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Existence of solutions for an infinite system of tempered fractional order boundary value problems in the spaces of tempered sequences

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Abstract: This paper deals with infinite system of nonlinear two-point tempered fractional order boundary value problems

$$ {}^{\mathtt{RL}}_{\ 0}\mathbb{D}^{\delta_2,\ell}_{\mathtt{z}}\Big[\mathtt{p}_{\mathtt{j}}(\mathtt{z})\,{}^{\mathtt{RL}}_{\ 0}\mathbb{D}^{\delta_1,\ell}_{\mathtt{z}}\vartheta_{\mathtt{j}}(\mathtt{z})\Big] = \lambda_{\mathtt{j}}\varphi\big(\mathtt{z},\vartheta(\mathtt{z})\big),\ \mathtt{z}\in[0,\mathtt{T}], \delta_1,\delta_2\in(1,2), $$

$$ \vartheta_{\mathtt{j}}(0) = \lim_{\mathtt{z}\to 0}\Big[{}^{\mathtt{RL}}_{\ 0}\mathbb{D}^{\delta_1,\ell}_{\mathtt{z}}(e^{\ell\mathtt{z}}\vartheta_{\mathtt{j}}(\mathtt{z}))\Big] = 0, $$

$$ e^{\ell\mathtt{T}}\vartheta_{\mathtt{j}}(\mathtt{T}) = \lim_{\mathtt{z}\to \mathtt{T}}\Big[{}^{\mathtt{RL}}_{\ 0}\mathbb{D}^{\delta_1,\ell}_{\mathtt{z}}(e^{\ell\mathtt{z}}\vartheta_{\mathtt{j}}(\mathtt{z}))\Big] = 0, $$

where $\mathtt{j}\in\{1,2,3,\cdots\}$, $\ell\geq 0$, ${}^{\mathtt{RL}}_{\ 0}\mathbb{D}^{\star,\ell}_{\mathtt{z}}$ denotes the Riemann–Liouville tempered fractional derivative of order $\star\in\{\delta_1,\delta_2\}$, $\vartheta(\mathtt{z}) = (\vartheta_{\mathtt{j}}(\mathtt{z}))_{\mathtt{j}=1}^{\infty}$, $\varphi_{\mathtt{j}}:[0,\mathtt{T}]\to[0,\mathtt{T}]$ are continuous and we derive sufficient conditions for the existence of solutions to the system via the Hausdorff measure of noncompactness and Meir–Keeler fixed point theorem in tempered sequence spaces.

Key words: Tempered fractional derivative, tempered sequence space, iterative system, Meir–Keeler fixed point theorem, Hausdorff measure of noncompactness.

1. Introduction

Fractional differential equations with boundary conditions have occupied an important area in the fractional calculus domain, since these problems appear in various applications of sciences and engineering, such as mechanics, finance, control theory, electricity, chemistry, biology, chemistry, and economics [8, 10, 32, 35, 36]. The progress of this field has motivated researchers to investigate some interesting questions related to the existence, uniqueness, and stability of solutions [13, 23, 25–27, 31].

The tempered fractional derivative is one of the generalized forms of the fractional derivatives. Multiplying classical fractional derivative by an exponential factor leads to the tempered fractional derivative. This new fractional operator depends on a parameter ℓ, and the classical Riemann–Liouville and Caputo fractional derivatives are obtained in special cases for $\ell = 0$. Nowadays, the tempered fractional derivative has become a popular topic for investigation due to its application in physics, groundwater hydrology, poroelasticity, geophysical flow, finance [7, 11, 12, 17, 18, 28] and so on. In [40], Zaky studied the well-posedness of the

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solution to the following two-point nonlinear tempered fractional boundary value problem

$$
\begin{aligned}
{}^{\mathtt{C}}_{0}\mathbb{D}^{\alpha,\ell}_{\mathtt{z}}\vartheta(\mathtt{z}) &= \mathtt{g}(\mathtt{z},\vartheta(\mathtt{z}))\ \mathtt{z}\in[0,\mathtt{T}], \alpha\in(0,1),\\
a\vartheta(0)+be^{\ell\mathtt{T}}\vartheta(\mathtt{T}) &= c,
\end{aligned}
$$

and also derived and analyzed a Jacobi spectral-collocation method for the numerical solution. In [37], Yadav et al. discussed the numerical approximation to solve regular tempered fractional Sturm-Liouville problem

$$
\begin{aligned}
{}^{\mathtt{C}}_{\mathtt{z}}\mathbb{D}^{\alpha,\ell}_{b}\left[\mathtt{p}(\mathtt{z})\,{}^{\mathtt{C}}_{a}\mathbb{D}^{\alpha,\ell}_{\mathtt{z}}\vartheta(\mathtt{z})\right] &= \lambda\,\varphi(\mathtt{z})\vartheta(\mathtt{z}))\ \mathtt{z}\in[a,b], \alpha\in(0,1),\\
\vartheta(a)=0,\ \left[\mathtt{p}(\mathtt{z})\,{}^{\mathtt{C}}_{a}\mathbb{D}^{\alpha,\ell}_{\mathtt{z}}\vartheta(\mathtt{z})\right]\Big|_{\mathtt{z}=b} &= 0,
\end{aligned}
$$

by using finite difference method. Recently, Pandey et al. [24] studied the properties of eigenvalue for the regular tempered fractional Sturm-Liouville problem

$$
\begin{aligned}
{}^{\mathtt{C}}_{\mathtt{z}}\mathbb{D}^{\alpha,\ell}_{b}\left[\mathtt{p}(\mathtt{z})\,{}^{\mathtt{C}}_{a}\mathbb{D}^{\alpha,\ell}_{\mathtt{z}}\vartheta(\mathtt{z})\right]+\varphi(\mathtt{z})\vartheta(\mathtt{z}) &= \lambda\,\psi_{\lambda}(\mathtt{z})\vartheta(\mathtt{z})\ \mathtt{z}\in[a,b], \alpha\in(0,1),\\
\vartheta(a)=0,\ \vartheta(b) &= 0,
\end{aligned}
$$

by using a fractional variational approach.

In functional analysis, the measure of noncompactness plays an important role which was introduced by Kuratowski [15]. Recently, Srivastava et al. [34] studied the solvability of nonlinear functional integral equations of two variables by using the measure of noncompactness on $C([0,a]\times[0,a])$ and a fixed point theorem. Moreover, the idea of measure of noncompactness has been used by many researchers in obtaining the existence of solutions of infinite systems of integral and differential equations, see the monograph of [6]. In [21], Mursaleen and Muhiuddine established existence theorems for the infinite systems of differential equations in the space ℓ_p. Alotaibi et al. [2] discussed existence theorems for the infinite systems of linear equations in ℓ_1 and ℓ_p. In [33], Srivastava et al. studied the existence of solutions of infinite systems of n^{th} order differential equations in the spaces c_0 and ℓ_1 via the measure of noncompactness. On the other hand, Mursaleen et al. [20] considered the following infinite systems of three-point fractional order boundary value problems in the spaces c_0 and ℓ_p,

$$
\begin{aligned}
{}^{\mathtt{RL}}\mathbb{D}^{\alpha}_{0+}(\vartheta_{\mathtt{j}}(\mathtt{z}))+\mathtt{f}_{\mathtt{j}}(\mathtt{s},\vartheta_{\mathtt{j}}(\mathtt{z})),&\quad 0<\mathtt{z}<\mathtt{T},\ 1<\alpha<2,\\
\vartheta_{\mathtt{j}}(0)=0,\ \vartheta_{\mathtt{j}}(\mathtt{T})=a\vartheta_{\mathtt{j}}(\xi),&\quad a\xi^{\alpha-1}<\mathtt{T}^{\alpha-1},\quad \mathtt{j}\in\mathbb{N},
\end{aligned}
$$

and derived necessary conditions for the existence of solutions for the infinite system. Recently, Das et al. [9] studied the infinite system of fractional order two-point boundary value problem

$$
\begin{aligned}
{}^{\mathtt{RL}}\mathbb{D}^{\alpha}_{0+}(\vartheta_{\mathtt{j}}(\mathtt{z}))+\mathtt{f}_{\mathtt{j}}(\mathtt{z},\vartheta_{\mathtt{j}}(\mathtt{z})),\ 0<\mathtt{z}<\mathtt{T},\ 1<\alpha<2,\\
\vartheta_{\mathtt{j}}(0)=0,\ \vartheta_{\mathtt{j}}(\mathtt{T})=0,\ \mathtt{j}\in\mathbb{N},
\end{aligned}
$$

via Hausdorff measure of noncompactness in the tempered sequence spaces and established existence of solutions for the infinite system. Inspired by the aforementioned studies, in this paper, we study the following infinite system of nonlinear two-point tempered fractional order boundary value problems

$$\begin{cases} {}^{\mathrm{RL}}_{0}\mathbb{D}^{\delta_2,\ell}_{\mathrm{z}}\Big[\mathrm{p}_{\mathrm{j}}(\mathrm{z}){}^{\mathrm{RL}}_{0}\mathbb{D}^{\delta_1,\ell}_{\mathrm{z}}\vartheta_{\mathrm{j}}(\mathrm{z})\Big] = \lambda_{\mathrm{j}}\varphi\big(\mathrm{z},\vartheta(\mathrm{z})\big),\ \mathrm{z}\in[0,\mathrm{T}],\delta_1,\delta_2\in(1,2), \\ \vartheta_{\mathrm{j}}(0) = \lim\limits_{\mathrm{z}\to 0}\left[{}^{\mathrm{RL}}_{0}\mathbb{D}^{\delta_1,\ell}_{\mathrm{z}}(e^{\ell\mathrm{z}}\vartheta_{\mathrm{j}}(\mathrm{z}))\right] = 0, \\ e^{\ell\mathrm{T}}\vartheta_{\mathrm{j}}(\mathrm{T}) = \lim\limits_{\mathrm{z}\to\mathrm{T}}\left[{}^{\mathrm{RL}}_{0}\mathbb{D}^{\delta_1,\ell}_{\mathrm{z}}(e^{\ell\mathrm{z}}\vartheta_{\mathrm{j}}(\mathrm{z}))\right] = 0, \end{cases} \tag{1.1}$$

where $\mathrm{j}\in\{1,2,3,\cdots\}$, $\ell\geq 0$, ${}^{\mathrm{RL}}_{0}\mathbb{D}^{*,\ell}_{\mathrm{z}}$ denotes the Riemann–Liouville tempered fractional derivative of order $\star\in\{\delta_1,\delta_2\}$, $\vartheta(\mathrm{z}) = \big(\vartheta_{\mathrm{j}}(\mathrm{z})\big)_{\mathrm{j}=1}^{\infty}$, $\varphi_{\mathrm{j}}:[0,\mathrm{T}]\to[0,\mathrm{T}]$ are continuous and we establish necessary conditions for the existence of solutions for the system via the concept of Hausdorff measure of noncompactness and Meir–Keeler fixed point theorem in a tempered sequence space.

2. Preliminaries

In this section, we first give the definitions and some properties of the tempered fractional calculus. Denote $L([a,b])$ as the integrable space which includes the Lebesgue measurable functions on the finite interval $[a,b]$. Let $AC[a,b]$ be the space of real values functions $\vartheta(\mathrm{z})$ which are absolutely continuous on $[a,b]$. For $n\in\mathbb{N}^+$, we denote $AC^n[a,b]$ as the space of real values functions $\vartheta(\mathrm{z})$ which have continuous derivatives up to order $n-1$ on $[a,b]$ such that $\frac{d^{n-1}\vartheta(\mathrm{z})}{d\mathrm{z}^{n-1}}\in AC[a,b]$.

Definition 2.1 ([16, 30]) *Suppose that the real function $\vartheta(\mathrm{z})$ is piecewise continuous on (a,b) and $\vartheta(\mathrm{z})\in L([a,b])$, $\theta>0$, $\ell\geq 0$. The Riemann-Liouville tempered fractional integral of order θ is defined as*

$${}^{\mathrm{RL}}_{a}\mathbb{I}^{\theta,\ell}_{\mathrm{z}}\vartheta(\mathrm{z}) = e^{-\ell\mathrm{z}}\,{}^{\mathrm{RL}}_{a}\mathbb{I}^{\theta}_{\mathrm{z}}\big(e^{\ell\mathrm{z}}\vartheta(\mathrm{z})\big) = \frac{1}{\Gamma(\theta)}\int_a^{\mathrm{z}} e^{-\ell(\mathrm{z}-\mathrm{y})}(\mathrm{z}-\mathrm{y})^{\theta-1}\vartheta(\mathrm{y})d\mathrm{y},$$

where ${}^{\mathrm{RL}}_{a}\mathbb{I}^{\theta}_{\mathrm{z}}$ denotes the Riemann–Liouville fractional integral [14]

$${}^{\mathrm{RL}}_{a}\mathbb{I}^{\theta}_{\mathrm{z}}\vartheta(\mathrm{z}) = \frac{1}{\Gamma(\theta)}\int_a^{\mathrm{z}}(\mathrm{z}-\mathrm{y})^{\theta-1}\vartheta(\mathrm{y})d\mathrm{y}.$$

Definition 2.2 ([16, 30]) *For $n-1<\theta<n$, $n\in\mathbb{N}^+$, $\ell\geq 0$. The Riemann–Liouville tempered fractional derivative of order θ is defined as*

$${}^{\mathrm{RL}}_{a}\mathbb{D}^{\theta,\ell}_{\mathrm{z}}\vartheta(\mathrm{z}) = e^{-\ell\mathrm{z}}\,{}^{\mathrm{RL}}_{a}\mathbb{D}^{\theta}_{\mathrm{z}}\big(e^{\ell\mathrm{z}}\vartheta(\mathrm{z})\big) = \frac{e^{-\ell\mathrm{z}}}{\Gamma(n-\theta)}\frac{d^n}{d\mathrm{y}^n}\int_a^{\mathrm{z}}\frac{e^{\ell\mathrm{y}}\vartheta(\mathrm{y})}{(\mathrm{z}-\mathrm{y})^{\theta-n+1}}d\mathrm{y},$$

where ${}^{\mathrm{RL}}_{a}\mathbb{D}^{\theta}_{\mathrm{z}}$ denotes the Riemann–Liouville fractional derivative [14]

$${}^{\mathrm{RL}}_{a}\mathbb{D}^{\theta}_{\mathrm{z}}\vartheta(\mathrm{z}) = \frac{1}{\Gamma(n-\theta)}\frac{d^n}{d\mathrm{y}^n}\int_a^{\mathrm{z}}\frac{\vartheta(\mathrm{y})}{(\mathrm{z}-\mathrm{y})^{\theta-n+1}}d\mathrm{y}.$$

Lemma 2.3 *(Composite property[16]) Let $\vartheta(\mathrm{z})\in AC^n[a,b]$ and $n-1<\theta<n$. Then the Riemann–Liouville tempered fractional derivative and fractional integral have the composite property:*

$${}^{\mathrm{RL}}_{a}\mathbb{I}^{\theta,\ell}_{\mathrm{z}}\left[{}^{\mathrm{RL}}_{a}\mathbb{D}^{\theta,\ell}_{\mathrm{z}}\vartheta(\mathrm{z})\right] = \vartheta(\mathrm{z}) - \sum_{k=0}^{n-1}\frac{e^{-\ell\mathrm{z}}(\mathrm{z}-a)^{\theta-k-1}}{\Gamma(\theta-k)}\left[{}^{\mathrm{RL}}_{a}\mathbb{D}^{\theta-k-1}_{\mathrm{z}}\big(e^{\ell\mathrm{z}}\vartheta(\mathrm{z})\big)\Big|_{\mathrm{z}=a}\right] \tag{2.1}$$

and

$$ {}^{\mathtt{RL}}_{a}\mathbb{D}^{\theta,\ell}_{\mathtt{z}}\left[{}^{\mathtt{RL}}_{a}\mathbb{I}^{\theta,\ell}_{\mathtt{z}}\vartheta(\mathtt{z})\right]=\vartheta(\mathtt{z}). \tag{2.2}$$

Remark 2.4 *([3]) For $\eta>-1,$ we have*

$$ {}^{\mathtt{RL}}_{a}\mathbb{D}^{\theta}_{\mathtt{z}}\,\mathtt{z}^{\eta}=\frac{\Gamma(\eta+1)}{\Gamma(\eta-\theta+1)}\,\mathtt{z}^{\eta-\theta}, $$

giving inparticular ${}^{\mathtt{RL}}_{a}\mathbb{D}^{\theta}_{\mathtt{z}}\,\mathtt{z}^{\theta-m}=0,\ m=1,2,\cdots,N,$ *where N is the smallest integer greater than or equal to* θ.

