 2\lambda^2 \ . \end{array}$$

After that, we find

$$\begin{aligned} p(z) &= \frac{1}{\theta_1\theta_2}\left[\theta_1\psi_4(z) \bigtriangleup f_6(z) - \theta_2 f_6(z) \bigtriangledown \psi_4(z)\right] \\ &= e^{4z}\left[\frac{e^{4z} - e^{3z} + e^{2z} - e^{z} + 1}{\theta_1} - \frac{e^{5z} - e^{4z} + e^{3z} - e^{2z} + e^{z} - 1}{\theta_2}\right]. \\ r(z) &= -\frac{1}{\theta_1\theta_2}\left[\theta_1\varphi_4(z) \bigtriangleup f_6(z) - \theta_2 f_6(z) \bigtriangledown \varphi_4(z)\right] \\ &= e^{3z}\left[\frac{e^{6z} - e^{5z} + e^{4z} - e^{3z} + e^{2z} - e^{z} + 1}{\theta_1} - \frac{e^{7z} - e^{6z} + e^{5z} - e^{4z} + e^{3z} - e^{2z} + e^{z} - 1}{\theta_2}\right] \end{aligned}$$

from (3.2), and the Jost solution of M

$$J_n(z) = \begin{cases} e^{6z}\left[p^{(1)}(z)\varphi_n(z) + r^{(1)}(z)\psi_n(z)\right] & ; \ n = 0,1,2,3,4 \\ e^{nz} & ; \ n = 6,7,8,\ldots \end{cases},$$

where

$$\begin{aligned} p^{(1)}(z) &= \frac{\lambda^2 - \lambda - 1}{\theta_1} - \frac{\left(\lambda^2 - 1\right)\left(e^{z} - 1\right)}{\theta_2}, \\ r^{(1)}(z) &= \frac{\lambda\left(\lambda^2 - \lambda - 2\right) + 1}{\theta_1} - \frac{\lambda\left(\lambda^2 - 2\right)\left(e^{z} - 1\right)}{\theta_2}. \end{aligned}$$

By using (2.8), the scattering function of M is

$$\begin{aligned} S(z) &= \frac{\theta_1\varphi_4(-z) \bigtriangleup f_6(-z) - \theta_2 f_6(-z) \bigtriangledown \varphi_4(-z)}{\theta_1\varphi_4(z) \bigtriangleup f_6(z) - \theta_2 f_6(z) \bigtriangledown \varphi_4(z)} \\ &= e^{-6z}\left[\frac{\theta_2\left(e^{-6z} - e^{-5z} + e^{-4z} - e^{-3z} + e^{-2z} - e^{-z} + 1\right)}{\theta_2\left(e^{6z} - e^{5z} + e^{4z} - e^{3z} + e^{2z} - e^{z} + 1\right) - \theta_1\left(e^{7z} - e^{6z} + e^{5z} - e^{4z} + e^{3z} - e^{2z} + e^{z} - 1\right)}\right. \\ &\quad \left. - \frac{\theta_1\left(e^{-7z} - e^{-6z} + e^{-5z} - e^{-4z} + e^{-3z} - e^{-2z} + e^{-z} - 1\right)}{\theta_2\left(e^{6z} - e^{5z} + e^{4z} - e^{3z} + e^{2z} - e^{z} + 1\right) - \theta_1\left(e^{7z} - e^{6z} + e^{5z} - e^{4z} + e^{3z} - e^{2z} + e^{z} - 1\right)}\right] \end{aligned}$$

for $z \in T_0 \setminus \{i\pi\}$.

Moreover, we obtain the eigenvalues of M

$$\sigma_d(M) = \{\lambda \in \mathbb{C} : \lambda = 2\cosh z, \ z \in T_-, \ r(z) = 0\}.$$

Since $\lambda = 2\cosh z$ and $r(z) = 0$ in $\sigma_d(M)$, then we get

$$\frac{e^{6z} - e^{5z} + e^{4z} - e^{3z} + e^{2z} - e^{z} + 1}{e^{7z} - e^{6z} + e^{5z} - e^{4z} + e^{3z} - e^{2z} + e^{z} - 1} = \frac{\theta_1}{\theta_2}.$$

If $\theta_2 = \alpha$ for $\alpha \neq 0$, then

$$e^{7z} - (\alpha+1)\left(e^{6z} - e^{5z} + e^{4z} - e^{3z} + e^{2z} - e^{z} + 1\right) = 0 \tag{3.3}$$

can be obtained. From (3.3), it is easily seen that $\alpha \neq 1$; i.e. $\theta_1 \neq \theta_2$.

Case 1: For $\alpha = 1$, we find

$$e^z \approx 1.30699 \ ,$$

$$\begin{array}{ll} e^z \approx 0.917886 - i0.707524 & , \ e^z \approx 0.917886 + i0.707524 \ , \\ e^z \approx 0.106125 - i1.049262 & , \ e^z \approx 0.106125 + i1.049262 \ , \\ e^z \approx -0.677506 - i0.751906 & , \ e^z \approx -0.677506 + i0.751906. \end{array}$$

Since these equations do not have any root in T_-, then $\sigma_d(M) = \varnothing$.

Case 2: For $\alpha = 999999$, then

$$e^z \approx 999999 \ ,$$

$$\begin{array}{ll} e^z \approx 0.900969 - i0.433883 & , \ e^z \approx 0.900969 + i0.433883 \ , \\ e^z \approx 0.222521 - i0.974928 & , \ e^z \approx 0.222521 + i0.974928 \ , \\ e^z \approx -0.623489 - i0.781831 & , \ e^z \approx -0.623489 + i0.781831. \end{array}$$

Since there are two roots for the last phrase $z_1 \approx -2.53669\times10^{-7}+i4.03919$ and $z_2 \approx -2.53669\times10^{-7}+i2.24399$ in T_-, then the set of eigenvalues of M is $\sigma_d(M) = \{\lambda \in \mathbb{C} : \lambda = 2\cosh z_k;\ k = 1,2\}$. In additon, we can note that, the eigenvalues of M will appear for big enough α; i.e. $\alpha \to \infty$.

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