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Tensor products of graded-simple $\mathfrak{sl}_2(\mathbb{C})$-modules

Yuri BAHTURIN[1,*], **Abdallah SHIHADEH**[2]

[1]Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, NL, Canada
[2]Department of Mathematics, Faculty of Science, The Hashemite University, Zarqa, Jordan

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Abstract: In our paper [3] we have constructed the first example of simple graded torsion-free $\mathfrak{sl}_2(\mathbb{C})$-module denoted by M_λ^C. Here we examine tensor product of M_λ^C with finite dimensional simple $\mathfrak{sl}_2(\mathbb{C})$-modules.

Key words: Graded Lie algebras, enveloping algebras, simple modules, graded modules

1. Introduction

Let G be a group. A G*-grading* on a vector space V over a field $\mathbb{F}$ is the direct sum decomposition of the form

$$V = \bigoplus_{g\in G} V_g. \tag{1.1}$$

The subspaces V_g are allowed to be zero. The subset $S \subseteq G$ consisting of those $g \in G$ for which $V_g \neq \{0\}$ is called the *support* of the grading Γ and denoted by $\operatorname{Supp}\Gamma$ or $\operatorname{Supp} V$.

The subspaces V_g are called the *homogeneous components* of Γ, and the nonzero elements in V_g are called *homogeneous of degree* g (with respect to Γ). An $\mathbb{F}$-subspace U in V is called *graded* in V (or in Γ) if $U = \bigoplus_{g\in G} U \cap V_g$.

Now let Γ and $\Gamma' : V = \bigoplus_{g'\in G'} V'_{g'}$ be two gradings on V with supports S and S', respectively. We say that Γ is a *refinement* of Γ' (or Γ' is a *coarsening* of Γ), if for any $s \in S$ there exists $s' \in S'$ such that $V_s \subseteq V'_{s'}$. The refinement is *proper* if this inclusion is strict for at least one $s \in S$. A grading is called *fine* if it does not have proper refinements.

A left module M over a G*-graded* associative algebra A is called *graded* if M is a G-graded vector space and

$$A_g M_h \subset M_{g+h} \text{ for all } g, h \in G.$$

(Later on, the multiplicative notation for the graded modules will be used, as well).

A left G-graded A-module M is called *graded-simple* if M has no graded submodules different from $\{0\}$ and M. Graded modules and graded-simple modules over a graded Lie algebra L are defined in the same way.

If a Lie algebra L is G-graded then its universal enveloping algebra $U(G)$ is also G-graded. Every graded L-module is a left $U(L)$-graded module, and *vice versa*. The same is true for simple graded L-modules

*Correspondence: bahturin@mun.ca
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and graded-simple L-modules.

An $\mathfrak{sl}_2(\mathbb{C})$-module is called *torsion-free* if it is torsion-free as an $\mathbb{C}[h]$-module. Every simple module is either torsion-free or torsion, also called *weight* module. Torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules have been studied in [4, 6, 12–14]. In 1992 Bavula constructed a family of simple torsion-free modules [5]. Simple torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules of rank 1 have been classified in [13]. In [14], the author produced torsion-free modules of arbitrary finite rank.

The gradings on the torsion-free $\mathfrak{sl}_2(\mathbb{C})$-module of rank 1 have been dealt with in [1]; it was shown that the torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules of rank 1 cannot be $\mathbb{Z}$- or $\mathbb{Z}_2^2$-graded. In [1] one also considered the gradings of the family of modules constructed in [5].

In [3], we constructed the first simple torsion-free $\mathbb{Z}_2^2$-graded module of rank 2, which we denoted by M_λ^C, where $\lambda \in \mathbb{C}$, see its definition in Section 3

$$M_\lambda^C := U(I_\lambda)/U(I_\lambda)C.$$

The main result is that the graded module M_λ^C is simple when λ is not an even integer. Otherwise, the module M_λ^C contains a proper graded-simple torsion-free submodule of rank 2.

It was interesting to see if using tensor products one can produce more graded simple modules. The main goal in this paper is to study the tensor products of graded-simple $\mathfrak{sl}_2(\mathbb{C})$-modules, in particular, tensor products of M_λ^C with simple graded finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-modules.

