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On bounded solutions of a second-order iterative boundary value problem

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Abstract: In this article, we investigate a second-order iterative differential equation with boundary conditions. The use of the principle of contraction mappings and the Schauder's fixed point theorem allows us to prove some existence and uniqueness results. Finally, an example is given to check the validity of our findings, which are new, and complete some published manuscripts to some degree.

Key words: Continuous dependence, bounded solutions, iterative functional differential equation, fixed point theorem

1. Introduction

Due to their promising applications in many areas, the study of iterative differential equations, which have a history of more than a century [13] and can be regarded as a particular class of functional differential equations with delays that depend on time and the state variable, has received a lot of attention. We can find them in infectious disease transmission models in epidemiology, population dynamics models in ecology, hematopoiesis models in biology, and so on (see [3, 8, 11, 20–22]). To the best of our knowledge, in spite of their importance in applications, the iterative terms make them very difficult to study and, hence, the papers published on this topic are very scarce. For more detailed works on first-order iterative problems, we refer the readers to [3, 8, 12, 24] and the references cited therein. For second-order iterative problems, we cite the following works:

In [5], the authors employed Schauder's fixed point theorem to prove some existence results on periodic solutions of an equation of the form

$$\begin{aligned}\frac{d^2}{dt^2}x(t) + p(t)\frac{d}{dt}x(t) + q(t)x(t)\\ = \frac{d}{dt}g\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right) + f\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right),\end{aligned}$$

where p and q are two T-periodic continuous functions, $f(t, x_1, x_2, ..., x_n)$ and $g(t, x_1, x_2, ..., x_n)$ are supposed to be periodic in t with period T and globally Lipschitz in $x_1, x_2, ..., x_n$.

By means of the same last fixed point theorem, Kaufmann [17] studied the following problem:

$$x''(t) = h\left(t, x(t), x^{[2]}(t)\right), \ a \leq t \leq b,$$

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with

$$x(a) = a, \quad x(b) = b,$$

or

$$x(a) = b, \quad x(b) = a.$$

In [7], by virtue of the Krasnoselskii's fixed point theorem, the authors proved the existence of positive periodic solutions for the following second-order iterative differential equation;

$$\frac{d^2}{dt^2}x(t) + p(t)\frac{d}{dt}x(t) + q(t)x(t) = \frac{d}{dt}f\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right) + \sum_{i=1}^{n} c_i(t)\, x^{[i]}(t).$$

For third-order iterative problems, there are only three works by the second and the third authors of this paper [6, 10, 19], and, so far, no work has been done on fourth order or higher order ones.

Motivated by the above works and for the purpose to contribute to filling these gaps in the literature, we focus on second order iterative differential equations of the form

$$y''(t) + h\left(t, y(t), y^{[2]}(t), ..., y^{[n]}(t)\right) = 0, \quad -b \leq t \leq b, \tag{1.1}$$

$$y(-b) = \eta_1, \ y(b) = \eta_2, \ \eta_1, \eta_2 \in [-b, b], \tag{1.2}$$

where $y^{[2]}(t) = y(y(t)), ..., y^{[n]}(y) = y^{[n-1]}(y(t))$ and h is a continuous function from $[-b, b] \times \mathbb{R}^n$ to $\mathbb{R}$.

This manuscript is planned as follows: Section 2 is dedicated to present some preliminaries that play a key role in establishing our main results. In Sections 3, we convert our iterative problem into an equivalent integral equation, and we also prove two important properties of the Green's kernel for using them with Schauder's fixed point theorem to establish the existence results. Uniqueness and the continuous dependence of the solution will be proved in Section 4 and Section 5. In Section 6, an example is displayed. Finally, we draw a brief conclusion in the last section.

2. Preliminaries

In order to make the iterates well defined, we will define an appropriate subset of $\mathcal{C}[-b, b]$, where $b > 0$.