In order to study BVP (1.1), we first consider the following linear boundary value problem,

$$ {}^{\mathtt{RL}}_{0}\mathbb{D}^{\delta_1,\ell}_{\mathtt{z}}\vartheta_{\mathtt{j}}(\mathtt{z})+\mathtt{f}(\mathtt{z})=0,\ \mathtt{z}\in[0,\mathtt{T}],\delta_1\in(1,2), \tag{2.3}$$

$$ \vartheta_{\mathtt{j}}(0)=e^{\ell\mathtt{T}}\vartheta_{\mathtt{j}}(\mathtt{T})=0, \tag{2.4}$$

where $\mathtt{f}\in C[0,\mathtt{T}]$ is a given function.

Lemma 2.5 *For every $\mathtt{f}\in C[0,\mathtt{T}]$, the linear boundary value problem* (2.3)–(2.4) *has a unique solution*

$$ \vartheta_{\mathtt{j}}(\mathtt{z})=\int_0^{\mathtt{T}}\aleph_{\delta_1}(\mathtt{z},\mathtt{y})e^{-\ell(\mathtt{z}-\mathtt{y})}\mathtt{f}(\mathtt{y})d\mathtt{y}, \tag{2.5}$$

where

$$ \aleph_{\delta_1}(\mathtt{z},\mathtt{y})=\frac{1}{\Gamma(\delta_1)}\begin{cases}\dfrac{\mathtt{z}^{\delta_1-1}(\mathtt{T}-\mathtt{y})^{\delta_1-1}}{\mathtt{T}^{\delta_1-1}}-(\mathtt{z}-\mathtt{y})^{\delta_1-1}, & \mathtt{y}\leq\mathtt{z},\\ \dfrac{\mathtt{z}^{\delta_1-1}(\mathtt{T}-\mathtt{y})^{\delta_1-1}}{\mathtt{T}^{\delta_1-1}}, & \mathtt{z}\leq\mathtt{y}.\end{cases} $$

Proof Applying the Rieman–Liouville tempered fractional integral operator ${}^{\mathtt{RL}}_{0}\mathbb{I}^{\delta_1,\ell}_{\mathtt{z}}$ on both sides of the first equation of (2.3) and using composite property (2.1), we get

$$ \vartheta_{\mathtt{j}}(\mathtt{z})=c_0e^{-\ell\mathtt{z}}\mathtt{z}^{\delta_1-1}+c_1e^{-\ell\mathtt{z}}\mathtt{z}^{\delta_1-2}-\frac{1}{\Gamma(\delta_1)}\int_0^{\mathtt{z}}e^{-\ell(\mathtt{z}-\mathtt{y})}(\mathtt{z}-\mathtt{y})^{\delta_1-1}\mathtt{f}(\mathtt{y})d\mathtt{y}, \tag{2.6}$$

where $c_0=\frac{1}{\Gamma(\delta_1)}{}^{\mathtt{RL}}_{0}\mathbb{D}^{\theta-1}_{\mathtt{z}}\big(e^{\ell\mathtt{z}}\vartheta(\mathtt{z})\big)\Big|_{\mathtt{z}=0}$ and $c_1=\frac{1}{\Gamma(\delta_1-1)}{}^{\mathtt{RL}}_{0}\mathbb{D}^{\theta-2}_{\mathtt{z}}\big(e^{\ell\mathtt{z}}\vartheta(\mathtt{z})\big)\Big|_{\mathtt{z}=0}$. Using boundary conditions, we get $c_1=0$ and

$$ c_0=\frac{1}{\mathtt{T}^{\delta_1-1}\Gamma(\delta_1)}\int_0^{\mathtt{T}}e^{\ell\mathtt{y}}(\mathtt{T}-\mathtt{y})^{\delta_1-1}\mathtt{f}(\mathtt{y})d\mathtt{y}. $$

Plugging c_0 and c_1 into (2.5), we obtain

$$ \begin{aligned}\vartheta_{\mathtt{j}}(\mathtt{z})&=\frac{1}{\Gamma(\delta_1)}\int_0^{\mathtt{T}}e^{-\ell(\mathtt{z}-\mathtt{y})}\frac{\mathtt{z}^{\delta_1-1}(\mathtt{T}-\mathtt{y})^{\delta_1-1}}{\mathtt{T}^{\delta_1-1}}\mathtt{f}(\mathtt{y})d\mathtt{y}\\ &\qquad-\frac{1}{\Gamma(\delta_1)}\int_0^{\mathtt{z}}e^{-\ell(\mathtt{z}-\mathtt{y})}(\mathtt{z}-\mathtt{y})^{\delta_1-1}\mathtt{f}(\mathtt{y})d\mathtt{y} \qquad (2.7)\\ &=\int_0^{\mathtt{T}}\aleph_{\delta_1}(\mathtt{z},\mathtt{y})e^{-\ell(\mathtt{z}-\mathtt{y})}\mathtt{f}(\mathtt{y})d\mathtt{y}\end{aligned} $$

Next we show that (2.5) is the solution of (1.1). Applying the operator ${}^{\mathtt{RL}}_{0}\mathbb{D}^{\delta_1,\ell}_{\mathtt{z}}$ to both sides of (2.7) and using composite property (2.2), we get

$$\begin{aligned}{}^{\mathtt{RL}}_{0}\mathbb{D}^{\delta_1,\ell}_{\mathtt{z}}\vartheta_{\mathtt{j}}(\mathtt{z}) &= \frac{1}{\Gamma(\delta_1)}\int_0^{\mathtt{T}} e^{-\ell\mathtt{y}}\ {}^{\mathtt{RL}}_{0}\mathbb{D}^{\delta_1,\ell}_{\mathtt{z}}\left(e^{-\ell\mathtt{z}}\mathtt{z}^{\delta_1-1}\right)\frac{(\mathtt{T}-\mathtt{y})^{\delta_1-1}}{\mathtt{T}^{\delta_1-1}}\mathtt{f}(\mathtt{y})d\mathtt{y} - {}^{\mathtt{RL}}_{0}\mathbb{D}^{\delta_1,\ell}_{\mathtt{z}}\left({}^{\mathtt{RL}}_{0}\mathbb{I}^{\delta_1,\ell}_{\mathtt{z}}\mathtt{f}(\mathtt{z})\right)\\ &= -\mathtt{f}(\mathtt{z}),\end{aligned}$$

where we use the fact

$${}^{\mathtt{RL}}_{0}\mathbb{D}^{\delta_1,\ell}_{\mathtt{z}}\left(e^{-\ell\mathtt{z}}\mathtt{z}^{\delta_1-1}\right) = e^{-\ell\mathtt{z}}\ {}^{\mathtt{RL}}_{0}\mathbb{D}^{\delta_1}_{\mathtt{z}}\left(\mathtt{z}^{\delta_1-1}\right) = 0$$

by Remark 2.4. Now from (2.7), it is clear that $\vartheta_{\mathtt{j}}(0) = e^{\ell\mathtt{T}}\vartheta_{\mathtt{j}}(\mathtt{T}) = 0$. This completes the proof. □

Lemma 2.6 *The kernel* $\aleph_{\delta_1}(\mathtt{z},\mathtt{y})$ *has the following properties:*

(*i*) $\aleph_{\delta_1}(\mathtt{z},\mathtt{y})$ *is nonnegative and continuous on* $[0,\mathtt{T}]\times[0,\mathtt{T}]$.

(*ii*) $\aleph_{\delta_1}(\mathtt{z},\mathtt{y}) \le \aleph_{\delta_1}(\mathtt{y},\mathtt{y}) \le \dfrac{\mathtt{T}^{\delta_1-1}}{\Gamma(\delta_1)}$ *for* $\mathtt{z},\mathtt{y}\in[0,\mathtt{T}]\times[0,\mathtt{T}]$.

(*iii*) $\displaystyle\max_{\mathtt{z}\in[0,\mathtt{T}]}\int_0^{\mathtt{T}}\aleph_{\delta_1}(\mathtt{z},\mathtt{y})d\mathtt{y} = \frac{\mathtt{T}^{\delta_1}}{\Gamma(\delta_1)}$.

Proof It is clear from the definition of $\aleph_{\delta_1}(\mathtt{z},\mathtt{y})$ that $\aleph_{\delta_1}(\mathtt{z},\mathtt{y})$ is continuous on $[0,\mathtt{T}]\times[0,\mathtt{T}]$. For $0\le\mathtt{y}\le\mathtt{z}\le\mathtt{T}$, we have

$$\begin{aligned}\aleph_{\delta_1}(\mathtt{z},\mathtt{y}) &= \frac{1}{\Gamma(\delta_1)}\left[\frac{\mathtt{z}^{\delta_1-1}(\mathtt{T}-\mathtt{y})^{\delta_1-1}}{\mathtt{T}^{\delta_1-1}} - (\mathtt{z}-\mathtt{y})^{\delta_1-1}\right]\\ &= \frac{1}{\Gamma(\delta_1)}\left[\mathtt{z}^{\delta_1-1}\left(1-\frac{\mathtt{y}}{\mathtt{T}}\right)^{\delta_1-1} - \mathtt{z}^{\delta_1-1}\left(1-\frac{\mathtt{y}}{\mathtt{z}}\right)^{\delta_1-1}\right]\\ &\ge \frac{\mathtt{z}^{\delta_1-1}}{\Gamma(\delta_1)}\left[\left(1-\frac{\mathtt{y}}{\mathtt{T}}\right)^{\delta_1-1} - \left(1-\frac{\mathtt{y}}{\mathtt{z}}\right)^{\delta_1-1}\right] \ge 0.\end{aligned}$$

For $0\le\mathtt{z}\le\mathtt{y}\le\mathtt{T}$, it is obvious that $\aleph_{\delta_1}(\mathtt{z},\mathtt{y})\ge 0$. Thus, $\aleph_{\delta_1}(\mathtt{z},\mathtt{y})\ge 0$ for all $\mathtt{z},\mathtt{y}\in[0,\mathtt{T}]$. Since, for $0\le\mathtt{y}\le\mathtt{z}\le\mathtt{T}$,

$$\begin{aligned}\frac{\partial\aleph_{\delta_1}(\mathtt{z},\mathtt{y})}{\partial\mathtt{z}} &= \frac{1}{\Gamma(\delta_1)}\left[\frac{(\delta_1-1)\mathtt{z}^{\delta_1-2}(\mathtt{T}-\mathtt{y})^{\delta_1-1}}{\mathtt{T}^{\delta_1-1}} - (\delta_1-1)(\mathtt{z}-\mathtt{y})^{\delta_1-2}\right]\\ &= \frac{\mathtt{z}^{\delta_1-1}}{\Gamma(\delta_1-1)}\left[\left(1-\frac{\mathtt{y}}{\mathtt{T}}\right)^{\delta_1-2} - \left(1-\frac{\mathtt{y}}{\mathtt{z}}\right)^{\delta_1-2}\right]\\ &\le 0,\ 1<\delta_1\le 2,\end{aligned}$$

$\aleph_{\delta_1}(\mathtt{z},\mathtt{y})$ is nonincreasing with respect to $\mathtt{z}$ on $[\mathtt{y},\mathtt{T}]$. It follows that

$$\aleph_{\delta_1}(\mathtt{z},\mathtt{y}) \le \aleph_{\delta_1}(\mathtt{y},\mathtt{y}) = \frac{\mathtt{z}^{\delta_1-1}(\mathtt{T}-\mathtt{y})^{\delta_1-1}}{\mathtt{T}^{\delta_1-1}\Gamma(\delta_1)} \le \frac{\mathtt{T}^{\delta_1-1}}{\Gamma(\delta_1)}.$$

Since, for $0\le\mathtt{z}\le\mathtt{y}\le\mathtt{T}$,

$$\frac{\partial\aleph_{\delta_1}(\mathtt{z},\mathtt{y})}{\partial\mathtt{z}} = \frac{1}{\Gamma(\delta_1)}\left[\frac{(\delta_1-1)\mathtt{z}^{\delta_1-2}(\mathtt{T}-\mathtt{y})^{\delta_1-1}}{\mathtt{T}^{\delta_1-1}}\right] \ge 0,$$

$\aleph_{\delta_1}(\mathtt{z},\mathtt{y})$ is nondecreasing with respect to $\mathtt{z}$ on $[0,\mathtt{y}]$. It follows that

$$\aleph_{\delta_1}(\mathtt{z},\mathtt{y}) \le \aleph_{\delta_1}(\mathtt{y},\mathtt{y}) = \frac{\mathtt{z}^{\delta_1-1}(\mathtt{T}-\mathtt{y})^{\delta_1-1}}{\mathtt{T}^{\delta_1-1}\Gamma(\delta_1)} \le \frac{\mathtt{T}^{\delta_1-1}}{\Gamma(\delta_1)}.$$