2. Group gradings on simple $\mathfrak{sl}_2(\mathbb{C})$-modules

In this section we restrict our attention to the modules over the Lie algebra of the type A_1, which can be realized as $\mathfrak{sl}_2(\mathbb{C})$.

2.1. Group gradings of $\mathfrak{sl}_2(\mathbb{C})$

All group gradings on $\mathfrak{sl}_2(\mathbb{C})$ are well-known, see e.g., [7]. We will use the canonical bases:

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \tag{2.1}$$

and

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = h, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{2.2}$$

Up to equivalence, there are precisely two fine gradings on $\mathfrak{sl}_2(\mathbb{C})$ (see [7, Theorem 3.55]):

- *Cartan grading* with the universal group $\mathbb{Z}$,

$$\begin{aligned} \Gamma^1_{\mathfrak{sl}_2} : \mathfrak{sl}_2(\mathbb{C}) &= L_{-1} \oplus L_0 \oplus L_1 \\ \text{where } L_0 &= \langle h \rangle,\, L_1 = \langle x \rangle,\, L_{-1} = \langle y \rangle; \end{aligned}$$

- *Pauli grading* with the universal group $\mathbb{Z}_2^2$,

$$\begin{aligned} \Gamma^2_{\mathfrak{sl}_2} : \mathfrak{sl}_2(\mathbb{C}) &= L_{(1,0)} \oplus L_{(0,1)} \oplus L_{(1,1)} \\ \text{where } L_{(1,0)} &= \langle A \rangle,\, L_{(0,1)} = \langle B \rangle,\, L_{(1,1)} = \langle C \rangle. \end{aligned}$$

Up to isomorphism, any G-grading on $\mathfrak{sl}_2(\mathbb{C})$ is a coarsening of one of the two gradings: Cartan or Pauli.

Note that any grading Γ of a Lie algebra L uniquely extends to a grading $U(\Gamma)$ of its universal enveloping algebra $U(L)$. The grading $U(\Gamma)$ is a grading in the sense of associative algebras but also as L-modules where $U(L)$ is either a (left) regular L-module or an adjoint L-module.

2.2. Simple finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-modules

In this section we recall simple finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-modules (see e.g., [9, §7.2]).

Let $V(n)$ be a space with the basis $\{v_i \mid i = 0, 1, \dots n\}$. We set

$$\begin{aligned} h.v_i &= (n-2i)v_i;\\ y.v_i &= (i+1)v_{i+1};\\ x.v_i &= \begin{cases} (n-(i-1))v_{i-1} & \text{if} \quad i \geq 1;\\ 0 & \text{if} \quad i = 0. \end{cases} \end{aligned} \tag{2.3}$$

Then the following is true.

Theorem 2.1 *Let $n \in \mathbb{N}_0$, then*

1. *The module $V(n)$ is a simple $\mathfrak{sl}_2(\mathbb{C})$-module of dimension $n+1$. Moreover,*

$$V(n) = V_n \oplus V_{n-2} \oplus \cdots \oplus V_{-(n-2)} \oplus V_{-n}.$$

where $V_i = < v_i >$ for $i = n, n-2, \dots, -n$.

2. *Any simple finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-module of dimension $n+1$ is isomorphic to the module $V(n)$.* □

For the proof of Theorem 2.1, see e.g., [9, §7.2] or [12, Theorem 1.22].

2.3. Gradings on simple finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-modules

Every simple finite-dimensional module of $\mathfrak{sl}_2(\mathbb{C})$ is a weight module, that is, decomposes as the direct sum of weight spaces and this decomposition is a grading compatible with the Cartan grading on $\mathfrak{sl}_2(\mathbb{C})$. In [8], the authors gave a complete classification of the highest weights λ for any graded finite-dimensional simple Lie algebra L such that the simple highest weight module $V(\lambda)$ admits a grading compatible with the grading of L. In particular, it was shown that a simple module $V(n)$ for $L = \mathfrak{sl}_2(\mathbb{C})$, equipped with the Pauli grading, can be made a graded module if and only if n is an even number. The proof in [8] does not provide the explicit form of the grading on $V(n)$. We recall the result from [1] below.