Consider the Banach space $\mathcal{X} = (\mathcal{C}([-b, b], \mathbb{R}), \|.\|)$ with the norm

$$\|x\|_{\mathcal{X}} = \sup_{t \in [-b, b]} |x(t)|.$$

For $\alpha \in [0, b]$ and $\beta \geq 0$, we define the set $\mathcal{CB}(\alpha, \beta)$ as follows:

$$\mathcal{CB}(\alpha, \beta) = \{y \in \mathcal{X}, \ -\alpha \leq y \leq \alpha, \ |y(t_2) - y(t_1)| \leq \beta |t_2 - t_1|, \ \forall t_1, t_2 \in [-b, b]\}.$$

Then, $\mathcal{CB}(\alpha, \beta)$ is a convex bounded and closed subset of $\mathcal{X}$.

Lemma 2.1 *[24] If $x, y \in \mathcal{CB}(\alpha, \beta)$, then*

$$\left\|x^{[k]} - y^{[k]}\right\| \leq \sum_{j=0}^{k-1} \beta^j \|x - y\|, \ k = 1, 2, ...$$

Remark 2.2 *It follows from Arzelà-Ascoli theorem that $\mathcal{CB}(\alpha, \beta) \subset \mathcal{X}$ is compact.*

3. Existence results

We are interested in the existence of solutions of (1.1)–(1.2). For reaching this aim, we will rewrite our problem as an equivalent integral equation before using Schauder's fixed point theorem.

We assume that

$$\left|h\left(t, y_1, ..., y_n\right)-h\left(t, z_1, ..., z_n\right)\right| \leq \sum_{i=1}^n a_i\left\|y_i-z_i\right\|, \tag{3.1}$$

where $a_1, a_2, ..., a_n$ are positive constants.

The following lemma will be needed for defining the integral operator in the sequel.

Lemma 3.1 *If $b \neq 0$, then $y \in \mathcal{CB}(\alpha, \beta) \cap \mathcal{C}^2([-b, b], \mathbb{R})$ is a solution of (1.1)-(1.2) if and only if $y \in \mathcal{CB}(\alpha, \beta)$ is a solution of the following integral equation:*

$$y(t)=\eta_1+\frac{\eta_2-\eta_1}{2 b}(t+b)+\int_{-b}^b G(t, s) h\left(s, y(s), y^{[2]}(s), ..., y^{[n]}(s)\right) d s, \tag{3.2}$$

where

$$G(t, s)=\frac{1}{2 b}\left\{\begin{array}{ll}(t-b)(s+b), & -b \leq s \leq t \leq b, \\ (t+b)(s-b), & -b \leq t \leq s \leq b.\end{array}\right. \tag{3.3}$$

Proof Let $y \in \mathcal{CB}(\alpha, \beta) \cap \mathcal{C}^2([-b, b], \mathbb{R})$ be a solution of (1.1)-(1.2). An integration of equation (1.1) from $-b$ to t gives

$$\int_{-b}^t y''(s)\, d s=\int_{-b}^t h\left(s, y(s), y^{[2]}(s), ..., y^{[n]}(s)\right) d s.$$

Therefore,

$$y'(t)=y'(-b)+\int_{-b}^t h\left(s, y(s), y^{[2]}(s), ..., y^{[n]}(s)\right) d s.$$

Integrating again from $-b$ to t, we obtain

$$\int_{-b}^t y'(s)\, d s=\int_{-b}^t y'(-b)\, d s+\int_{-b}^t\left[\int_{-b}^x h\left(s, y(s), y^{[2]}(s), ..., y^{[n]}(s)\right) d s\right] d x,$$

which gives

$$y(t)=y(-b)+y'(-b)(t+b)+\int_{-b}^t(t-s) h\left(s, y(s), y^{[2]}(s), ..., y^{[n]}(s)\right) d s.$$

Using (1.2), we get

$$y(t)=\eta_1+y'(-b)(t+b)+\int_{-b}^t(t-s) h\left(s, y(s), y^{[2]}(s), ..., y^{[n]}(s)\right) d s, \tag{3.4}$$

and

$$y(b)=\eta_2=\eta_1+y'(-b)\, 2 b+\int_{-b}^b(b-s) h\left(s, y(s), y^{[2]}(s), ..., y^{[n]}(s)\right) d s.$$

So

$$y'(-b) = \frac{\eta_2 - \eta_1}{2b} - \frac{1}{2b}\int_{-b}^{b} (b-s)\, h\left(s, y(s), y^{[2]}(s), ..., y^{[n]}(s)\right) ds. \tag{3.5}$$