The assertion (iii) is evident. □

Lemma 2.7 *Let* $\mathtt{g} \in \mathtt{C}[0,\mathtt{T}]$. *Then the boundary value problem*

$$ {}^{\mathtt{RL}}_{0}\mathbb{D}^{\delta_2,\ell}_{\mathtt{z}}\left[\mathtt{p_j}(\mathtt{z})\,{}^{\mathtt{RL}}_{0}\mathbb{D}^{\delta_1,\ell}_{\mathtt{z}}\vartheta_{\mathtt{j}}(\mathtt{z})\right] = \mathtt{g}(\mathtt{z}),\ \mathtt{z}\in[0,\mathtt{T}], \delta_1,\delta_2\in(1,2), \tag{2.8}$$

$$\begin{cases} \vartheta_{\mathtt{j}}(0) = \lim\limits_{\mathtt{z}\to 0}\left[{}^{\mathtt{RL}}_{0}\mathbb{D}^{\delta_1,\ell}_{\mathtt{z}}(e^{\ell\mathtt{z}}\vartheta_{\mathtt{j}}(\mathtt{z}))\right] = 0,\\ e^{\ell\mathtt{T}}\vartheta_{\mathtt{j}}(\mathtt{T}) = \lim\limits_{\mathtt{z}\to \mathtt{T}}\left[{}^{\mathtt{RL}}_{0}\mathbb{D}^{\delta_1,\ell}_{\mathtt{z}}(e^{\ell\mathtt{z}}\vartheta_{\mathtt{j}}(\mathtt{z}))\right] = 0, \end{cases} \tag{2.9}$$

has a unique solution

$$\vartheta_{\mathtt{j}}(\mathtt{z}) = \int_0^{\mathtt{T}} \aleph_{\delta_1}(\mathtt{z},\mathtt{y})e^{-\ell\mathtt{z}}\left[\int_0^{\mathtt{T}}\frac{\aleph_{\delta_2}(\mathtt{y},\mathtt{x})e^{\ell\mathtt{x}}}{\mathtt{p_j}(\mathtt{x})}\mathtt{g}(\mathtt{x})d\mathtt{x}\right]d\mathtt{y}, \tag{2.10}$$

where $\aleph_{\delta_1}(\mathtt{z},\mathtt{y})$ *is defined in Lemma 2.5 and*

$$\aleph_{\delta_2}(\mathtt{y},\mathtt{x}) = \frac{1}{\Gamma(\delta_2)}\begin{cases} \dfrac{\mathtt{y}^{\delta_2-1}(\mathtt{T}-\mathtt{x})^{\delta_2-1}}{\mathtt{T}^{\delta_2-1}} - (\mathtt{y}-\mathtt{x})^{\delta_1-1}, & \mathtt{x}\le\mathtt{y},\\ \dfrac{\mathtt{y}^{\delta_2-1}(\mathtt{T}-\mathtt{x})^{\delta_1-1}}{\mathtt{T}^{\delta_1-1}}, & \mathtt{y}\le\mathtt{x}. \end{cases}$$

Proof Let $\vartheta_{\mathtt{j}}(\mathtt{z}) = -{}^{\mathtt{RL}}_{0}\mathbb{D}^{\delta_1,\ell}_{\mathtt{z}}\vartheta_{\mathtt{j}}(\mathtt{z})$ for $\mathtt{z}\in[0,\mathtt{T}]$. Then the boundary value problem

$$ {}^{\mathtt{RL}}_{0}\mathbb{D}^{\delta_2,\ell}_{\mathtt{z}}\left[\mathtt{p_j}(\mathtt{z})\,{}^{\mathtt{RL}}_{0}\mathbb{D}^{\delta_1,\ell}_{\mathtt{z}}\vartheta_{\mathtt{j}}(\mathtt{z})\right] = \mathtt{g}(\mathtt{z}),\ \mathtt{z}\in[0,\mathtt{T}],\ \delta_1,\delta_2\in(1,2)$$
$$\vartheta_{\mathtt{j}}(0) = \lim_{\mathtt{z}\to 0}\left[{}^{\mathtt{RL}}_{0}\mathbb{D}^{\delta_1,\ell}_{\mathtt{z}}(e^{\ell\mathtt{z}}\vartheta_{\mathtt{j}}(\mathtt{z}))\right] = 0,$$
$$e^{\ell\mathtt{T}}\vartheta_{\mathtt{j}}(\mathtt{T}) = \lim_{\mathtt{z}\to \mathtt{T}}\left[{}^{\mathtt{RL}}_{0}\mathbb{D}^{\delta_1,\ell}_{\mathtt{z}}(e^{\ell\mathtt{z}}\vartheta_{\mathtt{j}}(\mathtt{z}))\right] = 0,$$

is equivalent to the problem

$$\begin{cases} {}^{\mathtt{RL}}_{0}\mathbb{D}^{\delta_2,\ell}_{\mathtt{z}}\vartheta_{\mathtt{j}}(\mathtt{z}) + \dfrac{\mathtt{g}(\mathtt{z})}{\mathtt{p_j}(\mathtt{z})} = 0,\ \mathtt{p_j}(\mathtt{z})\neq 0,\ \mathtt{z}\in[0,\mathtt{T}], \delta_1\in(1,2),\\ \vartheta_{\mathtt{j}}(0) = e^{\ell\mathtt{T}}\vartheta_{\mathtt{j}}(\mathtt{T}) = 0, \end{cases} \tag{2.11}$$

By Lemma 2.5, the boundary value problem (2.11) has unique solution

$$\vartheta_{\mathtt{j}}(\mathtt{z}) = \int_0^{\mathtt{T}} \aleph_{\delta_2}(\mathtt{z},\mathtt{y})e^{-\ell(\mathtt{z}-\mathtt{y})}\frac{\mathtt{g}(\mathtt{y})}{\mathtt{p_j}(\mathtt{y})}d\mathtt{y}.$$

That is

$$ {}^{\mathtt{RL}}_{0}\mathbb{D}^{\delta_1,\ell}_{\mathtt{z}}\vartheta_{\mathtt{j}}(\mathtt{z}) + \int_0^{\mathtt{T}}\frac{\aleph_{\delta_2}(\mathtt{z},\mathtt{y})e^{-\ell(\mathtt{z}-\mathtt{y})}}{\mathtt{p_j}(\mathtt{y})}\mathtt{g}(\mathtt{y})d\mathtt{y} = 0. \tag{2.12}$$

Again by Lemma 2.5, the differential equation (2.12) with boundary conditions

$$\vartheta_{\mathtt{j}}(0) = 0 \quad \text{and} \quad e^{\ell \mathtt{T}}\vartheta_{\mathtt{j}}(\mathtt{T}) = 0$$

has a unique solution

$$\begin{aligned}\vartheta_{\mathtt{j}}(\mathtt{z}) &= \int_0^{\mathtt{T}} \aleph_{\delta_1}(\mathtt{z},\mathtt{y})e^{-\ell(\mathtt{z}-\mathtt{y})}\left[\int_0^{\mathtt{T}} \frac{\aleph_{\delta_2}(\mathtt{y},\mathtt{x})e^{-\ell(\mathtt{y}-\mathtt{x})}}{\mathtt{p}_{\mathtt{j}}(\mathtt{x})}\mathtt{g}(\mathtt{x})d\mathtt{x}\right]d\mathtt{y}\\ &= \int_0^{\mathtt{T}} \aleph_{\delta_1}(\mathtt{z},\mathtt{y})e^{-\ell \mathtt{z}}\left[\int_0^{\mathtt{T}} \frac{\aleph_{\delta_2}(\mathtt{y},\mathtt{x})e^{\ell \mathtt{x}}}{\mathtt{p}_{\mathtt{j}}(\mathtt{x})}\mathtt{g}(\mathtt{x})d\mathtt{x}\right]d\mathtt{y}.\end{aligned}$$

This completes the proof. □

From the above Lemma 2.7, it can be seen that solution of the BVP (1.1) is the solution of the follwoing integral equation

$$\vartheta_{\mathtt{j}}(\mathtt{z}) = \lambda_{\mathtt{j}}\int_0^{\mathtt{T}} \aleph_{\delta_1}(\mathtt{z},\mathtt{y})e^{-\ell \mathtt{z}}\left[\int_0^{\mathtt{T}} \frac{\aleph_{\delta_2}(\mathtt{y},\mathtt{x})e^{\ell \mathtt{x}}}{\mathtt{p}_{\mathtt{j}}(\mathtt{x})}\varphi_{\mathtt{j}}(\mathtt{x},\vartheta(\mathtt{x}))d\mathtt{x}\right]d\mathtt{y} \tag{2.13}$$

and vice versa.

Lemma 2.8 *The kernel* $\aleph_{\delta_2}(\mathtt{z},\mathtt{x})$ *has the following properties:*

(i) $\aleph_{\delta_2}(\mathtt{y},\mathtt{x})$ *is nonnegative and continuous on* $[0,\mathtt{T}]\times[0,\mathtt{T}]$.

(ii) $\aleph_{\delta_2}(\mathtt{y},\mathtt{x}) \le \aleph_{\delta_2}(\mathtt{x},\mathtt{x}) \le \dfrac{\mathtt{T}^{\delta_2-1}}{\Gamma(\delta_2)}$ *for* $\mathtt{y},\mathtt{x}\in[0,\mathtt{T}]\times[0,\mathtt{T}]$.

(iii) $\displaystyle\max_{\mathtt{y}\in[0,\mathtt{T}]}\int_0^{\mathtt{T}} \aleph_{\delta_2}(\mathtt{y},\mathtt{x})d\mathtt{x} = \frac{\mathtt{T}^{\delta_2}}{\Gamma(\delta_2)}$.

Proof The proof follows from Lemma 2.6. □

Definition 2.9 ([15]) *Let* $(\mathtt{Y},\mathrm{d})$ *be a metric space and* $\mathtt{X}\subset\mathtt{Y}$. *Then the Kurtowski measure of noncompactness of* $\mathtt{X}$ *is denoted by* $\mathcal{K}(\mathtt{X})$, *if defined as*

$$\mathcal{K}(\mathtt{X}) = \inf\left\{\varepsilon>0 : \mathtt{X}\subset\bigcup_{\mathtt{j}=1}^{n}\mathtt{E}_{\mathtt{j}},\ \mathtt{E}_{\mathtt{j}}\subset\mathtt{Y},\ diam(\mathtt{E}_{\mathtt{j}})<\varepsilon,\ n\in\mathbb{N}\right\},$$

here $\mathcal{K}$ *is called Kuratowski measure of noncompactness. It is evident that*

$$\mathcal{K}(\mathtt{X}) \le diam(\mathtt{X}) \quad \forall \quad \mathtt{X}\subset\mathtt{Y}.$$

Let $\mathtt{B}$ *be a real Banach space with the norm* $\|\cdot\|$ *and* $\mathcal{A}(a,a_0)$ *be a closed ball in* $\mathtt{B}$ *centered at* a *and radius* a_0. *If* $\mathtt{F}$ *is a nonempty subset of* $\mathtt{B}$ *then* $\overline{\mathtt{F}}$ *and* $conv(\mathtt{F})$ *represents closure and convex closure of* $\mathtt{F}$. *Furthermore, let* $\mathscr{M}_{\mathtt{B}}$ *denote the family of all nonempty and bounded subsets of* $\mathtt{B}$ *and* $\mathscr{N}_{\mathtt{B}}$ *its subfamily consisting of all relatively compact sets.*

Definition 2.10 ([4]) *A function* $\mu : \mathscr{M}_{\mathtt{B}} \to [0,1)$ *is called a measure of noncompactness, iff*

($\mathtt{C}_1$) *the family* $ker\mu = \{\mathtt{F} \in \mathscr{M}_{\mathtt{B}} : \mu(\mathtt{F}) = 0\}$ *is nonempty and* $ker\mu \subset \mathscr{N}_{\mathtt{B}}$.

($\mathtt{C}_2$) $\mathtt{F} \subset \mathtt{E}$ *then* $\mu(\mathtt{F}) \leq \mu(\mathtt{E})$.

($\mathtt{C}_3$) $\mu(\overline{\mathtt{F}}) = \mu(\mathtt{F})$.

($\mathtt{C}_4$) $\mu(conv\mathtt{F}) = \mu(\mathtt{F})$.

($\mathtt{C}_5$) $\mu(\lambda\mathtt{E} + (1-\lambda)\mathtt{F}) \leq \lambda\mu(\mathtt{E}) + (1-\lambda)\mu(\mathtt{F})$ *for* $\lambda \in [0,1]$.

($\mathtt{C}_6$) *if* $\mathtt{E}_n \in \mathscr{M}_{\mathtt{B}}$, $X_n = \overline{\mathtt{E}}_n$, $\mathtt{E}_{n+1} \subset \mathtt{E}_n$ *for* $n = 1,2,3,\cdots$ *and* $\lim_{n\to\infty}\mu(\mathtt{E}_n) = 0$ *then* $\cap_{n=1}^{\infty}\mathtt{E}_n \neq \emptyset$.

Definition 2.11 ([6]) *Let* $(\mathtt{Y}, \mathrm{d})$ *be a metric space,* $\mathtt{X}$ *be a bounded subset* $\mathtt{Y}$ *and* $\mathcal{A}(a, a_0) = \{b \in \mathtt{Y} : \mathrm{d}(a,b) < a_0\}$. *Then the Hausdorff measure of noncompactness* $\chi(\mathtt{X})$ *of* $\mathtt{X}$ *is defined by*

$$\chi(\mathtt{X}) = \inf\left\{\varepsilon > 0 : \mathtt{X} \subset \bigcup_{\mathtt{j}=1}^{n} \mathcal{A}(a_{\mathtt{j}}, a_{0_{\mathtt{j}}}), a_{\mathtt{j}} \in \mathtt{Y}, a_{0_{\mathtt{j}}} < \varepsilon, n \in \mathbb{N}\right\}.$$

Next, let $\mathtt{c}_0$ and $\mathtt{c}$ be Banach spaces with sup norms, which are defined as

$$\mathtt{c}_0 = \left\{a \in \omega : \lim_{\mathtt{j}\to\infty} a_{\mathtt{j}} = 0, \quad \|a\|_{\mathtt{c}_0} = \sup_{\mathtt{j}} |a_{\mathtt{j}}|\right\},$$

$$\mathtt{c} = \left\{a \in \omega : \lim_{\mathtt{j}\to\infty} a_{\mathtt{j}} = l, \quad l \in \mathbb{C}, \quad \|a\|_{\mathtt{c}} = \sup_{\mathtt{j}} |a_{\mathtt{j}}|\right\},$$

respectively.