Let $V = V(n)$ be a simple $\mathfrak{sl}_2(\mathbb{C})$-module with an even highest weight n and the canonical basis $\{v_0, v_1, \dots, v_n\}$ (see subsection 2.2).

To construct a $\mathbb{Z}_2^2$-grading on V, we first define a new basis of V as follows.

Set $e_i = v_i + v_{n-i}$, for $i = 0, 1, \dots, \frac{n}{2}$, and $d_i = v_i - v_{n-i}$ for $i = 0, 1, \dots, \frac{n}{2} - 1$. Then $\left\{e_0, e_1, \dots, e_{\frac{n}{2}}, d_0, d_1, \dots, d_{\frac{n}{2}-1}\right\}$ is a basis of V and the module action is given by:

$$h.e_i = (n-2i)d_i\,, \text{ for } i = 0,1,\ldots,\tfrac{n}{2}\,;$$

$$B.e_i = \begin{cases} (n-i+1)e_{i-1} + (i+1)e_{i+1}, & \text{if } i = 0,1,\ldots,\frac{n}{2}-1; \\ 2(\frac{n}{2}+1)e_{\frac{n}{2}-1}, & \text{if } i = \frac{n}{2}; \end{cases}$$

$$C.e_i = \begin{cases} (n-i+1)d_{i-1} - (i+1)d_{i+1}, & \text{if } i = 0,1,\ldots,\frac{n}{2}-1; \\ 2(\frac{n}{2}+1)d_{\frac{n}{2}-1}, & \text{if } i = \frac{n}{2}; \end{cases}$$

$$h.d_i = (n-2i)e_i\,, \text{ for } i = 0,1,\ldots,\tfrac{n}{2}-1;$$

$$B.d_i = (n-i+1)d_{i-1} + (i+1)d_{i+1}\,, \text{ if } i = 0,1,\ldots,\tfrac{n}{2}-1\,;$$

$$C.d_i = (n-i+1)e_{i-1} - (i+1)e_{i+1}\,, \text{ if } i = 0,1,\ldots,\tfrac{n}{2}-1\,.$$

Let $V_{(0,0)} = \langle e_i \mid i \text{ is even}\rangle$, $V_{(0,1)} = \langle e_i \mid i \text{ is odd}\rangle$, $V_{(1,0)} = \langle d_i \mid i \text{ is even }\rangle$, and $V_{(1,1)} = \langle d_i \mid i \text{ is odd}\rangle$. Then the following is true.

Proposition 2.2 ([1]) *The above formulae provide a $\mathbb{Z}_2^2$-grading $\Gamma : V = \bigoplus_{g\in\mathbb{Z}_2^2} V_g$ on V compatible with the Pauli grading on $\mathfrak{sl}_2(\mathbb{C})$.* □

2.4. Algebras $U(I_\lambda)$

Let $c \in U(\mathfrak{sl}_2(\mathbb{C}))$ be the Casimir element for $\mathfrak{sl}_2(\mathbb{C})$. With respect to the basis $\{h,x,y\}$ of $\mathfrak{sl}_2(\mathbb{C})$, this element can be written as

$$c = (h+1)^2 + 4yx = h^2 + 1 + 2xy + 2yx. \tag{2.4}$$

It is well-known that the center of $U(\mathfrak{sl}_2(\mathbb{C}))$ is the polynomial ring $\mathbb{C}[c]$. Note that c is a homogeneous element of degree zero, with respect to the Cartan grading of $U(\mathfrak{sl}_2(\mathbb{C}))$. One can write the Casimir element with respect to the basis $\{h,B,C\}$ of $\mathfrak{sl}_2(\mathbb{C})$.