By substituting (3.4) in (3.5), we get

$$\begin{aligned} y(t) &= \eta_1 + \frac{\eta_2 - \eta_1}{2b}(t+b) - \frac{1}{2b}\int_{-b}^{b} (t+b)(b-s)\, h\left(s, y(s), y^{[2]}(s), ..., y^{[n]}(s)\right) ds \\ &+ \int_{-b}^{t} (t-s)\, h\left(s, y(s), y^{[2]}(s), ..., y^{[n]}(s)\right) ds \\ &= \eta_1 + \frac{\eta_2 - \eta_1}{2b}(t+b) - \frac{1}{2b}\int_{-b}^{t} (b-t)(b+s)\, h\left(s, y(s), y^{[2]}(s), ..., y^{[n]}(s)\right) ds \\ &- \frac{1}{2b}\int_{t}^{b} (t+b)(b-s)\, h\left(s, y(s), y^{[2]}(s), ..., y^{[n]}(s)\right) ds. \end{aligned}$$

Consequently,

$$y(t) = \eta_1 + \frac{\eta_2 - \eta_1}{2b}(t+b) + \int_{-b}^{b} G(t,s)\, h\left(s, y(s), y^{[2]}(s), ..., y^{[n]}(s)\right) ds.$$

On the other hand, it is easy to obtain (1.1)–(1.2) by a direct derivation of (3.2). □

Lemma 3.2 *Green's function* (3.3) *satisfies*

$$\int_{-b}^{b} |G(t,s)|\, ds \leq \frac{1}{2}b^2. \tag{3.6}$$

Proof We have

$$\int_{-b}^{b} |G(t,s)|\, ds \leq \frac{1}{2b}\int_{-b}^{t} |(t-b)(s+b)|\, ds + \frac{1}{2b}\int_{t}^{b} |(t+b)(s-b)|\, ds.$$

Since $(t-b)(s+b) \leq 0$ for $-b \leq s \leq t$ and $(t+b)(s-b) \leq 0$ for $t \leq s \leq b$, then

$$|(t-b)(s+b)| = (b-t)(b+s) \text{ and } |(t+b)(s-b)| = (b-s)(b+t).$$

Hence,

$$\begin{aligned} \int_{-b}^{b} |G(t,s)|\, ds &= \frac{1}{2b}\int_{-b}^{t} (b-t)(b+s)\, ds + \frac{1}{2b}\int_{t}^{b} (b-s)(b+t)\, ds \\ &= \frac{1}{2}b^2 - \frac{1}{2}t^2 \\ &\leq \frac{1}{2}b^2. \end{aligned}$$

Thus, the lemma is proved. □

Now, to apply Schauder's fixed point theorem, we construct an operator $\digamma$ from $\mathcal{CB}(\alpha,\beta)$ to $\mathcal{X}$ as follows:

$$(\digamma\psi)(t)=\eta_1+\frac{\eta_2-\eta_1}{2b}(t+b)+\int_{-b}^{b}G(t,s)\,h\left(s,\psi(s),\psi^{[2]}(s),...,\psi^{[n]}(s)\right)ds. \tag{3.7}$$

By Lemma 3.1, ψ is a solution of (1.1)-(1.2) if and only if ψ is a fixed point of $\digamma$. So, our main goal here is to show that $\digamma$ has at least one fixed point.

Lemma 3.3 *If condition (3.1) holds. Then, the operator $\digamma$ given by (3.7) is continuous.*

Proof Clearly, $\digamma$ is well defined and for $\varphi,\psi\in\mathcal{CB}(\alpha,\beta)$, we get

$$|(\digamma\varphi)(t)-(\digamma\psi)(t)|\le\int_{-b}^{b}|G(t,s)|\left|h\left(s,\varphi(s),...,\varphi^{[n]}(s)\right)-h\left(s,\psi(s),...,\psi^{[n]}(s)\right)\right|ds.$$

Using (3.1) and Lemma 2.1, we obtain

$$\begin{aligned}\left|h\left(s,\varphi(s),...,\varphi^{[n]}(s)\right)-h\left(s,\psi(s),...,\psi^{[n]}(s)\right)\right|&\le\sum_{i=1}^{n}a_i\left\|\varphi^{[i]}-\psi^{[i]}\right\|\\&\le\sum_{i=1}^{n}a_i\sum_{j=0}^{i-1}\beta^j\|\varphi-\psi\|.\end{aligned} \tag{3.8}$$