Definition 2.12 ([6]) *The Hausdorff measure of noncompactness* χ *on the Banach space* $(\mathtt{c}_0, \|\cdot\|_{\mathtt{c}_0)}$ *is defined by*

$$\chi(\mathcal{A}) = \lim_{\mathtt{j}\to\infty}\left\{\sup_{a(\mathtt{z})\in\mathcal{A}}\left[\max_{m\geq\mathtt{j}}|a_m|\right]\right\}, \quad \mathcal{A} \in \mathscr{M}_{\mathtt{c}_0}.$$

Definition 2.13 ([22]) *The (regular)measure of noncompactness* μ *on the Banach space* $(\mathtt{c}, \|\cdot\|_{\mathtt{c})}$ *is defined by*

$$\mu(\mathcal{A}) = \lim_{\mathtt{j}\to\infty}\left\{\sup_{a(\mathtt{z})\in\mathcal{A}}\left[\sup_{m\geq\mathtt{j}}\left|a_m - \lim_{n\to\infty} a_n\right|\right]\right\}, \quad \mathcal{A} \in \mathscr{M}_{\mathtt{c}}.$$

Definition 2.14 ([5]) *Let* $\sigma = (\sigma_{\mathtt{j}})$ *be such that* $\sigma_{\mathtt{j}}$ *is positive and nonincreasing sequence, Then* σ *is called tempering sequence. Let* $\mathcal{G}$ *be set consisting of all real sequences* $a = (a_{\mathtt{j}})_{\mathtt{j}=1}^{\infty}$ *such that* $\sigma_{\mathtt{j}}a_{\mathtt{j}} \to 0$ *as* $\mathtt{j} \to \infty$. *It is clear that* $\mathcal{G}$ *forms a linear space over the field of real numbers. We denote the space by* $\mathtt{c}_0^{\sigma}$. *Moreover, it can be seen that* $\mathtt{c}_0^{\sigma}$ *is a Banach space with the norm*

$$\|a\|_{\mathtt{c}_0^{\sigma}} = \sup\left\{\sigma_{\mathtt{j}}|a_{\mathtt{j}}|\right\}.$$

Similarly, let $\mathcal{S}$ be a set consisting of all real sequences $a = (a_{\mathtt{j}})_{\mathtt{j}=1}^{\infty}$ such that $\sigma_{\mathtt{j}} a_{\mathtt{j}}$ converges to finite limit. It is clear that $\mathcal{S}$ forms a linear space over the field of real numbers. We denote the space by $\mathtt{c}^{\sigma}$. Moreover, it can be seen that $\mathtt{c}^{\sigma}$ is a Banach space with the norm

$$\|a\|_{\mathtt{c}^{\sigma}} = \sup\left\{\sigma_{\mathtt{j}}|a_{\mathtt{j}}|\right\}.$$

Next, consider the function spaces $\mathcal{C}(\mathcal{J}, \mathtt{c}_0^{\sigma})$ and $\mathcal{C}(\mathcal{J}, \mathtt{c}^{\sigma})$, where $\mathcal{J} = [0, \mathtt{T}]$, $\mathtt{T} > 0$ the spaces of all continuous functions on $\mathcal{J}$ with values in $\mathtt{c}_0^{\sigma}$ and the spaces of all continuous functions on $\mathcal{J}$ with values in c^{β} respectively. Then $\mathcal{C}(\mathcal{J}, \mathtt{c}_0^{\sigma})$ and $\mathcal{C}(\mathcal{J}, \mathtt{c}^{\sigma})$ are Banach spaces with respect to the norms,

$$\|a\|_{\mathcal{C}(\mathcal{J}, \mathtt{c}_0^{\sigma})} = \max\left\{\|a(\mathtt{z})\|_{\mathtt{c}_0^{\sigma}} : \mathtt{z} \in \mathcal{J}\right\}, \quad a \in \mathcal{C}(\mathcal{J}, \mathtt{c}_0^{\sigma}),$$

$$\|a\|_{\mathcal{C}(\mathcal{J}, \mathtt{c}^{\sigma})} = \max\left\{\|a(\mathtt{z})\|_{\mathtt{c}^{\sigma}} : \mathtt{z} \in \mathcal{J}\right\}, \quad a \in \mathcal{C}(\mathcal{J}, \mathtt{c}^{\sigma}),$$

respectively (see [9]).

For any nonempty, closed, bounded, and convex subset $\mathtt{Y}$ of $\mathcal{C}(\mathcal{J}, \mathtt{c}_0^{\sigma})$ or $\mathcal{C}(\mathcal{J}, \mathtt{c}^{\sigma})$ and $\mathtt{z} \in \mathcal{J}$, we have

$$\chi_{\mathcal{C}(\mathcal{J}, \mathtt{c}_0^{\sigma})}(\mathtt{Y}) = \sup\left\{\chi_{\mathtt{c}_0^{\sigma}}(\mathtt{Y}(\mathtt{z})) : t \in \mathcal{J}\right\}$$

and

$$\mu_{\mathcal{C}(\mathcal{J}, \mathtt{c}^{\sigma})}(\mathtt{Y}) = \sup\left\{\mu_{\mathtt{c}^{\sigma}}(\mathtt{Y}(\mathtt{z})) : t \in \mathcal{J}\right\}.$$

Note that $\chi_{\mathcal{C}(\mathcal{J}, \mathtt{c}_0^{\sigma})}$ and $\mu_{\mathcal{C}(\mathcal{J}, \mathtt{c}^{\sigma})}$ satisfy all the axioms of measure of noncompactness on $\mathcal{C}(\mathcal{J}, \mathtt{c}_0^{\sigma})$ and $\mathcal{C}(\mathcal{J}, \mathtt{c}^{\sigma})$, respectively (see [9]).

Definition 2.15 ([19]) *Let $(\mathtt{Y}, d)$ be a metric space. Then a mapping $\mathcal{T}$ on $\mathtt{Y}$ is said to be a Meir-Keeler contraction if for any $\varepsilon > 0$, there exists $\mathcal{K} > 0$ such that*

$$\varepsilon \le d(x, y) < \varepsilon + \mathcal{K} \implies d(Tx, Ty) < \varepsilon, \quad \forall x, y \in \mathtt{Y}.$$

Theorem 2.16 ([19]) *Let $(\mathtt{Y}, d)$ be a complete metric space. If $\mathcal{T} : \mathtt{Y} \to \mathtt{Y}$ is a Meir-Keeler contraction, then $\mathcal{T}$ has a unique fixed point.*

Definition 2.17 ([1]) *Let $\mathcal{C}$ be a nonempty subset of a Banach space $\mathtt{E}$ and let μ be an arbitrary measure of noncompactness on $\mathtt{E}$. We say that an operator $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is a Meir-Keeler condensing operator if for any $\varepsilon > 0$, there exists $\mathcal{K} > 0$ such that*

$$\varepsilon \le \mu(\mathtt{Y}) < \varepsilon + \mathcal{K} \implies \mu(\mathcal{T}(\mathtt{Y})) < \varepsilon$$

for any bounded subset $\mathtt{Y}$ of $\mathcal{C}$.

Theorem 2.18 ([1]) *Let $\mathcal{C}$ be a nonempty, bounded, closed, and convex subset of a Banach space $\mathtt{E}$ and let μ be an arbitrary measure of noncompactness on $\mathtt{E}$. If $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is a continuous and Meir-Keeler condensing operator, then $\mathcal{T}$ has at least one fixed point and the set of all fixed points of $\mathcal{T}$ in $\mathcal{C}$ is compact.*

3. Existence of solutions for tempered fractional BVP (1.1) in $\mathcal{C}(\mathcal{J}, \mathtt{c}_0^{\sigma})$

In this section, we investigate the existence of solutions for infinite system of tempered fractional order two-point boundary value problems (1.1) in the sequences spaces $\mathcal{C}(\mathcal{J}, \mathtt{c}_0^{\sigma})$.

Suppose

$(\mathtt{G}_1)$ $\varphi_{\mathtt{j}} : \mathcal{J} \to \mathbb{R}^{\infty}$ and define an operator $\Phi : \mathcal{J} \times \mathtt{c}_0^{\sigma} \to \mathtt{c}_0^{\sigma}$ as

$$(\mathtt{z}, \vartheta(\mathtt{z})) \to (\Phi\vartheta)(\mathtt{z}) = \big(\varphi_{\mathtt{j}}(\mathtt{z}, \vartheta(\mathtt{z}))\big)_{\mathtt{j}=1}^{\infty}$$

is the class of all functions $\big((\Phi\vartheta)(\mathtt{z})\big)_{\mathtt{z}\in\mathcal{J}}$ and equicontinuous on $\mathtt{c}_0^{\sigma}$.

$(\mathtt{G}_2)$ $\xi_{\mathtt{j}}(\mathtt{z}), \zeta_{\mathtt{j}}(\mathtt{z}) : \mathcal{J} \to \mathbb{R}$ are continuous functions such that the sequence $\sigma_{\mathtt{j}}\xi_{\mathtt{j}}(\mathtt{z})$ converges uniformly to zero on $\mathcal{J}$ and the sequence $(\zeta_{\mathtt{j}}(\mathtt{z}))$ is equibounded on $\mathcal{J}$, so we take $\zeta(\mathtt{z}) = \sup\{\zeta_{\mathtt{j}}(\mathtt{z}) : \mathtt{j} \in \mathbb{N}\}$, $\zeta^{\star} = \sup\{\zeta(\mathtt{z}) : \mathtt{z} \in \mathcal{J}\}$, $\sigma^{\star} = \sup\{\sigma_{\mathtt{j}}\xi_{\mathtt{j}}(\mathtt{z}) : \mathtt{j} \in \mathbb{N}, \mathtt{z} \in \mathcal{J}\}$ and

$$\big|\varphi_{\mathtt{j}}(\mathtt{z}, \vartheta(\mathtt{z}))\big| \le \xi_{\mathtt{j}}(\mathtt{z}) + \zeta_{\mathtt{j}}(\mathtt{z})\big|\vartheta_{\mathtt{j}}(\mathtt{z})\big|, \quad \vartheta_{\mathtt{j}} \in \mathtt{c}_0^{\sigma},\ \mathtt{z} \in \mathcal{J}, \mathtt{j} \in \mathbb{N}.$$

Theorem 3.1 *Suppose* $\frac{\mathtt{T}^{\delta_1+\delta_2}e^{\ell\mathtt{T}}\zeta^{\star}}{\Gamma(\delta_1)\Gamma(\delta_2)} < 1$ *and* $(\mathtt{G}_1) - (\mathtt{G}_2)$ *hold, then the infinite system of tempered fractional order boundary value problems* (1.1) *has at least one solution* $\vartheta(\mathtt{z}) = \big(\vartheta_{\mathtt{j}}(\mathtt{z})\big)_{\mathtt{j}=1}^{\infty}$ *in* $\mathcal{C}(\mathcal{J}, \mathtt{c}_0^{\sigma})$ *provided*

$$0 < \lambda_{\mathtt{j}} < \frac{1}{\mathtt{p}_{\mathtt{j}}^{\star}},\ \mathtt{j} \in \mathbb{N},$$

where $\mathtt{p}_{\mathtt{j}}^{\star} = \sup\{\mathtt{p}_{\mathtt{j}}(\mathtt{z}) : \mathtt{z} \in \mathcal{J}\}$.

Proof Since $\sup\{\sigma_{\mathtt{j}}|\vartheta_{\mathtt{j}}(\mathtt{z})|\} < +\infty$ for all $\vartheta(\mathtt{z}) = \big(\vartheta_{\mathtt{j}}(\mathtt{z})\big)_{\mathtt{j}=1}^{\infty} \in \mathcal{C}(\mathcal{J}, \mathtt{c}_0^{\sigma})$ and $\mathtt{z} \in \mathcal{J}$, there exists $\kappa > 0$ such that $\sup\{\sigma_{\mathtt{j}}|\vartheta_{\mathtt{j}}(\mathtt{z})|\} < \kappa$. From $(\mathtt{G}_2)$ and (2.3), we get

$$\begin{aligned}
\|\vartheta(\mathtt{z})\|_{\mathtt{c}_0^{\sigma}} &= \sup_{\mathtt{j}\in\mathbb{N}} \left\{ \sigma_{\mathtt{j}} \left| \lambda_{\mathtt{j}} \int_0^{\mathtt{T}} \aleph_{\delta_1}(\mathtt{z}, \mathtt{y}) e^{-\ell\mathtt{z}} \left[\int_0^{\mathtt{T}} \frac{\aleph_{\delta_2}(\mathtt{y}, \mathtt{x}) e^{\ell\mathtt{x}}}{\mathtt{p}_{\mathtt{j}}(\mathtt{x})} \varphi_{\mathtt{j}}(\mathtt{x}, \vartheta(\mathtt{x})) d\mathtt{x} \right] d\mathtt{y} \right| \right\} \\
&\le \sup_{\mathtt{j}\in\mathbb{N}} \left\{ \frac{\sigma_{\mathtt{j}}\lambda_{\mathtt{j}} e^{\ell\mathtt{T}}}{\mathtt{p}_{\mathtt{j}}^{\star}} \int_0^{\mathtt{T}} |\aleph_{\delta_1}(\mathtt{z}, \mathtt{y})| \left[\int_0^{\mathtt{T}} |\aleph_{\delta_2}(\mathtt{y}, \mathtt{x})| |\varphi_{\mathtt{j}}(\mathtt{x}, \vartheta(\mathtt{x}))| d\mathtt{x} \right] d\mathtt{y} \right\} \\
&\le \sup_{\mathtt{j}\in\mathbb{N}} \left\{ \frac{\sigma_{\mathtt{j}}\lambda_{\mathtt{j}} e^{\ell\mathtt{T}}}{\mathtt{p}_{\mathtt{j}}^{\star}} \int_0^{\mathtt{T}} |\aleph_{\delta_1}(\mathtt{y}, \mathtt{y})| \left[\int_0^{\mathtt{T}} |\aleph_{\delta_2}(\mathtt{x}, \mathtt{x})| |\varphi_{\mathtt{j}}(\mathtt{x}, \vartheta(\mathtt{x}))| d\mathtt{x} \right] d\mathtt{y} \right\} \\
&\le \frac{\mathtt{T}^{\delta_1+\delta_2-1} e^{\ell\mathtt{T}}}{\Gamma(\delta_1)\Gamma(\delta_2)} \sup_{\mathtt{j}\in\mathbb{N}} \left\{ \frac{\sigma_{\mathtt{j}}\lambda_{\mathtt{j}}}{\mathtt{p}_{\mathtt{j}}^{\star}} \int_0^{\mathtt{T}} \left[\xi_{\mathtt{j}}(\mathtt{x}) + \zeta_{\mathtt{j}}(\mathtt{x}) \big|\vartheta_{\mathtt{j}}(\mathtt{x})\big| \right] d\mathtt{x} \right\} \\
&\le \frac{\mathtt{T}^{\delta_1+\delta_2-1} e^{\ell\mathtt{T}}}{\Gamma(\delta_1)\Gamma(\delta_2)} \sup_{\mathtt{j}\in\mathbb{N}} \left\{ \int_0^{\mathtt{T}} [\sigma^{\star} + \zeta^{\star}\kappa] \, d\tau \right\} \\
&\le \frac{(\sigma^{\star} + \zeta^{\star}\kappa) \mathtt{T}^{\delta_1+\delta_2} e^{\ell\mathtt{T}}}{\Gamma(\delta_1)\Gamma(\delta_2)} := a.
\end{aligned}$$

Therefore,

$$\max_{\mathtt{z}\in\mathcal{J}}\|\vartheta(\mathtt{z})\|_{\mathtt{c}_0^\sigma}\le a,\quad i.e.,\ \|\vartheta(\mathtt{z})\|_{\mathcal{C}(\mathcal{J},\mathtt{c}_0^\sigma)}\le a.$$

Now, let the closed ball $\mathcal{A}=\mathcal{A}(\vartheta^0(\mathtt{z}),a)$ centered at $\vartheta^0(\mathtt{z})=\left(\vartheta^0(\mathtt{z})\right)_{\mathtt{j}=1}^{\infty}$, $\vartheta^0(\mathtt{z})=0$, $\forall\,\mathtt{z}\in\mathcal{J}$, $\mathtt{j}\in\mathbb{N}$ and radius a. Thus, $\mathcal{A}$ is a nonempty bounded closed convex subset of $\mathcal{C}(\mathcal{J},\mathtt{c}_0^\sigma)$.