Namely,

$$\begin{aligned} c &= 2xy + 2yx + h^2 + 1 = 2\left(\frac{B+C}{2}\right)\left(\frac{B-C}{2}\right) \\ &+ 2\left(\frac{B-C}{2}\right)\left(\frac{B+C}{2}\right) + h^2 + 1 \\ &= \frac{1}{2}(B^2 + CB - BC - C^2) + \frac{1}{2}(B^2 + BC - CB - C^2) + h^2 + 1, \end{aligned}$$

and so

$$c = B^2 - C^2 + h^2 + 1 = A^2 + B^2 - C^2 + 1.$$

It follows that c is also homogeneous, of degree (0,0), with respect to the Pauli grading of $U(\mathfrak{sl}_2(\mathbb{C}))$. Note that this graded basis of the center of the universal enveloping algebra is a very particular case of the graded bases given in [2].

Let R be an associative algebra (or just an associative ring), and V be a left R-module. The annihilator of V, denoted by $\mathrm{Ann}_R(V)$, is the set of all elements r in R such that, for all v in V, $r.v = 0$:

$$\mathrm{Ann}_R(V) = \{r \in R \mid r.v = 0 \text{ for all } v \in V\}.$$

Given $\lambda \in \mathbb{C}$, let I_λ be the two-sided ideal of $U(\mathfrak{sl}_2(\mathbb{C}))$, generated by the central element $c - (\lambda+1)^2$.

Theorem 2.3 *[12, Theorem 4.7] For any simple $U(\mathfrak{sl}_2(\mathbb{C}))$-module M, there exists $\lambda \in \mathbb{C}$ such that $I_\lambda \subset \mathrm{Ann}_{U(\mathfrak{sl}_2(\mathbb{C}))}(M)$.* □

Clearly, if R be a graded algebra and M be a graded R-module, then $\mathrm{Ann}_R(M)$ is a graded ideal.

Proposition 2.4 ([1]) *The ideal I_λ is both $\mathbb{Z}-$ and $\mathbb{Z}_2^2-$graded.*

□

Theorem 2.5 *[12, Theorem 4.26] For any nonzero left ideal $I \subset U(I_\lambda)$, the $U(I_\lambda)$-module $U(I_\lambda)/I$ has finite length.* □

Now for any $\lambda \in \mathbb{C}$, we write $U(I_\lambda) := U(\mathfrak{sl}_2(\mathbb{C}))/I_\lambda$. Using Proposition 2.4, it follows that $U(I_\lambda)$ has natural $\mathbb{Z}$- and $\mathbb{Z}_2^2$-gradings induced from $\mathfrak{sl}_2(\mathbb{C})$. It is well-known (see e.g., [12]) that the algebra $U(I_\lambda)$ is a free $\mathbb{C}[h]$-module with basis $\mathcal{B}_0 = \{1, x, y, x^2, y^2, \ldots\}$, and so it is a vector space over $\mathbb{C}$ with basis $\mathcal{B} = \{1, h, h^2, \ldots\}.\mathcal{B}_0$. Note that the basis $\mathcal{B}$ is a basis of $U(I_\lambda)$ consisting of homogeneous element with respect to the Cartan grading by $\mathbb{Z}$.

A basis of $U(I_\lambda)$ over $\mathbb{C}$ consisting of homogeneous element with respect to the Pauli grading by $\mathbb{Z}_2^2$ can be computed as follows. Set $\widehat{B}_0 = \{1, B, C, BC, B^2, B^2C, B^3, B^3C, \ldots\}$. Then easy calculations, using induction by the degree in B and the relation $C^2 = h^2 + B^2 - \lambda^2 - 2\lambda$ show that the set $\widehat{B} = \{1, h, h^2, \ldots\} \cdot \widehat{B}_0$ is a $\mathbb{Z}_2^2$-homogeneous basis of $U(I_\lambda)$.