From (3.6) and (3.8), it yields that

$$\begin{aligned}|(\digamma\varphi)(t)-(\digamma\psi)(t)|&\le\sum_{i=1}^{n}a_i\sum_{j=0}^{i-1}\beta^j\|\varphi-\psi\|\int_{-b}^{b}|G(t,s)|\,ds\\&\le\frac{b^2}{2}\sum_{i=1}^{n}a_i\sum_{j=0}^{i-1}\beta^j\|\varphi-\psi\|,\end{aligned}$$

which proves that the operator $\digamma$ is continuous. □

Lemma 3.4 *If condition (3.1) holds and if*

$$2|\eta_1|+|\eta_2|+\frac{b^2}{2}\left(h_0+\alpha\sum_{i=1}^{n}a_i\sum_{j=0}^{i-1}\beta^j\right)\le\alpha, \tag{3.9}$$

where

$$h_0=\max_{s\in[0,b]}|h(s,0,...,0)|,$$

then $-\alpha\le(\digamma\psi)(t)\le\alpha$ for all $t\in[-b,b]$ and $\psi\in\mathcal{CB}(\alpha,\beta)$.

Proof If $\psi\in\mathcal{CB}(\alpha,\beta)$, we have

$$\begin{aligned}|(\digamma\psi)(t)|&=\left|\eta_1+\frac{\eta_2-\eta_1}{2b}(t+b)+\int_{-b}^{b}G(t,s)\,h\left(s,\psi(s),\psi^{[2]}(s),...,\psi^{[n]}(s)\right)ds\right|\\&\le2|\eta_1|+|\eta_2|+\int_{-b}^{b}|G(t,s)|\left|h\left(s,\psi(s),\psi^{[2]}(s),...,\psi^{[n]}(s)\right)\right|ds.\end{aligned}$$

By virtue of (3.1) and Lemma 2.1, we arrive at

$$\left|h\left(s,\psi^{[1]}\left(s\right),...,\psi^{[n]}\left(s\right)\right)\right| \leq h_0+\alpha\sum_{i=1}^{n}a_i\sum_{j=0}^{i-1}\beta^j. \tag{3.10}$$

Further, it follows from (3.6), (3.9) and (3.10) that

$$\left|\left(\digamma\psi\right)(t)\right| \leq 2\left|\eta_1\right|+\left|\eta_2\right|+\frac{b^2}{2}\left(h_0+\alpha\sum_{i=1}^{n}a_i\sum_{j=0}^{i-1}\beta^j\right)\leq\alpha.$$

Consequently, $-\alpha\leq\left(\digamma\psi\right)(t)\leq\alpha$ for all $t\in[-b,b]$ and $\psi\in\mathcal{CB}\left(\alpha,\beta\right).$ □

Lemma 3.5 *Green's function* (3.3) *satisfies*

$$\int_{-b}^{b}\left|G\left(t_2,s\right)-G\left(t_1,s\right)\right|\leq 5b\left|t_2-t_1\right|.$$

Proof Let $t_1,t_2\in[-b,b]$ with $t_1\leq t_2$ and we set

$$G_1\left(t,s\right)=\frac{1}{2b}\left(t-b\right)\left(s+b\right),$$
$$G_2\left(t,s\right)=\frac{1}{2b}\left(t+b\right)\left(s-b\right).$$

We have

$$\begin{aligned}\int_{-b}^{b}\left|G\left(t_2,s\right)-G\left(t_1,s\right)\right|ds &= \int_{-b}^{t_1}\left|G_1\left(t_2,s\right)-G_1\left(t_1,s\right)\right|ds\\ &+\int_{t_1}^{t_2}\left|G_1\left(t_2,s\right)-G_2\left(t_1,s\right)\right|ds\\ &+\int_{t_2}^{b}\left|G_2\left(t_2,s\right)-G_2\left(t_1,s\right)\right|ds.\end{aligned}$$