For fixed $\mathtt{z}\in\mathcal{J}$, define an operator $\Xi=(\Xi_\mathtt{j})_{\mathtt{j}=1}^{\infty}:\mathcal{C}(\mathcal{J},\mathtt{c}_0^\sigma)\to\mathcal{C}(\mathcal{J},\mathtt{c}_0^\sigma)$ as

$$(\Xi\vartheta)(\mathtt{z})=\{(\Xi_\mathtt{j}\vartheta)(\mathtt{z})\}_{\mathtt{j}=1}^{\infty}=\left\{\lambda_\mathtt{j}\int_0^{\mathtt{T}}\aleph_{\delta_1}(\mathtt{z},\mathtt{y})e^{-\ell\mathtt{z}}\left[\int_0^{\mathtt{T}}\frac{\aleph_{\delta_2}(\mathtt{y},\mathtt{x})e^{\ell\mathtt{x}}}{\mathtt{p}_\mathtt{j}(\mathtt{x})}\varphi_\mathtt{j}(\mathtt{x},\vartheta(\mathtt{x}))d\mathtt{x}\right]d\mathtt{y}\right\}_{\mathtt{j}=1}^{\infty}.$$

Since $\left(\varphi_\mathtt{j}(\mathtt{z},\vartheta(\mathtt{z}))\right)_{\mathtt{j}=1}^{\infty}\in\mathtt{c}_0^\sigma$, for $\mathtt{z}\in\mathcal{J}$, it follows that

$$\begin{aligned}\lim_{\mathtt{j}\to\infty}\{\sigma_\mathtt{j}(\Xi_\mathtt{j}\vartheta)(\mathtt{z})\}&=\lim_{\mathtt{j}\to\infty}\left\{\sigma_\mathtt{j}\lambda_\mathtt{j}\int_0^{\mathtt{T}}\aleph_{\delta_1}(\mathtt{z},\mathtt{y})e^{-\ell\mathtt{z}}\left[\int_0^{\mathtt{T}}\frac{\aleph_{\delta_2}(\mathtt{y},\mathtt{x})e^{\ell\mathtt{x}}}{\mathtt{p}_\mathtt{j}(\mathtt{x})}\varphi_\mathtt{j}(\mathtt{x},\vartheta(\mathtt{x}))d\mathtt{x}\right]d\mathtt{y}\right\}\\&\le\frac{\mathtt{T}^{\delta_1+\delta_2-1}e^{\ell\mathtt{T}}}{\Gamma(\delta_1)\Gamma(\delta_2)}\int_0^{\mathtt{T}}\lim_{\mathtt{j}\to\infty}[\sigma_\mathtt{j}\lambda_\mathtt{j}\mathtt{p}_\mathtt{j}^\star\varphi_\mathtt{j}(\mathtt{x},\vartheta(\mathtt{x}))]d\mathtt{x}\\&\le\frac{\mathtt{T}^{\delta_1+\delta_2-1}e^{\ell\mathtt{T}}}{\Gamma(\delta_1)\Gamma(\delta_2)}\int_0^{\mathtt{T}}\lim_{\mathtt{j}\to\infty}[\sigma_\mathtt{j}\varphi_\mathtt{j}(\mathtt{x},\vartheta(\mathtt{x}))]d\mathtt{x}\\&=0.\end{aligned}$$

Thus, $(\Xi\vartheta)(\mathtt{z})\in\mathcal{C}(\mathcal{J},\mathtt{c}_0^\sigma)$. Moreover, it can be seen that $(\Xi_\mathtt{j}\vartheta)(\mathtt{z})$ satisfies boundary conditions, i.e.

$$\begin{aligned}(\Xi_\mathtt{j}\vartheta(0))&=\lim_{\mathtt{z}\to0}\left[{}^{\mathtt{RL}}_{\ 0}\mathbb{D}_\mathtt{z}^{\delta_1,\ell}(e^{\ell\mathtt{z}}\left(\Xi_\mathtt{j}\vartheta(\mathtt{z})\right))\right]=0,\\e^{\ell\mathtt{T}}(\Xi_\mathtt{j}\vartheta(\mathtt{T}))&=\lim_{\mathtt{z}\to\mathtt{T}}\left[{}^{\mathtt{RL}}_{\ 0}\mathbb{D}_\mathtt{z}^{\delta_1,\ell}(e^{\ell\mathtt{z}}(\Xi_\mathtt{j}\vartheta(\mathtt{z})))\right]=0.\end{aligned}$$

For fixed $\mathtt{z}\in\mathtt{T}$ and $\vartheta(\mathtt{z})\in\mathcal{A}$, we get

$$\begin{aligned}\|(\Xi\vartheta)(\mathtt{z})-\vartheta^0(\mathtt{z})\|_{\mathtt{c}_0^\sigma}\le a&\implies\max_{\mathtt{z}\in\mathcal{J}}\|(\Xi\vartheta)(\mathtt{z})-\vartheta^0(\mathtt{z})\|_{\mathtt{c}_0^\sigma}\le a\\&\implies\|(\Xi\vartheta)(\mathtt{z})-\vartheta^0(\mathtt{z})\|_{\mathcal{C}(\mathcal{J},\mathtt{c}_0^\sigma)}\le a,\end{aligned}$$

which proves that Ξ is self mapping on $\mathcal{A}$. From $(\mathtt{G}_1)$, for any $\vartheta(\mathtt{z})=\left(\vartheta_\mathtt{j}(\mathtt{z})\right)_{\mathtt{j}=1}^{\infty}\in\mathcal{A}$ and for any $\varepsilon>0$ there exists $\delta>0$ such that $\|(\Phi\vartheta)(\mathtt{z})-(\Phi\vartheta)(\mathtt{z})\|_{\mathtt{c}_0^\sigma}<\frac{\varepsilon\Gamma(\delta_1)\Gamma(\delta_2)}{\mathtt{T}^{\delta_1+\delta_2}e^{\ell\mathtt{T}}}$ for each $\vartheta(\mathtt{z})\in\mathcal{A}$, whenever $|\vartheta(\mathtt{z})-\vartheta(\mathtt{z})|\le\delta$, where $\mathtt{z}\in\mathcal{J}$. Thus, for $\mathtt{z}\in\mathcal{J}$, we have

$$\begin{aligned}\|(\Xi\vartheta)(\mathtt{z})-(\Xi\vartheta)(\mathtt{z})\|_{\mathtt{c}_0^\sigma}&=\sup_{\mathtt{j}\in\mathbb{N}}\left\{\left|\sigma_\mathtt{j}\lambda_\mathtt{j}\int_0^{\mathtt{T}}\aleph_{\delta_1}(\mathtt{z},\mathtt{y})e^{-\ell\mathtt{z}}\left[\int_0^{\mathtt{T}}\frac{\aleph_{\delta_2}(\mathtt{y},\mathtt{x})e^{\ell\mathtt{x}}}{\mathtt{p}_\mathtt{j}(\mathtt{x})}[\varphi_\mathtt{j}(\mathtt{x},\vartheta(\mathtt{x}))-\varphi_\mathtt{j}(\mathtt{x},\vartheta(\mathtt{x}))]d\mathtt{x}\right]d\mathtt{y}\right|\right\}\\&\le\frac{\mathtt{T}^{\delta_1+\delta_2-1}e^{\ell\mathtt{T}}}{\Gamma(\delta_1)\Gamma(\delta_2)}\sup_{\mathtt{j}\in\mathbb{N}}\left\{\sigma_\mathtt{j}\lambda_\mathtt{j}\mathtt{p}_\mathtt{j}^\star\int_0^{\mathtt{T}}|\varphi_\mathtt{j}(\mathtt{x},\vartheta(\mathtt{x}))-\varphi_\mathtt{j}(\mathtt{x},\vartheta(\mathtt{x}))|d\mathtt{x}\right\}\\&\le\frac{\mathtt{T}^{\delta_1+\delta_2-1}e^{\ell\mathtt{T}}}{\Gamma(\delta_1)\Gamma(\delta_2)}\,\frac{\varepsilon\Gamma(\delta_1)\Gamma(\delta_2)}{\mathtt{T}^{\delta_1+\delta_2}e^{\ell\mathtt{T}}}\mathtt{T}<\varepsilon.\end{aligned}$$

It shows that Ξ is continuous on $\mathcal{A}$ for every $\mathtt{z} \in \mathcal{J}$.
Now, we have

$$\begin{aligned}
\chi(\Xi\mathcal{A}) &= \lim_{\mathtt{j}\to\infty}\left\{\sup_{\vartheta(\mathtt{z})\in\mathcal{A}}\sup_{m\geq \mathtt{j}}\left[\sigma_m\left|\lambda_m\int_0^{\mathtt{T}}\aleph_{\delta_1}(\mathtt{z},\mathtt{y})e^{-\ell\mathtt{z}}\left[\int_0^{\mathtt{T}}\frac{\aleph_{\delta_2}(\mathtt{y},\mathtt{x})e^{\ell\mathtt{x}}}{\mathtt{p}_m(\mathtt{x})}\varphi_m(\mathtt{x},\vartheta(\mathtt{x}))d\mathtt{x}\right]d\mathtt{y}\right|\right]\right\}\\
&\leq \frac{\mathtt{T}^{\delta_1+\delta_2-1}e^{\ell\mathtt{T}}}{\Gamma(\delta_1)\Gamma(\delta_2)}\lim_{\mathtt{j}\to\infty}\left\{\sup_{\vartheta(\mathtt{z})\in\mathcal{A}}\sup_{m\geq \mathtt{j}}\left[\int_0^{\mathtt{T}}\left(\sigma_m\xi_m(\mathtt{x})+\sigma_m\zeta_m(\mathtt{x})\left|\vartheta_m(\mathtt{x})\right|\right)d\mathtt{x}\right]\right\}\\
&\leq \frac{\mathtt{T}^{\delta_1+\delta_2}e^{\ell\mathtt{T}}\zeta^\star}{\Gamma(\delta_1)\Gamma(\delta_2)}\chi(\mathcal{A}).
\end{aligned}$$

Thus,

$$\sup_{\mathtt{z}\in\mathcal{J}}\chi(\Xi\mathcal{A}) \leq \frac{\mathtt{T}^{\delta_1+\delta_2}e^{\ell\mathtt{T}}\zeta^\star}{\Gamma(\delta_1)\Gamma(\delta_2)}\sup_{\mathtt{z}\in\mathcal{J}}\chi(\mathcal{A}).$$

It follows that

$$\chi_{\mathcal{C}(\mathcal{J},\mathtt{c}_0^\sigma)}(\Xi\mathcal{A}) \leq \frac{\mathtt{T}^{\delta_1+\delta_2}e^{\ell\mathtt{T}}\zeta^\star}{\Gamma(\delta_1)\Gamma(\delta_2)}\chi_{\mathcal{C}(\mathcal{J},\mathtt{c}_0^\sigma)}(\mathcal{A}) < \varepsilon.$$

That is

$$\chi_{\mathcal{C}(\mathcal{J},\mathtt{c}_0^\sigma)}(\mathcal{A}) < \frac{\varepsilon\Gamma(\delta_1)\Gamma(\delta_2)}{\mathtt{T}^{\delta_1+\delta_2}e^{\ell\mathtt{T}}\zeta^\star}.$$

Letting $\delta = \dfrac{\varepsilon}{\mathtt{T}^{\delta_1+\delta_2}e^{\ell\mathtt{T}}\zeta^\star}\left[\Gamma(\delta_1)\Gamma(\delta_2) - \mathtt{T}^{\delta_1+\delta_2}e^{\ell\mathtt{T}}\zeta^\star\right]$, we obtain $\varepsilon \leq \chi_{\mathcal{C}(\mathcal{J},\mathtt{c}_0^\sigma)}(\mathcal{A}) < \varepsilon + \delta$.

Therefore, Ξ is a Meir-Keeler condensing operator on $\mathcal{A}$. Moreover, Ξ satisfies all the conditions of Theorem 2.18, i.e. Ξ has a fixed point in $\mathcal{A}$. Hence, the infinite system (1.1) has a solution in $\mathcal{C}(\mathcal{J},\mathtt{c}_0^\sigma)$. □

4. Existence of solutions for tempered fractional BVP (1.1) in $\mathcal{C}(\mathcal{J},\mathtt{c}^\sigma)$

In this section, we derive the sufficient conditions for the existence of solutions for infinite system of tempered fractional order boundary value problems (1.1) in the sequences spaces $\mathcal{C}(\mathcal{J},\mathtt{c}^\sigma)$.
Suppose

($\mathtt{H}_1$) $\varphi_{\mathtt{j}} : \mathcal{J} \to \mathbb{R}^\infty$ and define an operator $\Phi : \mathcal{J}\times\mathtt{c}^\sigma \to \mathtt{c}^\sigma$ as

$$(\mathtt{z},\vartheta(\mathtt{z})) \to (\Phi\vartheta)(\mathtt{z}) = \left(\varphi_{\mathtt{j}}(\mathtt{z},\vartheta(\mathtt{z}))\right)_{\mathtt{j}=1}^\infty$$

is the class of all functions $\left((\Phi\vartheta)(\mathtt{z})\right)_{\mathtt{z}\in\mathcal{J}}$ and equicontinuous on $\mathtt{c}^\sigma$.

($\mathtt{H}_2$) $\eta_{\mathtt{j}}(\mathtt{z}), \theta_{\mathtt{j}}(\mathtt{z}) : \mathcal{J} \to \mathbb{R}$ are continuous functions such that the sequence $\sigma_{\mathtt{j}}\eta_{\mathtt{j}}(\mathtt{z})$ converges uniformly to zero on $\mathcal{J}$ and the sequence $(\theta_{\mathtt{j}}(\mathtt{z}))$ is convergence on $\mathcal{J}$, so we take $\theta(\mathtt{z}) = \sup\{\theta_{\mathtt{j}}(\mathtt{z}) : \mathtt{j}\in\mathbb{N}\}$, $\theta^\star = \sup\{\theta(\mathtt{z}) : \mathtt{z}\in\mathcal{J}\}$, $\sigma^\star = \sup\{\sigma_{\mathtt{j}}\eta_{\mathtt{j}}(\mathtt{z}) : \mathtt{j}\in\mathbb{N}, \mathtt{z}\in\mathcal{J}\}$ and

$$\varphi_{\mathtt{j}}(\mathtt{z},\vartheta(\mathtt{z})) \leq \eta_{\mathtt{j}}(\mathtt{z}) + \theta_{\mathtt{j}}(\mathtt{z})\vartheta_{\mathtt{j}}(\mathtt{z}), \quad \vartheta_{\mathtt{j}} \in \mathtt{c}^\sigma,\ \mathtt{z}\in\mathcal{J}, \mathtt{j} = 1,2,3,\cdots.$$

Theorem 4.1 *Suppose* $\frac{T^{\delta_1+\delta_2}e^{\ell T}\theta^\star}{\Gamma(\delta_1)\Gamma(\delta_2)} < 1$ *and* (H_1)–(H_2) *hold, then the infinite systems of fractional order boundary value problems* (1.1) *has at least one solution* $\vartheta(z) = \big(\vartheta_j(z)\big)$ *in* $\mathcal{C}(\mathcal{J}, c^\sigma)$ *provided*

$$0 < \lambda_j < \frac{1}{p_j^\star},\ j \in \mathbb{N},$$

where $p_j^\star = \sup\{p_j(z) : z \in \mathcal{J}\}$.