Now let $p(t) = \frac{1}{4}((\lambda^2 + 2\lambda) - 2t - t^2) \in \mathbb{C}[t]$. Then, inside $U(I_\lambda)$ for any $q(t) \in \mathbb{C}[t]$, we have the following relations:

$$x^k q(h) = q(h-2k)x^k$$
$$y^j q(h) = q(h+2j)y^j$$

If $k \geq j$ then

$$x^k y^j = p(h-2k)\cdots p(h-2(k-j+1))x^{k-j}$$
$$y^j x^k = p(h+2(j-1))\cdots p(h)x^{k-j}.$$

If $j \geq k$ then

$$x^k y^j = p(h-2k)\cdots p(h-2)y^{j-k}$$
$$y^j x^k = p(h+2(j-1))\cdots p(h+2(j-k))y^{j-k}.$$

Proposition 2.6 ([3]) *Set $\widehat{B}_0 = \{1, B, C, BC, B^2, B^2C, B^3, B^3C, \ldots\}$. Then $\widehat{B} = \{1, h, h^2, \ldots\} \cdot \widehat{B}_0$ is a $\mathbb{Z}_2^2$-homogeneous basis of $U(I_\lambda)$.*

2.5. $\mathfrak{sl}_2(\mathbb{C})$-modules: torsion-free modules

Definition 2.7 *Following [12], an* $\mathfrak{sl}_2(\mathbb{C})$ *-module* M *is called torsion if for any* $m \in M$ *there exists nonzero* $p(t) \in \mathbb{C}[t]$ *such that* $p(h).m = 0$. *In other words,* h *has an eigenvector in* M. *We call* M torsion-free *if* $M \neq 0$ *and* $p(h).m \neq 0$ *for all* $0 \neq m \in M$ *and all nonzero* $p(t) \in \mathbb{C}[t]$. *If* M *is finite generated as* $\mathbb{C}[h]$ *-module, then* M *is a free* $\mathfrak{sl}_2(\mathbb{C})$ *-module. If the rank of this module is* n, *we say that* M *is a torsion-free* $\mathfrak{sl}_2(\mathbb{C})$ *-module of rank* n. *In many recent papers, torsion-free modules are also called* $U(H)$ *-free modules.*

Theorem 2.8 *[12, Theorem 6.3] A simple* $\mathfrak{sl}_2(\mathbb{C})$ *-module is either a weight or a torsion-free module.* □

Theorem 2.8 means that if h has at least one eigenvector on M, then M is a weight module.

As a consequence of Theorem 2.3, it is sufficient to describe simple torsion-free $U(I_\lambda)$-modules instead of simple $U(\mathfrak{sl}_2(\mathbb{C}))$-modules (see e.g., [12]).

The classification of simple torsion free $U(I_\lambda)$-module is given in the seminal paper [6]. In [4] the author uses a localization $\mathbb{A}$ of $U(I_\lambda)$ to obtain the following.

Theorem 2.9 *[4, Proposition 3] Let* M *be a simple torsion-free* $U(I_\lambda)$ *-module, then* $M \cong U(I_\lambda)/(U(I_\lambda) \cap \mathbb{A}\alpha)$, *for some* $\alpha \in U(I_\lambda)$ *which is irreducible as an element of* $\mathbb{A}$.

Theorem 2.10 ([1]) *Torsion free* $\mathfrak{sl}_2(\mathbb{C})$ *-modules of rank* 1 *cannot be* $\mathbb{Z}$ *or* $\mathbb{Z}_2^2$ *-graded.*

3. Simple $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-modules of rank 2

In [3] the authors study simple torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules of rank 2. In particular, they construct the first family of simple $\mathbb{Z}_2^2$-graded torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules of rank 2.

Given $\lambda \in \mathbb{C}$, we consider an $U(I_\lambda)$-module $M_\lambda^C = U(I_\lambda)/U(I_\lambda)C$. For $u, v \in U(\mathfrak{sl}_2(\mathbb{C}))$, we say that $u \equiv v$ and say that u is equivalent to v, if and only if $u + U(I_\lambda)C = v + U(I_\lambda)C$. All elements of the form $h^k B^l C$ are equivalent to 0. Moreover $B^2 \equiv \mu - h^2$ where $\mu = \lambda^2 + 2\lambda$. Hence

$$h^k B^2 \equiv h^k(\mu - h^2) = \mu h^k - h^{k+2},$$

which implies that any element of M_λ^C can be written as a linear combination of elements of the form $h^k B^m$ where $m = 0, 1$. This means that M_λ^C is a torsion-free $\mathfrak{sl}_2(\mathbb{C})$-module of rank 2, with basis $\{1, B\}$ as a $\mathbb{C}[h]$-module, that is, M_λ^C can be written as $M_\lambda^C = \mathbb{C}[h] \oplus \mathbb{C}[h]B$, which is graded module since $U(I_\lambda)C$ is a graded left ideal. Furthermore,