Since

$$\begin{aligned}\int_{-b}^{t_1}\left|G_1\left(t_2,s\right)-G_1\left(t_1,s\right)\right|ds &= \int_{-b}^{t_1}\left|\frac{1}{2b}\left(t_2-b\right)\left(s+b\right)-\frac{1}{2b}\left(t_1-b\right)\left(s+b\right)\right|ds\\ &\leq\frac{1}{2b}\left|t_2-t_1\right|\int_{-b}^{t_1}\left(b+s\right)ds\\ &=\frac{1}{4b}\left|t_2-t_1\right|\left(b+t_1\right)^2\\ &\leq b\left|t_2-t_1\right|,\end{aligned}$$

$$\int_{t_1}^{t_2} |G_1(t_2,s) - G_2(t_1,s)|\, ds = \int_{t_1}^{t_2} \left| \frac{1}{2b}(t_2-b)(s+b) - \frac{1}{2b}(t_1+b)(s-b) \right| ds$$
$$\leq \frac{1}{2b}\int_{t_1}^{t_2}(b-t_2)(s+b)\, ds + \frac{1}{2b}\int_{t_1}^{t_2}(t_1+b)(b-s)\, ds$$
$$\leq 3b\,|t_2-t_1|,$$

and

$$\int_{t_2}^{b} |G_2(t_2,s) - G_2(t_1,s)|\, ds = \int_{t_2}^{b} \left| \frac{1}{2b}(t_2+b)(s-b) - \frac{1}{2b}(t_1+b)(s-b) \right| ds$$
$$\leq \frac{1}{2b}|t_2-t_1| \int_{t_2}^{b}(b-s)\, ds$$
$$= \frac{1}{4b}|t_2-t_1|(b-t_2)^2$$
$$\leq b\,|t_2-t_1|.$$

then,

$$\int_{-b}^{b} |G(t_2,s) - G(t_1,s)|\, ds \leq 5b\,|t_2-t_1|.$$

The proof is now complete. □

Lemma 3.6 *If condition (3.1) holds and if*

$$\frac{|\eta_1-\eta_2|}{2|b|} + 5b\left(h_0 + \alpha\sum_{i=1}^{n} a_i \sum_{j=0}^{i-1}\beta^j\right) \leq \beta, \tag{3.11}$$

then $|(F\psi)(t_2) - (F\psi)(t_1)| \leq \beta\,|t_2-t_1|$ *for all* $t_1,t_2 \in [-b,b]$ *and* $\psi \in \mathcal{CB}(\alpha,\beta)$.

Proof If $t_1,t_2 \in [-b,b]$ and $\psi \in \mathcal{CB}(\alpha,\beta)$ with $t_1 \leq t_2$, we have

$$|(F\psi)(t_2) - (F\psi)(t_1)| \leq \left|\eta_1 + \frac{\eta_2-\eta_1}{2b}(t_2+b) - \eta_1 - \frac{\eta_2-\eta_1}{2b}(t_1+b)\right|$$
$$+ \int_{-b}^{b} |G(t_2,s) - G(t_1,s)| \left| h\left(s, \psi(s), \psi^{[2]}(s), ..., \psi^{[n]}(s)\right)\right| ds.$$

From Lemma 3.5 and (3.10), we deduce that

$$|(F\psi)(t_2) - (F\psi)(t_1)| \leq \frac{|\eta_1-\eta_2|}{2|b|}|t_2-t_1| + 5b\left(\alpha\sum_{i=1}^{n} a_i \sum_{j=0}^{i-1}\beta^j + h_0\right)|t_2-t_1|$$
$$= \left(\frac{|\eta_1-\eta_2|}{2|b|} + 5b\left(\alpha\sum_{i=1}^{n} a_i \sum_{j=0}^{i-1}\beta^j + h_0\right)\right)|t_2-t_1|.$$

Thanks to (3.11), we have

$$|(\mathcal{F}\psi)(t_2) - (\mathcal{F}\psi)(t_1)| \leq \beta |t_2 - t_1|,$$

which completes the proof. □

Remark 3.7 *It follows from Lemmas 3.4 and 3.6 that $\mathcal{F}$ maps $\mathcal{CB}(\alpha,\beta)$ into itself, i.e.* $\mathcal{F}(\mathcal{CB}(\alpha,\beta)) \subset \mathcal{CB}(\alpha,\beta)$.