Proof Since $\sup\{\sigma_j|\vartheta_j(z)|\} < +\infty$ for all $\vartheta(z) = \big(\vartheta_j(z)\big)_{j=1}^\infty \in \mathcal{C}(\mathcal{J}, c^\sigma)$ and $z \in \mathcal{J}$, there exists $\varrho > 0$ such that $\sup\{\sigma_j|\vartheta_j(z)|\} < \varrho$. From (H_2) and (2.3), we get

$$\begin{aligned}
\|\vartheta(z)\|_{c^\sigma} &= \sup_{j\in\mathbb{N}}\left\{\sigma_j\left|\lambda_j\int_0^T \aleph_{\delta_1}(z,y)e^{-\ell z}\left[\int_0^T \frac{\aleph_{\delta_2}(y,x)e^{\ell x}}{p_j(x)}\varphi_j(x,\vartheta(x))dx\right]dy\right|\right\}\\
&\le \sup_{j\in\mathbb{N}}\left\{\sigma_j\lambda_j e^{\ell T}p_j^\star\int_0^T |\aleph_{\delta_1}(z,y)|\left[\int_0^T |\aleph_{\delta_2}(y,x)||\varphi_j(x,\vartheta(x))|dx\right]dy\right\}\\
&\le \sup_{j\in\mathbb{N}}\left\{\sigma_j\lambda_j e^{\ell T}p_j^\star\int_0^T |\aleph_{\delta_1}(y,y)|\left[\int_0^T |\aleph_{\delta_2}(x,x)||\varphi_j(x,\vartheta(x))|dx\right]dy\right\}\\
&\le \frac{T^{\delta_1+\delta_2-1}e^{\ell T}}{\Gamma(\delta_1)\Gamma(\delta_2)}\sup_{j\in\mathbb{N}}\left\{\sigma_j\lambda_j p_j^\star\int_0^T \big[\eta_j(x)+\theta_j(x)\big|\vartheta_j(x)\big|\big]\,dx\right\}\\
&\le \frac{T^{\delta_1+\delta_2-1}e^{\ell T}}{\Gamma(\delta_1)\Gamma(\delta_2)}\sup_{j\in\mathbb{N}}\left\{\int_0^T [\sigma^\star+\theta^\star\varrho]\,d\tau\right\}\\
&\le \frac{(\sigma^\star+\theta^\star\varrho)T^{\delta_1+\delta_2}e^{\ell T}}{\Gamma(\delta_1)\Gamma(\delta_2)} := b.
\end{aligned}$$

Therefore,

$$\max_{z\in\mathcal{J}}\|\vartheta(z)\|_{c^\sigma} \le b, \quad i.e.,\ \|\vartheta(z)\|_{\mathcal{C}(\mathcal{J},c^\sigma)} \le b,$$

Now, let the closed ball $\mathcal{B} = \mathcal{B}(\vartheta^0(z), r_1)$ centered at $\vartheta^0(z) = \big(\vartheta^0(z)\big)_{j=1}^\infty$, $\vartheta^0(z) = 0\,\forall\, z \in \mathcal{J}$, $j \in \mathbb{N}$ and radius b. Therefore, $\mathcal{B}$ is a nonempty bounded closed convex subset of $\mathcal{C}(\mathcal{J}, c^\sigma)$.
For fixed $z \in \mathcal{J}$, define an operator $\Upsilon = (\Upsilon_j)_{j=1}^\infty : \mathcal{C}(\mathcal{J}, c_0^\sigma) \to \mathcal{C}(\mathcal{J}, c_0^\sigma)$ as

$$(\Upsilon\vartheta)(z) = \{(\Upsilon_j\vartheta)(z)\}_{j=1}^\infty = \left\{\lambda_j\int_0^T \aleph_{\delta_1}(z,y)e^{-\ell z}\left[\int_0^T \frac{\aleph_{\delta_2}(y,x)e^{\ell x}}{p_j(x)}\varphi_j(x,\vartheta(x))dx\right]dy\right\}_{j=1}^\infty.$$

Since $\left(\varphi_{\mathtt{j}}(\mathtt{z},\vartheta(\mathtt{z}))\right)_{\mathtt{j}=1}^{\infty} \in \mathtt{c}^{\sigma}$ and $p_{\mathtt{j}}(\mathtt{z}) = p(\mathtt{z})$ for $\mathtt{z} \in \mathcal{J}$, it follows that $\aleph_{\mathtt{j}}(\mathtt{z},\tau) = \aleph_{\mathtt{i}}(\mathtt{z},\tau)$ for any $k,l \in \mathbb{N}$. Therefore, we have

$$
\begin{aligned}
&|\sigma_{\mathtt{j}}(\Upsilon_{\mathtt{j}}\vartheta)(\mathtt{z}) - \sigma_{\mathtt{i}}(\Upsilon_{\mathtt{i}}\vartheta)(\mathtt{z})| \\
&= \Bigg|\sigma_{\mathtt{j}}\lambda_{\mathtt{j}} \int_0^{\mathtt{T}} \aleph_{\delta_1}(\mathtt{z},\mathtt{y}) e^{-\ell\mathtt{z}} \left[\int_0^{\mathtt{T}} \frac{\aleph_{\delta_2}(\mathtt{y},\mathtt{x}) e^{\ell\mathtt{x}}}{\mathtt{p}_{\mathtt{j}}(\mathtt{x})} \varphi_{\mathtt{j}}(\mathtt{x},\vartheta(\mathtt{x})) d\mathtt{x}\right] d\mathtt{y} \\
&\qquad\qquad - \sigma_{\mathtt{i}}\lambda_{\mathtt{i}} \int_0^{\mathtt{T}} \aleph_{\delta_1}(\mathtt{z},\mathtt{y}) e^{-\ell\mathtt{z}} \left[\int_0^{\mathtt{T}} \frac{\aleph_{\delta_2}(\mathtt{y},\mathtt{x}) e^{\ell\mathtt{x}}}{\mathtt{p}_{\mathtt{i}}(\mathtt{x})} \varphi_{\mathtt{i}}(\mathtt{x},\vartheta(\mathtt{x})) d\mathtt{x}\right] d\mathtt{y}\Bigg| \\
&= \Bigg| \int_0^{\mathtt{T}} \aleph_{\delta_1}(\mathtt{z},\mathtt{y}) \left[\int_0^{\mathtt{T}} \aleph_{\delta_2}(\mathtt{y},\mathtt{x}) e^{-\ell(\mathtt{z}-\mathtt{x})} \frac{\sigma_{\mathtt{j}}\lambda_{\mathtt{j}}}{\mathtt{p}_{\mathtt{j}}(\mathtt{x})} [\eta_{\mathtt{j}}(\mathtt{x}) + \theta_{\mathtt{j}}(\mathtt{x})\vartheta_{\mathtt{j}}(\mathtt{x})] d\mathtt{x}\right] d\mathtt{y} \\
&\qquad - \int_0^{\mathtt{T}} \aleph_{\delta_1}(\mathtt{z},\mathtt{y}) \left[\int_0^{\mathtt{T}} \aleph_{\delta_2}(\mathtt{y},\mathtt{x}) e^{-\ell(\mathtt{z}-\mathtt{x})} \frac{\sigma_{\mathtt{i}}\lambda_{\mathtt{i}}}{\mathtt{p}_{\mathtt{i}}(\mathtt{x})} [\eta_{\mathtt{i}}(\mathtt{x}) + \theta_{\mathtt{i}}(\mathtt{x})\vartheta_{\mathtt{i}}(\mathtt{x})] d\mathtt{x}\right] d\mathtt{y}\Bigg| \\
&\le \int_0^{\mathtt{T}} |\aleph_{\delta_1}(\mathtt{z},\mathtt{y})| \left[\int_0^{\mathtt{T}} |\aleph_{\delta_2}(\mathtt{y},\mathtt{x})||e^{-\ell(\mathtt{z}-\mathtt{x})}| \left|\frac{\sigma_{\mathtt{j}}\lambda_{\mathtt{j}}\eta_{\mathtt{j}}(\mathtt{x})}{\mathtt{p}_{\mathtt{j}}(\mathtt{x})} - \frac{\sigma_{\mathtt{i}}\lambda_{\mathtt{i}}\eta_{\mathtt{i}}(\mathtt{x})}{\mathtt{p}_{\mathtt{i}}(\mathtt{x})}\right| d\mathtt{x}\right] d\mathtt{y} \\
&\quad + \int_0^{\mathtt{T}} |\aleph_{\delta_1}(\mathtt{z},\mathtt{y})| \left[\int_0^{\mathtt{T}} |\aleph_{\delta_2}(\mathtt{y},\mathtt{x})||e^{-\ell(\mathtt{z}-\mathtt{x})}| \left|\frac{\sigma_{\mathtt{j}}\lambda_{\mathtt{j}}\theta_{\mathtt{j}}(\mathtt{x})\vartheta_{\mathtt{j}}(\mathtt{x})}{\mathtt{p}_{\mathtt{j}}(\mathtt{x})} - \frac{\sigma_{\mathtt{i}}\lambda_{\mathtt{i}}\theta_{\mathtt{i}}(\mathtt{x})\vartheta_{\mathtt{i}}(\mathtt{x})}{\mathtt{p}_{\mathtt{i}}(\mathtt{x})}\right| d\mathtt{x}\right] d\mathtt{y} \\
&\le \int_0^{\mathtt{T}} |\aleph_{\delta_1}(\mathtt{z},\mathtt{y})| \left[\int_0^{\mathtt{T}} |\aleph_{\delta_2}(\mathtt{y},\mathtt{x})||e^{-\ell(\mathtt{z}-\mathtt{x})}| \left|\frac{\sigma_{\mathtt{j}}\lambda_{\mathtt{j}}\eta_{\mathtt{j}}(\mathtt{x})}{\mathtt{p}_{\mathtt{j}}(\mathtt{x})} - \frac{\sigma_{\mathtt{i}}\lambda_{\mathtt{i}}\eta_{\mathtt{i}}(\mathtt{x})}{\mathtt{p}_{\mathtt{i}}(\mathtt{x})}\right| d\mathtt{x}\right] d\mathtt{y} \\
&\quad + \int_0^{\mathtt{T}} |\aleph_{\delta_1}(\mathtt{z},\mathtt{y})| \Bigg[\int_0^{\mathtt{T}} |\aleph_{\delta_2}(\mathtt{y},\mathtt{x})||e^{-\ell(\mathtt{z}-\mathtt{x})}| \Bigg[\frac{\sigma_{\mathtt{j}}\lambda_{\mathtt{j}}\vartheta_{\mathtt{j}}(\mathtt{x})}{\mathtt{p}_{\mathtt{j}}(\mathtt{x})} |\theta_{\mathtt{j}}(\mathtt{x}) - \theta_{\mathtt{i}}(\mathtt{x})| \\
&\qquad\qquad + |\theta_{\mathtt{i}}(\mathtt{x})| \left|\frac{\sigma_{\mathtt{j}}\lambda_{\mathtt{j}}\vartheta_{\mathtt{j}}(\mathtt{x})}{\mathtt{p}_{\mathtt{j}}(\mathtt{x})} - \frac{\sigma_{\mathtt{i}}\lambda_{\mathtt{i}}\vartheta_{\mathtt{i}}(\mathtt{x})}{\mathtt{p}_{\mathtt{i}}(\mathtt{x})}\right|\Bigg] d\mathtt{x}\Bigg] d\mathtt{y}
\end{aligned}
$$

Since $(\lambda_{\mathtt{j}})$, $(\theta_{\mathtt{j}})$, $(\sigma_{\mathtt{j}}\eta_{\mathtt{j}})$ are convergent on $\mathcal{J}$ and $\frac{\lambda_{\mathtt{j}}\vartheta_{\mathtt{j}}}{\mathtt{p}_{\mathtt{j}}} \in \mathtt{c}^{\sigma}$, it follows that

$$
|\sigma_{\mathtt{j}}(\Upsilon_{\mathtt{j}}\vartheta)(\mathtt{z}) - \sigma_{\mathtt{i}}(\Upsilon_{\mathtt{i}}\vartheta)(\mathtt{z})| \to 0 \quad \text{as} \quad \mathtt{j},\mathtt{i} \to \infty.
$$

Hence, $(\Upsilon\vartheta)(\mathtt{z}) \in \mathcal{C}(\mathcal{J},\mathtt{c}^{\sigma})$. Moreover, it can be seen that $(\Upsilon_{\mathtt{j}}\vartheta)(\mathtt{z})$ satisfies boundary conditions, i.e.

$$
\begin{aligned}
(\Upsilon_{\mathtt{j}}\vartheta(0)) &= \lim_{\mathtt{z}\to 0} \left[{}^{\mathtt{RL}}_{\ 0}\mathbb{D}^{\delta_1,\ell}_{\mathtt{z}}(e^{\ell\mathtt{z}}\left(\Upsilon_{\mathtt{j}}\vartheta(\mathtt{z})\right))\right] = 0, \\
e^{\ell\mathtt{T}}(\Upsilon_{\mathtt{j}}\vartheta(\mathtt{T})) &= \lim_{\mathtt{z}\to \mathtt{T}} \left[{}^{\mathtt{RL}}_{\ 0}\mathbb{D}^{\delta_1,\ell}_{\mathtt{z}}(e^{\ell\mathtt{z}}(\Upsilon_{\mathtt{j}}\vartheta(\mathtt{z})))\right] = 0.
\end{aligned}
$$