$$M_\lambda^C = (M_\lambda^C)_{(0,0)} \oplus (M_\lambda^C)_{(0,1)} \oplus (M_\lambda^C)_{(1,0)} \oplus (M_\lambda^C)_{(1,1)}, \tag{3.1}$$

where

$$\begin{aligned}
(M_\lambda^C)_{(0,0)} &= \langle h^{2k} \mid k \in \mathbb{N} \rangle \\
(M_\lambda^C)_{(1,0)} &= \langle h^{2k+1} \mid k \in \mathbb{N} \rangle \\
(M_\lambda^C)_{(0,1)} &= \langle h^{2k} B \mid k \in \mathbb{N} \rangle \\
(M_\lambda^C)_{(1,1)} &= \langle h^{2k+1} B \mid k \in \mathbb{N} \rangle .
\end{aligned} \tag{3.2}$$

The following theorems are the main results in [3].

Theorem 3.1 **([3])** *Let $\lambda \in \mathbb{C} \setminus 2\mathbb{Z}$, then M_λ^C is a simple $\mathfrak{sl}_2(\mathbb{C})$-module .*

Theorem 3.2 **([3])** *Let $\lambda \in 2\mathbb{Z}$, then M_λ^C has a unique maximal (graded) submodule N_λ^C such that $N_\lambda^C = P \oplus Q$, where P and Q are simple torsion-free $\mathfrak{sl}_2(\mathbb{C})$-module of rank 1.*

4. Tensor products of M_λ^C and $V(2)$

We start by studying the tensor product of the module M_λ^C and $V(2)$. We chose $V(2)$ because the simple decomposition $M_\lambda^C \otimes V(2)$ could provide us with new simple graded $\mathfrak{sl}_2(\mathbb{C})$-modules, in addition to M_λ^C. Without loss of generality, we will consider the module $V(2)$ as the adjoint module of $\mathfrak{sl}_2(\mathbb{C})$. Clearly, this inherits a $\mathbb{Z}_2^2$-grading from $\mathfrak{sl}_2(\mathbb{C})$ itself. Actually, each $V(2n)$ can be endowed by a $\mathbb{Z}_2$-grading, as shown in [8]. In subsection 2.3 we gave an explicit description of these gradings.

Recall that we use the term "Casimir constants" to refer to the Casimir eigenvalues.

We will use the standard basis $\{x, h, y\}$ of $\mathfrak{sl}_2(\mathbb{C})$ as the basis of $V(2)$. If $\lambda \in \mathbb{C}$, then we define the module

$$L_{(\lambda,2)} := M_\lambda^C \otimes V(2). \tag{4.1}$$

Hence, any element $z \in L_{(\lambda,2)}$, z can be written as

$$\begin{aligned} z = f_1(h) \otimes x + & f_2(h)B \otimes x + f_3(h) \otimes h + f_4(h)B \otimes h \\ & + f_5(h) \otimes y + f_6(h)B \otimes y. \end{aligned} \tag{4.2}$$

For simplicity, we will write the element z in (4.2) as

$$z = (\ f_1(h), f_2(h), f_3(h), f_4(h), f_5(h), f_6(h)\ ).$$

Occasionally, we will use both notations, as needed.

An important fact which we will be using is the following theorem due to B. Kostant [11, Theorem 5.1].