Theorem 3.8 *Suppose that conditions (3.1), (3.9) and (3.10) hold. Then, problem (1.1)–(1.2) admits at least one bounded solution y in $\mathcal{CB}(\alpha,\beta)$.*

Proof Thanks to Remark 2.2, $\mathcal{CB}(\alpha,\beta)$ is compact. Furthermore, from Lemma 3.3 and Remark 3.7, all requirements of Schauder's fixed point theorem are fulfilled. Then, $\mathcal{F}$ has a fixed point y in $\mathcal{CB}(\alpha,\beta)$ such that $\mathcal{F}y = y$, which means that y is a solution of (1.1)–(1.2). □

4. Uniqueness

Now, we state and prove the uniqueness result. To be more precise, by taking into account the previous assumptions and under an additional condition, we prove that $\mathcal{F}$ is a contraction mapping. So, by virtue of the principle of contraction mappings, (1.1)–(1.2) has a unique solution.

Theorem 4.1 *Besides the hypotheses of Theorem 3.8, we suppose that*

$$\frac{1}{2}b^2\sum_{i=1}^{n} a_i \sum_{j=0}^{i-1}\beta^j < 1. \tag{4.1}$$

Then, problem (1.1)–(1.2) has a unique solution in $\mathcal{CB}(\alpha,\beta)$.

Proof Let $\varphi,\psi \in \mathcal{CB}(\alpha,\beta)$. Proceeding as in the proof of Lemma 3.3, we may show that

$$|(\mathcal{F}\varphi)(t) - (\mathcal{F}\psi)(t)| \leq \left(\frac{b^2}{2}\sum_{i=1}^{n} a_i \sum_{j=0}^{i-1}\beta^j\right)\|\varphi-\psi\|.$$

According to (4.1), we conclude by the principle of contraction mappings that $\mathcal{F}$ has a unique fixed point in $\mathcal{CB}(\alpha,\beta)$, which is exactly the unique solution of (1.1)–(1.2). □

5. Continuous dependence

Our goal here is to establish the continuous dependence of the unique solution upon the function h.

Theorem 5.1 *Under the hypotheses of Theorem 4.1, the unique solution of (1.1)–(1.2) depends continuously on h.*

Proof Let

$$y_1(t) = \frac{\eta}{2b}(t+b) + \int_{-b}^{b} |G(t,s)|\, h_1\left(s, y_1(s), y_1^{[2]}(s), ..., y_1^{[n]}(s)\right) ds,$$

and

$$y_2(t) = \frac{\eta}{2b}(t+b) + \int_{-b}^{b} |G(t,s)|\, h_2\left(s, y_2(s), y_2^{[2]}(s), ..., y_2^{[n]}(s)\right) ds,$$

are two solutions of (1.1)-(1.2) corresponding to the two different functions $h_1, h_1 : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$. We have

$$|y_2(t) - y_1(t)| \leq \int_{-b}^{b} |G(t,s)| \left| h_2\left(s, y_2(s), ..., y_2^{[n]}(s)\right) - h_1\left(s, y_1(s), ..., y_1^{[n]}(s)\right) \right| ds,$$

and taking into account conditions (3.1), (3.6) and Lemma 2.1, we obtain

$$\begin{aligned} &\left| h_2\left(s, y_2(s), ..., y_2^{[n]}(s)\right) - h_1\left(s, y_1(s), ..., y_1^{[n]}(s)\right) \right| \\ &\qquad \leq \left| h_2\left(s, y_2(s), ..., y_2^{[n]}(s)\right) - h_1\left(s, y_2(s), ..., y_2^{[n]}(s)\right) \right| \\ &\qquad + \left| h_1\left(s, y_2(s), ..., y_2^{[n]}(s)\right) - h_1\left(s, y_1(s), ..., y_1^{[n]}(s)\right) \right| \\ &\qquad \leq \|h_2 - h_1\| + \sum_{i=1}^{n} a_i \sum_{j=0}^{i-1} \beta^j \|y_2 - y_1\|. \end{aligned}$$

So

$$|y_2(t) - y_1(t)| \leq \frac{1}{2} b^2 \left(\|h_2 - h_1\| + \sum_{i=1}^{n} a_i \sum_{j=0}^{i-1} \beta^j \|y_2 - y_1\| \right).$$

Using (4.1) , we obtain

$$\|y_2 - y_1\| \leq \frac{\frac{1}{2} b^2}{1 - \frac{1}{2} b^2 \sum_{i=1}^{n} a_i \sum_{j=0}^{i-1} \beta^j} \|h_2 - h_1\|.$$

This completes the proof. □

6. Example

To show the validity of the foregoing findings, we propose an example.