For fixed $\mathtt{z} \in \mathtt{T}$ and $\vartheta(\mathtt{z}) \in \mathcal{B}$, we get

$$
\begin{aligned}
\|(\Upsilon\vartheta)(\mathtt{z}) - \vartheta^0(\mathtt{z})\|_{\mathtt{c}^{\sigma}} \le b &\implies \max_{\mathtt{z}\in\mathcal{J}} \|(\Upsilon\vartheta)(\mathtt{z}) - \vartheta^0(\mathtt{z})\|_{\mathtt{c}^{\sigma}} \le b \\
&\implies \|(\Upsilon\vartheta)(\mathtt{z}) - \vartheta^0(\mathtt{z})\|_{\mathcal{C}(\mathcal{J},\mathtt{c}^{\sigma})} \le b,
\end{aligned}
$$

which proves that Υ is self mapping on $\mathcal{B}$. From $(\mathtt{H}_1)$, for any $\vartheta(\mathtt{z}) = \left(\vartheta_{\mathtt{j}}(\mathtt{z})\right)_{\mathtt{j}=1}^{\infty} \in \mathcal{B}$ and for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|(\Phi\vartheta)(\mathtt{z}) - (\Phi\vartheta)(\mathtt{z})\|_{\mathtt{c}^{\sigma}} < \frac{\varepsilon\Gamma(\delta_1)\Gamma(\delta_2)}{\mathtt{T}^{\delta_1+\delta_2}e^{\ell\mathtt{T}}}\mathtt{T}$ for each $\vartheta(\mathtt{z}) \in \mathcal{B}$, whenever $|\vartheta(\mathtt{z}) - \vartheta(\mathtt{z})| \le \delta$, where $\mathtt{z} \in \mathcal{J}$. Thus, for $\mathtt{z} \in \mathcal{J}$, we have

$$\begin{aligned}
\|(\Upsilon\vartheta)(\mathtt{z}) - (\Upsilon\vartheta)(\mathtt{z})\|_{\mathtt{c}^\sigma} &= \sup_{\mathtt{j}\in\mathbb{N}} \left\{ \left| \sigma_\mathtt{j}\lambda_\mathtt{j} \int_0^\mathtt{T} \aleph_{\delta_1}(\mathtt{z},\mathtt{y}) e^{-\ell\mathtt{z}} \left[\int_0^\mathtt{T} \frac{\aleph_{\delta_2}(\mathtt{y},\mathtt{x}) e^{\ell\mathtt{x}}}{\mathtt{p}_\mathtt{j}(\mathtt{x})} [\varphi_\mathtt{j}(\mathtt{x},\vartheta(\mathtt{x})) - \varphi_\mathtt{j}(\mathtt{x},\vartheta(\mathtt{x}))] d\mathtt{x} \right] d\mathtt{y} \right| \right\} \\
&\leq \frac{\mathtt{T}^{\delta_1+\delta_2-1} e^{\ell\mathtt{T}}}{\Gamma(\delta_1)\Gamma(\delta_2)} \sup_{\mathtt{j}\in\mathbb{N}} \left\{ \sigma_\mathtt{j}\lambda_\mathtt{j}\mathtt{p}_\mathtt{j}^\star \int_0^\mathtt{T} |\varphi_\mathtt{j}(\mathtt{x},\vartheta(\mathtt{x})) - \varphi_\mathtt{j}(\mathtt{x},\vartheta(\mathtt{x}))| d\mathtt{x} \right\} \\
&\leq \frac{\mathtt{T}^{\delta_1+\delta_2-1} e^{\ell\mathtt{T}}}{\Gamma(\delta_1)\Gamma(\delta_2)} \frac{\varepsilon\Gamma(\delta_1)\Gamma(\delta_2)}{\mathtt{T}^{\delta_1+\delta_2} e^{\ell\mathtt{T}}} \mathtt{T} < \varepsilon.
\end{aligned}$$

Thus, Ω is continuous on $\mathcal{B}$ for every $\mathtt{z} \in \mathcal{J}$.
Now, we have

$$\begin{aligned}
\mu(\Omega\mathcal{B}) &= \lim_{\mathtt{j}\to\infty} \Bigg\{ \sup_{\vartheta(\mathtt{z})\in\mathcal{B}} \sup_{m\geq\mathtt{j}} \Bigg[\Bigg| \sigma_m\lambda_m \int_0^\mathtt{T} \aleph_{\delta_1}(\mathtt{z},\mathtt{y}) e^{-\ell\mathtt{z}} \left[\int_0^\mathtt{T} \frac{\aleph_{\delta_2}(\mathtt{y},\mathtt{x}) e^{\ell\mathtt{x}}}{\mathtt{p}_m(\mathtt{x})} \varphi_m(\mathtt{x},\vartheta(\mathtt{x})) d\mathtt{x} \right] d\mathtt{y} \\
&\qquad - \lim_{n\to\infty} \left(\sigma_n\lambda_n \int_0^\mathtt{T} \aleph_{\delta_1}(\mathtt{z},\mathtt{y}) e^{-\ell\mathtt{z}} \left[\int_0^\mathtt{T} \frac{\aleph_{\delta_2}(\mathtt{y},\mathtt{x}) e^{\ell\mathtt{x}}}{\mathtt{p}_n(\mathtt{x})} \varphi_n(\mathtt{x},\vartheta(\mathtt{x})) d\mathtt{x} \right] d\mathtt{y} \right) \Bigg| \Bigg] \Bigg\} \\
&\leq \frac{\mathtt{T}^{\delta_1+\delta_2-1} e^{\ell\mathtt{T}}}{\Gamma(\delta_1)\Gamma(\delta_2)} \lim_{\mathtt{j}\to\infty} \Bigg\{ \sup_{\vartheta(\mathtt{z})\in\mathcal{B}} \sup_{m\geq\mathtt{j}} \Bigg[\int_0^\mathtt{T} \Bigg| \frac{\sigma_m\lambda_m}{\mathtt{p}_m(\mathtt{x})} \varphi_m(\tau,\vartheta(\mathtt{x})) \\
&\qquad - \lim_{n\to\infty} \frac{\sigma_n\lambda_n}{\mathtt{p}_n(\mathtt{x})} \varphi_n(\mathtt{x},\vartheta(\mathtt{x})) \Bigg| d\mathtt{x} \Bigg] \Bigg\} \\
&\leq \frac{\mathtt{T}^{\delta_1+\delta_2-1} e^{\ell\mathtt{T}}}{\Gamma(\delta_1)\Gamma(\delta_2)} \lim_{\mathtt{j}\to\infty} \Bigg\{ \sup_{\vartheta(\mathtt{z})\in\mathcal{B}} \sup_{m\geq\mathtt{j}} \Bigg[\int_0^\mathtt{T} \Bigg| \frac{\sigma_m\lambda_m\theta_m(\mathtt{x})\vartheta_m(\mathtt{x})}{\mathtt{p}_m(\mathtt{x})} \\
&\qquad - \lim_{n\to\infty} \frac{\sigma_n\lambda_n\theta_n(\mathtt{x})\vartheta_n(\mathtt{x})}{\mathtt{p}_n(\mathtt{x})} \Bigg| d\mathtt{x} \Bigg] \Bigg\} \\
&\leq \frac{\mathtt{T}^{\delta_1+\delta_2-1} e^{\ell\mathtt{T}}}{\Gamma(\delta_1)\Gamma(\delta_2)} \lim_{\mathtt{j}\to\infty} \Bigg\{ \sup_{\vartheta(\mathtt{z})\in\mathcal{B}} \sup_{m\geq\mathtt{j}} \Bigg[\int_0^\mathtt{T} \Big(\frac{\lambda_m|\theta_m(\tau)|}{\mathtt{p}_m(\mathtt{x})} \big| \sigma_m\vartheta_m(\tau) - \lim_{n\to\infty} \sigma_n\vartheta_n(\tau) \big| \\
&\qquad - \big| \lim_{n\to\infty} \frac{\sigma_n\vartheta_n(\tau)}{\mathtt{p}_n(\mathtt{x})} \big(\lambda_m\theta_m(\tau) - \lambda_n\theta_n(\tau) \big) \big| \Big) d\tau \Bigg] \Bigg\} \\
&\leq \frac{\mathtt{T}^{\delta_1+\delta_2} e^{\ell\mathtt{T}}\theta^\star}{\Gamma(\delta_1)\Gamma(\delta_2)} \mu(\mathcal{B}).
\end{aligned}$$

Thus,

$$\sup_{\mathtt{z}\in\mathcal{J}} \mu(\Upsilon\mathcal{B}) \leq \frac{\mathtt{T}^{\delta_1+\delta_2} e^{\ell\mathtt{T}}\theta^\star}{\Gamma(\delta_1)\Gamma(\delta_2)} \sup_{\mathtt{z}\in\mathcal{J}} \mu(\mathcal{B}).$$

It follows that

$$\mu_{\mathcal{C}(\mathcal{J},\mathtt{c}^\sigma)}(\Upsilon\mathcal{B}) \leq \frac{\mathtt{T}^{\delta_1+\delta_2} e^{\ell\mathtt{T}}\theta^\star}{\Gamma(\delta_1)\Gamma(\delta_2)} \mu_{\mathcal{C}(\mathcal{J},\mathtt{c}^\sigma)}(\mathcal{B}) < \varepsilon \implies \mu_{\mathcal{C}(\mathcal{J},\mathtt{c}^\sigma)}(\mathcal{B}) < \frac{\varepsilon\Gamma(\delta_1)\Gamma(\delta_2)}{\mathtt{T}^{\delta_1+\delta_2} e^{\ell\mathtt{T}}\theta^\star}.$$

Letting $\delta = \frac{\varepsilon}{\mathtt{T}^{\delta_1+\delta_2} e^{\ell \mathtt{T}} \theta^\star}\left[\Gamma(\delta_1)\Gamma(\delta_2) - \mathtt{T}^{\delta_1+\delta_2} e^{\ell\mathtt{T}}\theta^\star\right]$, we obtain $\varepsilon \leq \mu_{\mathcal{C}(\mathcal{J},\mathtt{c}^\sigma)}(\mathcal{B}) < \varepsilon + \delta$.

Therefore, Υ is a Meir-Keeler condensing operator on $\mathcal{B}$. Since $\mathtt{z}$ is arbitrary, so for every $\mathtt{z} \in \mathcal{J}$, Υ satisfies all the conditions of Theorem 2.18, i.e. Υ has a fixed point in $\mathcal{B}$. Hence, the infinite systems (1.1) has a solution in $\mathcal{C}(\mathcal{J}, \mathtt{c}^\sigma)$. □

5. Applications

In this section, we present two examples to illustrate our main results.

Example 5.1 *Consider the following infinite system of tempered fractional order boundary value problems.*

$$\begin{cases} {}^{\mathtt{RL}}_{\;0}\mathbb{D}^{1.2,0.5}_{\mathtt{z}}\left[(\mathtt{z}^2+4)\left({}^{\mathtt{RL}}_{\;0}\mathbb{D}^{1.4,0.5}_{\mathtt{z}}(\vartheta_{\mathtt{j}}(\mathtt{z}))\right)\right] = \lambda_{\mathtt{j}}\left[\dfrac{e^{-\mathtt{j}\mathtt{z}^4}\cos(\mathtt{j}\mathtt{z})}{\mathtt{j}} + \displaystyle\sum_{\mathtt{i}=\mathtt{j}}^{\infty}\dfrac{\vartheta_{\mathtt{j}}(\mathtt{z})}{4\mathtt{i}^2}\right] \\ \vartheta_{\mathtt{j}}(0) = \lim\limits_{\mathtt{z}\to 0}\left[{}^{\mathtt{RL}}_{\;0}\mathbb{D}^{1.4,0.5}_{\mathtt{z}}(e^{0.5\mathtt{z}}\vartheta_{\mathtt{j}}(\mathtt{z}))\right] = 0, \\ e^{0.5}\vartheta_{\mathtt{j}}(1) = \lim\limits_{\mathtt{z}\to 1}\left[{}^{\mathtt{RL}}_{\;0}\mathbb{D}^{1.4,0.5}_{\mathtt{z}}(e^{0.5\mathtt{z}}\vartheta_{\mathtt{j}}(\mathtt{z}))\right] = 0,\ \mathtt{j}\in\mathbb{N}. \end{cases} \tag{5.1}$$

Thus, $\ell = \frac{1}{2}, \delta_1 = 1.4, \delta_2 = 1.2,\ \mathtt{p}_{\mathtt{j}}(\mathtt{z}) = \mathtt{z}^2+4,\ \mathtt{T} = 1,\ \mathcal{J} = [0,1]$, and $\varphi_{\mathtt{j}}(\mathtt{z}, \vartheta(\mathtt{z})) = \frac{e^{-\mathtt{j}\mathtt{z}^4}\cos(\mathtt{j}\mathtt{z})}{\mathtt{j}} + \sum_{\mathtt{i}=\mathtt{j}}^{\infty}\frac{\vartheta_{\mathtt{j}}(\mathtt{z})}{4\mathtt{i}^2}$. Let $\sigma_{\mathtt{j}} = \frac{1}{\mathtt{j}^2}$ for all $\mathtt{j}\in\mathbb{N}$. Now, for $\vartheta(\mathtt{z}) \in \mathcal{C}(\mathcal{J}, \mathtt{c}_0^\sigma)$, we have

$$\lim_{\mathtt{j}\to\infty}\sigma_{\mathtt{j}}\varphi_{\mathtt{j}}(\mathtt{z},\vartheta(\mathtt{z})) = \lim_{\mathtt{j}\to\infty}\left[\frac{e^{-\mathtt{j}\mathtt{z}^4}\cos(\mathtt{j}\mathtt{z})}{\mathtt{j}^3} + \frac{1}{\mathtt{j}^2}\sum_{\mathtt{i}=\mathtt{j}}^{\infty}\frac{\vartheta_{\mathtt{j}}(\mathtt{z})}{4\mathtt{i}^2}\right] = 0.$$

Next, let for any $\nu(\mathtt{z}) = \left(\nu_{\mathtt{j}}(\mathtt{z})\right)_{\mathtt{j}=1}^{\infty} \in \mathcal{C}(\mathcal{J}, \mathtt{c}_0^\sigma)$. Let $\varepsilon > 0$ be given and $\delta = \dfrac{24\varepsilon}{\pi^2}$ such that $\|\vartheta(\mathtt{z}) - \nu(\mathtt{z})\|_{\mathcal{C}(\mathcal{J},\mathtt{c}_0^\sigma)} < \delta$. Then

$$\begin{aligned} \|(\Phi\vartheta)(\mathtt{z}) - (\Phi\nu)(\mathtt{z})\|_{\mathtt{c}_0^\sigma} &= \sup_{\mathtt{j}\in\mathbb{N}}\left\{\sigma_{\mathtt{j}}\left|\varphi_{\mathtt{j}}(\mathtt{z},\vartheta(\mathtt{z})) - \varphi_{\mathtt{j}}(\mathtt{z},\nu(\mathtt{z}))\right|\right\} \\ &= \sup_{\mathtt{j}\in\mathbb{N}}\left\{\frac{1}{\mathtt{j}^2}\sum_{\mathtt{i}=\mathtt{j}}^{\infty}\frac{1}{4\mathtt{i}^2}\left|\vartheta_{\mathtt{j}}(\mathtt{z}) - \nu_{\mathtt{j}}(\mathtt{z})\right|\right\} \\ &\leq \frac{\pi^2}{24}\|\vartheta_{\mathtt{j}}(\mathtt{z}) - \nu_{\mathtt{j}}(\mathtt{z})\|_{\mathtt{c}_0^\sigma} < \varepsilon. \end{aligned}$$