Theorem 4.1 *Let c be the Casimir element of $U(\mathfrak{sl}_2(\mathbb{C}))$, and let M be an $\mathfrak{sl}_2(\mathbb{C})$-module on which c acts as the scalar ρ. Then for any finite dimensional module $V(n)$, the element*

$$\prod_{\mu_i \in \{n, n-2, \ldots, -n\}} (c - (\sqrt{\rho} + \mu_i)^2)$$

annihilates the module $M \otimes V(n)$. □

4.1. Actions on $L_{(\lambda,2)}$

The action of the elements $h, x, y \in \mathfrak{sl}_2(\mathbb{C})$ on the module $L_{(\lambda,2)}$ are given by the following :

$$\begin{aligned}
& h.(f_1(h), f_2(h), f_3(h), f_4(h), f_5(h), f_6(h)) = \\
& h.(f_1(h) \otimes x + f_2(h)B \otimes x + f_3(h) \otimes h + f_4(h)B \otimes h \\
+ \quad & f_5(h) \otimes y + f_6(h)B \otimes y) \\
= \quad & hf_1(h) \otimes x + 2f_1(h) \otimes x + hf_2(h)B \otimes x \\
+ \quad & 2f_2(h)B \otimes x + hf_3(h) \otimes h + hf_4(h)B \otimes h \\
+ \quad & hf_5(h) \otimes y - 2f_5(h) \otimes y + hf_6(h)B \otimes y - 2f_6(h)B \otimes y \\
= \quad & (h+2)f_1(h) \otimes x + (h+2)f_2(h)B \otimes x + hf_3(h) \otimes h \\
+ \quad & hf_4(h)B \otimes h + (h-2)f_5(h) \otimes y + (h-2)f_6(h)B \otimes y \\
= \quad & ((h+2)f_1(h), (h+2)f_2(h), hf_3(h), hf_4(h), \\
& (h-2)f_5(h), (h-2)f_6(h)).
\end{aligned}$$

Next,

$$\begin{aligned}
& x.\,(f_1(h), f_2(h), f_3(h), f_4(h), f_5(h), f_6(h)) \\
= \quad & x.\,(f_1(h) \otimes x + f_2(h)B \otimes x + f_3(h) \otimes h + f_4(h)B \otimes h \\
+ \quad & f_5(h) \otimes y + f_6(h)B \otimes y)
\end{aligned}$$

$$\begin{aligned}
= \quad & \left(\frac{1}{2}(\mu - h^2 + 2h)f_2(h-2) - 2f_3(h)\right) \otimes x \\
+ \quad & \left(\frac{1}{2}f_1(h-2) - 2f_4(h)\right) B \otimes x \\
+ \quad & \left(\frac{1}{2}(\mu - h^2 + 2h)f_4(h-2) + f_5(h)\right) \otimes h. \\
+ \quad & \left(\frac{1}{2}f_3(h-2) + f_6(h)\right) B \otimes h \\
+ \quad & \left(\frac{1}{2}(\mu - h^2 + 2h)f_6(h-2)\right) \otimes y + \frac{1}{2}f_5(h-2))B \otimes y
\end{aligned}$$

$$\begin{aligned}
= \quad & \left(\frac{1}{2}(\mu - h^2 + 2h)f_2(h-2) - 2f_3(h), \frac{1}{2}f_1(h-2) - 2f_4(h)\right., \\
& \frac{1}{2}(\mu - h^2 + 2h)f_4(h-2) + f_5(h), \frac{1}{2}f_3(h-2) + f_6(h)\ , \\
& \left.\frac{1}{2}(\mu - h^2 + 2h)f_6(h-2), \frac{1}{2}f_5(h-2)\right).
\end{aligned}$$

Finally,

$$\begin{aligned}
&y.\,(\,f_1(h), f_2(h), f_3(h), f_4(h), f_5(h), f_6(h)\,) = \\
&\quad x.(f_1(h)\otimes x + f_2(h)B\otimes x + f_3(h)\otimes h + f_4(h)B\otimes h \\
&\quad + f_5(h)\otimes y + f_6(h)B\otimes y) \\
&= \frac{1}{2}(\mu - h^2 - 2h)f_2(h+2)\otimes x + \frac{1}{2}f_1(h+2)B\otimes x \\
&+ (\frac{1}{2}(\mu - h^2 - 2h)f_4(h+2) - f_1(h))\otimes h + (\frac{1}{2}f_3(h+2) - f_2(h))B\otimes h \\
&+ (\frac{1}{2}(\mu - h^2 - 2h)f_6(h+2) + 2f_3(h))\otimes y \\
&+ (\frac{1}{2}f_5(h+2) + 2f_4(h))B\otimes y
\end{aligned}$$

$$\begin{aligned}
&= (\,\frac{1}{2}(\mu - h^2 - 2h)f_2(h+2), \frac{1}{2}f_1(h+2), \\
&\quad \frac{1}{2}(\mu - h^2 - 2h)f_4(h+2) - f_1(h), \frac{1}{2}f_3(h+2) - f_2(h), \\
&\quad \frac{1}{2}(\mu - h^2 - 2h)f_6(h+2) + 2f_3(h), \frac{1}{2}f_5(h+2) + 2f_4(h)\,)\,. \qquad (4.3)
\end{aligned}$$