Consider the following problem:

$$y''(t) = \frac{1}{11}\sin t + \frac{1}{5} y(t) + \frac{1}{7} y^{[2]}(t) + \frac{1}{9} y^{[3]}(t), \quad -\frac{\pi}{30} \leq t \leq \frac{\pi}{30}, \tag{6.1}$$

$$y\left(-\frac{\pi}{30}\right) = \frac{1}{12\pi}, \; y\left(\frac{\pi}{30}\right) = \frac{1}{8\pi}, \tag{6.2}$$

where

$$h(t, x_1, x_2, x_3) = \frac{1}{11}\sin t + \frac{1}{5} x_1 + \frac{1}{7} x_2 + \frac{1}{9} x_3, \; b = \frac{\pi}{30}, \; \eta_1 = \frac{1}{12\pi} \text{ and } \eta_2 = \frac{1}{8\pi}.$$

So

$$a_1 = \frac{1}{5}, \; a_2 = \frac{1}{7}, \; a_3 = \frac{1}{9} \text{ and } h_0 = \frac{1}{11}.$$

Let $\mathcal{CB}(\alpha,\beta)=\mathcal{CB}\left(\frac{\pi}{31},\frac{\pi}{4}\right)$. We have

$$2|\eta_1|+|\eta_2|+\frac{b^2}{2}\left(h_0+\alpha\sum_{i=1}^{n}a_i\sum_{j=0}^{i-1}\beta^j\right)\simeq 0.093\,74\leq\alpha=\frac{\pi}{31}\simeq 0.101\,34,$$

$$\frac{|\eta_1-\eta_2|}{2|b|}+5b\left(h_0+\alpha\sum_{i=1}^{n}a_i\sum_{j=0}^{i-1}\beta^j\right)\simeq 0.14924\leq\beta=\frac{\pi}{4}\simeq 0.78540,$$

and

$$\frac{1}{2}b^2\sum_{i=1}^{n}a_i\sum_{j=0}^{i-1}\beta^j\simeq 0.0039587<1.$$

It is not hard to check that all the assumptions in Theorems 3.8, 4.1 and 5.1 are satisfied. Thus, equation (6.1)-(6.2) has a unique bounded solution in $\mathcal{CB}\left(\frac{\pi}{31},\frac{\pi}{4}\right)$, which depends continuously on the function h.

7. Concluding remarks

The key object of this work is to study a second-order iterative boundary value problem. To achieve the purpose of this work, we used a powerful technique that involves defining an appropriate Banach space, which allows us to bring our desired results such as the continuity and the boundedness of solutions, to apply Banach and Schauder's fixed point theorems and, at the same time, to overcome the hitches caused by the iterative terms. Next, under some suitable criteria and with the aid of the aforementioned fixed point theorems, Arzelà–Ascoli theorem and some properties of the obtained Green's kernel, we established the existence, uniqueness and continuous dependence of the bounded solution. Finally, a concrete example is given.

The highlights of this manuscript are listed as follows:

(a) A set of sufficient conditions that ensure the existence, uniqueness and continuous dependence of bounded solutions has been established by using two different fixed point theorems.

(b) Up till now, we find only few papers devoted to study iterative differential equations. So, our obtained results enrich some known investigations to some extent. For instance, the second order problems in [2, 9, 14, 15] were without delays, and the majority of the papers that dealt with retarded second-order boundary value problems have focused on constant delays [1, 16] or time-varying ones [4, 18, 23], while the iterative source term in our problem involves implicitly state and time dependent delays of the form $\tau_i(t,y(t))$.

Furthermore, the aforementioned work [17] investigated a second-order iterative differential equation of only second degree, while our problem is of degree n.

(c) The technique used in this work can be employed successfully to investigate numerous other iterative differential problems of higher order.

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