Thus, $((\Phi\vartheta)(\mathtt{z}))_{\mathtt{z}\in\mathcal{J}}$ is equicontinuous on $\mathtt{c}_0^\sigma$. Moreover, for $\mathtt{z}\in\mathcal{J}$ and $\mathtt{j}\in\mathbb{N}$, we have

$$\begin{aligned} |\varphi_{\mathtt{j}}(\mathtt{z},\vartheta(\mathtt{z}))| &\leq \frac{e^{-\mathtt{j}\mathtt{z}^4}|\cos(\mathtt{j}\mathtt{z})|}{\mathtt{j}} + \sum_{\mathtt{i}=\mathtt{j}}^{\infty}\frac{1}{4\mathtt{i}^2}|\vartheta_{\mathtt{j}}(\mathtt{z})| \\ &\leq \frac{e^{-\mathtt{j}\mathtt{z}^2}}{\mathtt{j}} + \frac{\pi^2}{24}|\vartheta_{\mathtt{j}}(\mathtt{z})|, \end{aligned}$$

where $\xi_j(z) = \dfrac{e^{-jz^4}}{j}$ and $\zeta_j(z) = \dfrac{\pi^2}{24}$. Thus, $\zeta^\star = \dfrac{\pi^2}{24}$. We note that $\big(\sigma\xi_j(z)\big) = \Big(e^{-jz^4}/j^3\Big)$ converges uniformly to zero on $\mathcal{J}$ and the sequence $\zeta_j(z)$ is equibounded on $\mathcal{J}$. Moreover,

$$\frac{T^{\delta_1+\delta_2}e^{\ell T}\zeta^\star}{\Gamma(\delta_1)\Gamma(\delta_2)} = \frac{\sqrt{e}\pi^2}{24\Gamma(1.4)\Gamma(1.2)} < 1.$$

Hence, by Theorem 3.1 the infinite systems (5.1) has a solution in $\mathcal{C}(\mathcal{J}, c_0^\sigma)$ provided

$$\lambda < \max_{z\in[0,1]} \frac{1}{z^2+4} = \frac{1}{4}.$$

Example 5.2 *Consider the system of tempered fractional order two-point boundary value problems.*

$$\begin{cases} {}^{\mathrm{RL}}_{\ 0}\mathbb{D}_z^{1.8,1}\Big[\sqrt{z^2+25}\big({}^{\mathrm{RL}}_{\ 0}\mathbb{D}_z^{1.1,1}(\vartheta_j(z))\big)\Big] = \lambda_j\left[\dfrac{e^{-jz}\sin(jz)}{j} + \displaystyle\sum_{i=j}^{\infty}\dfrac{\vartheta_j(z)}{7i^2}\right] \\ \vartheta_j(0) = \lim\limits_{z\to 0}\big[{}^{\mathrm{RL}}_{\ 0}\mathbb{D}_z^{1.1,1}(e^z\vartheta_j(z))\big] = 0, \\ e\vartheta_j(1) = \lim\limits_{z\to 1}\big[{}^{\mathrm{RL}}_{\ 0}\mathbb{D}_z^{1.1,1}(e^z\vartheta_j(z))\big] = 0,\ j\in\mathbb{N}. \end{cases} \tag{5.2}$$

Thus, $\ell = 1, \delta_1 = 1.1,\ \delta_2 = 1.8,\ p_j(z) = \sqrt{z^2+25},\ T = 1,\ \mathcal{J} = [0,1]$, and $\varphi_j(z,\vartheta(z)) = \frac{e^{-jz}\sin(jz)}{j} + \sum_{i=j}^{\infty}\frac{\vartheta_j(z)}{7i^2}$. Let $\sigma_j = \frac{1}{j^2}$ for all $j\in\mathbb{N}$. Now, for $\vartheta(z)\in\mathcal{C}(\mathcal{J}, c^\sigma)$, we have

$$\lim_{j\to\infty}\sigma_j\varphi_j(z,\vartheta(z)) = \lim_{j\to\infty}\left[\frac{\sin(jz)}{j^3} + \frac{1}{j^2}\sum_{i=j}^{\infty}\frac{\vartheta_j(z)}{7i^2}\right] = 0.$$

This shows that $\big(\varphi_j(z,\vartheta(z))\big)\in c^\sigma$. Next, let for any $\nu(z) = \big(\nu_j(z)\big)_{j=1}^{\infty}\in\mathcal{C}(\mathcal{J}, c^\sigma)$. Let $\varepsilon > 0$ be given and $\delta = \dfrac{12\varepsilon}{\pi^2}$ such that $\|\vartheta(z)-\vartheta(z)\|_{\mathcal{C}(\mathcal{J},c^\sigma)} < \delta$. Then

$$\begin{aligned} \|(\Phi\vartheta)(z) - (\nu\vartheta)(z)\|_{c^\sigma} &= \sup_{j\in\mathbb{N}}\Big\{\sigma_j\big|\varphi_j(z,\vartheta(z)) - \varphi_j(z,\nu(z))\big|\Big\} \\ &= \sup_{j\in\mathbb{N}}\left\{\frac{1}{j^2}\sum_{i=j}^{\infty}\frac{1}{7i^2}\big|\vartheta_j(z)-\nu_j(z)\big|\right\} \\ &\le \frac{\pi^2}{42}\|\vartheta_j(z)-\nu_j(z)\|_{c_0^\sigma} < \varepsilon. \end{aligned}$$

Thus, $((\Phi\vartheta)(z))_{z\in\mathcal{J}}$ is equicontinuous on c^σ. Moreover, for $z\in\mathcal{J}$ and $j\in\mathbb{N}$, we have $\eta_j(z) = \dfrac{\sin jz}{j}$ and $\theta_j(z) = \dfrac{\pi^2}{42}$. Thus, $\theta^\star = \dfrac{\pi^2}{42}$. We note that $\big(\sigma\eta_j(z)\big) = \left(\dfrac{\sin jz}{j^3}\right)$ converges uniformly to zero on $\mathcal{J}$ and the sequence $\theta_j(z)$ is convergent on $\mathcal{J}$. Moreover,

$$\frac{T^{\delta_1+\delta_2}e^{\ell T}\theta^{\star}}{\Gamma(\delta_1)\Gamma(\delta_2)} = \frac{e\pi^2}{42\Gamma(1.1)\Gamma(1.8)} < 1.$$

Hence, by Theorem 4.1 the infinite systems (5.2) has a solution in $\mathcal{C}(\mathcal{J}, c^{\sigma})$ provided

$$\lambda < \max_{z\in[0,1]} \frac{1}{\sqrt{t^2+25}} = \frac{1}{5}.$$

6. Conclusion and future work

In the present paper, we studied an infinite system of tempered fractional order boundary value problem. The fractional derivative used in our problem is the so-called tempered fractional Riemann–Liouville derivative, which generalizes well-known Riemann–Liouville fractional derivative. By applying the Hausdorff measure of noncompactness technique and using the Meir–Keeler fixed point theorem, we examined the existence of solution to this infinite system. This investigation has been performed in new sequence spaces $\mathcal{C}(\mathcal{J}, c_0^{\sigma})$ and $\mathcal{C}(\mathcal{J}, c^{\sigma})$ called tempered sequence spaces. Finally, numerical examples are also provided to illustrate our obtained results. In the future, the following aspects can be explored further:

(1) Further investigation is needed to study infinite system of singular temepered fractional Sturm–Liouville boundary value problem. Here taking singularities over $p_j(z)$ is challenging.

(2) The idea used in this paper can be further developed to study infinite systems of fractional difference equations [41] and dynamic equations on time scales [29, 38, 39] etc.

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References

[1] Aghajani A, Mursaleen M, Haghighi AS. Fixed point theorems for Meir-Keeler condensing operators via measure of noncompactness. Acta Mathematica Scientia 2015; 35 (3): 552-566.

[2] Alotaibi A, Mursaleen M, Mohiuddine SA. Application of measure of noncompactness to infinite system of linear equations in sequence spaces. Bulletin of the Iranian Mathematical Society 2015; 41: 519-527.

[3] Bai Z, Lu H. Positive solutions for a boundary value problem of nonlinear fractional differential equations. Journal of Mathematical Analysis and Applications 2005; 311: 495-505.

[4] Banas, J. On measures of noncompactness in Banach spaces. Commentationes Mathematicae Universitatis Carolinae 1980; 21: 131-143.

[5] Banas J, Krajewska M. Existence of solutions for infinite systems of differential equations in spaces of tempered sequences, Electronic Journal of Differential Equations, 2017; (2017)(60): 1-28.

[6] Banas J, Mursaleen M. Sequence spaces and measures of noncompactness with applications to differential and integral equations. New Delhi, India: Springer, 2014.

[7] Cartea A, Castillo-Negrete DD. Fractional diffusion models of option prices in markets with jumps. Physica A: Statistical Mechanics and its Applications 2007; 374 (2): 749-763.

[8] Cichon M, Salem HAH. On the lack of equivalence between differential and integral forms of the Caputo-type fractional problems. Journal of Pseudo-Differential Operators and Applications 2020; 11: 1869-1895.

[9] Das A, Hazarika B, Agarwal RP, Nashine HK. Solvability of infinite systems of fractional differential equations in the spaces of tempered sequences. Filomat 2019; 33(17): 5519-5530.

[10] Debanath L. Recent applications of fractional calculus to science and engineering. International Journal of Mathematics and Mathematical Sciences 2003; Article ID 753601, 30 pages. doi:10.1155/S0161171203301486

[11] Hanyga A. Wave propagation in media with singular memory. Mathematical and computer modelling 2001; 34 (12-13): 1399-1421.

[12] He JQ, Dong Y, Li ST, Liu HL, Yu YJ et al. Study on force distribution of the tempered glass based on laser interference technology. Optik 2015; 126 (24): 5276-5279.

[13] Khuddush M, Prasad KR. Infinitely many positive solutions for an iterative system of conformable fractional order dynamic boundary value problems on time scales. Turkish Journal of Mathematics 2021. doi: 10.3906/mat-2103-117

[14] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies 204, Elsevier, North Holland, 2006.

[15] Kuratowski K. Sur les espaces completes. Fundamenta Mathematicae 1930; 15: 301-309 (in French).

[16] Li C, Deng W, Zhao L. Well-posedness and numerical algorithm for the tempered fractional ordinary differential equations. Discrete and Continuous Dynamical Systems 2019; 24 (4): 1989-2015.

[17] Meerschaert MM, Zhang Y, Baeumer B. Tempered anomalous diffusion in heterogeneous systems. Geophysical Research Letters 2008; 35 (17): 1-5.

[18] Meerschaert MM, Sabzikar F, Phanikumar MS, Zeleke A. Tempered fractional time series model for turbulence in geophysical flows. Journal of Statistical Mechanics: Theory and Experiment 2014; 2014 (9): P09023.

[19] Meir A, Keeler E. A theorem on contraction mappings. Journal of Mathematical Analysis and Applications 1969; 28: 326-329.

[20] Mursaleen M, Bilal B, Rizvi SMH. Applications of measure of noncompactness to infinite system of fractional differential equations. Filomat 2017; 31 (11): 3421-3432.

[21] Mursaleen M, Mohiuddine SA. Applications of measures of noncompactness to the infinite system of differential equations in ℓ_p spaces. Nonlinear Analysis 2012; 75: 2111-2115.

[22] Mursaleen M, Rizvi SMH, Samet B. Solvability of a class of boundary value problems in the space of convergent sequences. Applicable Analysis 2018; 97 (11): 1829-1845.

[23] Muthaiah S, Baleanu D, Thangaraj NG. Existence and Hyers-Ulam type stability results for nonlinear coupled system of Caputo-Hadamard type fractional differential equations. AIMS Mathematics 2021; 6 (1): 168-194.

[24] Pandey PK, Pandey RK, Yadav S, Agrawal OP. Variational approach for tempered fractional Sturm-Liouville Problem. International Journal of Applied and Computational Mathematics 2021; 7 (51). doi: 10.1007/s40819-021-01000-x

[25] Prasad KR, Khuddush M, Veeraiah P. Countably many positive solutions for singular R-L fractional order bvp with R-S integral boundary conditions. Nonlinear Studies 2020; 27 (4): 1075-1089.

[26] Prasad KR, Khuddush M, Leela D. Existence of solutions for infinite systems of regular fractional Sturm-Liouville problems in the spaces of tempered sequences. Tbilisi Mathematical Journal 2020; 13 (4), 193-209. doi: 10.32513/tbilisi/1608606058

[27] Rabbani M, Das A, Hazarika B, Arab R. Measure of noncompactness of a new space of tempered sequences and its application on fractional differential equations. Chaos, Solitons and Fractals 2020; 140, 110221.

[28] Rosenau P. Tempered diffusion: A transport process with propagating fronts and inertial delay. Physical Review A 1992; 46 (12): R7371.

[29] Rosiak MN. Uncountably many nonoscillatory bounded solutions to second-order nonlinear neutral dynamic equations. Turkish Journal of Mathematics 2019; 43 (3): 1699-1711.

[30] Sabzikar F, Meerschaert MM, Chen J. Tempered fractional calculus. Journal of Computational Physics 2015; 293: 14-28.

[31] Salem A, Alshehri, HM. Almaghamsi, L. Measure of noncompactness for an infinite system of fractional Langevin equation in a sequence space. Advances in Difference Equations 2021, 132 (2021). doi: 10.1186/s13662-021-03302-2

[32] Srivastava HM. Diabetes and its resulting complications: Mathematical modeling via fractional calculus. Public Health Open Access 2020; 4 (3): 1-5.

[33] Srivastava HM, Das A, Hazarika B, Mohiuddine SA. Existence of solutions of infinite systems of differential equations of general order with boundary conditions in the spaces c_0 and ℓ_1 via the measure of noncompactness. Mathematical Methods in the Applied Sciences 2018; 41 (10): 3558-3569.

[34] Srivastava HM, Das A, Hazarika B, Mohiuddine SA. Existence of solution for non-linear functional integral equations of two variables in Banach algebra. Symmetry 2019; 11: Article ID 674, 1-16.

[35] Srivastava HM, Saad KM, Khader MM, An efficient spectral collocation method for the dynamic simulation of the fractional epidemiological model of the Ebola virus. Chaos, Solitons and Fractals 2020; 140: 1-7.

[36] Tarasov VE, Tarasova VV. Time dependent fractional dynamics with memory in quantum and economic physics. Annals of Physics 2017; 383: 579-599.

[37] Yadav S, Pandey RK, Pandey PK. Numerical approximation of tempered fractional Sturm-Liouville problem with application in fractional diffusion equation. International Journal of Numerical Methods in Fluids 2020. doi: 10.1002/fld.4901

[38] Yantir A, Kubiaczyk I, Sikorska-Nowak A. Nonlinear Sturm-Liouville dynamic equation with a measure of noncompactness in Banach spaces. Bulletin of the Belgian Mathematical Society Simon Stevin 2013; 20: 587-601.

[39] Yantir A, Kubiaczyk I, Sikorska-Nowak A. Carathéodory solutions of Sturm-Liouville dynamic equation with a measure of noncompactness in Banach spaces. Open Mathematics 2015; 13 (1).

[40] Zaky MA. Existence, uniqueness and numerical analysis of solutions of tempered fractional boundary value problems. Applied Numerical Mathods 2019; 145: 429-457.

[41] Zhou WX, Liu HZ. Existence solutions for boundary value problem of nonlinear fractional q-difference equations. Advances in Difference Equations 2013; 113 (2013). doi: 10.1186/1687-1847-2013-113