The action of yx is also needed, which helps us determine the action of the Casimir element c. Using our previous computations, we have:

$$\begin{aligned}
&4yx.\,(\,f_1(h), f_2(h), f_3(h), f_4(h), f_5(h), f_6(h)\,) \\
&\quad = (\,(\mu - h^2 - 2h)f_1(h) - 4(\mu - h^2 - 2h)f_4(h+2), \\
&\quad (\mu - h^2 - 2h)f_2(h) - 4f_3(h+2), \\
&\quad (\mu - h^2 - 2h + 8)f_3(h) + 2(\mu - h^2 - 2h)f_6(h+2) \\
&\quad - 2(\mu - h^2 + 2h)f_2(h-2), \qquad (4.4) \\
&\quad (\mu - h^2 - 2h + 8)f_4(h) + 2f_5(h+2) - 2f_1(h-2), \\
&\quad (\mu - h^2 - 2h + 8)f_5(h) + 4(\mu - h^2 + 2h)f_4(h-2), \\
&\quad (\mu - h^2 - 2h + 8)f_6(h) + 4f_3(h-2)\,)\,.
\end{aligned}$$

Hence

$$\begin{aligned}
&c.\,(\,f_1(h), f_2(h), f_3(h), f_4(h), f_5(h), f_6(h)\,) \qquad (4.5) \\
&\quad = (\,(4h + \mu + 9)f_1(h) - 4(\mu - h^2 - 2h)f_4(h+2), \\
&\quad (4h + \mu + 9)f_2(h) - 4f_3(h+2),
\end{aligned}$$

$$(\mu+9)f_3(h)+2(\mu-h^2-2h)f_6(h+2)$$
$$-2(\mu-h^2+2h)f_2(h-2),$$
$$(9+\mu)f_4(h)+2f_5(h+2)-2f_1(h-2),$$
$$(-4h+\mu+9)f_5(h)+4(\mu-h^2+2h)f_4(h-2),$$
$$(-4h+\mu+9)f_6(h)+4f_3(h-2)\ ).$$

Let L^α be a maximal submodule of $L_{(\lambda,2)}$ where c acts as a scalar α. Then for any $f_1(h),\dots,f_6(h)\in\mathbb{C}[h]$ such that

$$(f_1(h),f_2(h),f_3(h),f_4(h),f_5(h),f_6(h)\ )\in L^\alpha$$

we must have

$$(c-\alpha).(\ f_1(h),f_2(h),f_3(h),f_4(h),f_5(h),f_6(h)\ )=0. \tag{4.6}$$

Using (4.5) and (4.6) we have the following two systems of equations,

$$(4h+\mu+9-\alpha)f_1(h)-4(\mu-h^2-2h)f_4(h+2)=0, \tag{4.7}$$

$$(\mu+9-\alpha)f_4(h)+2f_5(h+2)-2f_1(h-2)=0, \tag{4.8}$$

$$(-4h+\mu+9-\alpha)f_5(h)+4(\mu-h^2+2h)f_4(h-2)=0, \tag{4.9}$$

and

$$(4h+\mu+9-\alpha)f_2(h)-4f_3(h+2)=0, \tag{4.10}$$

$$(\mu+9-\alpha)f_3(h)+2(\mu-h^2-2h)f_6(h+2)-2(\mu-h^2+2h)f_2(h-2)=0, \tag{4.11}$$

$$(-4h+\mu+9-\alpha)f_6(h)+4f_3(h-2)=0. \tag{4.12}$$

By Equation (4.7) we have

$$f_1(h-2)=\frac{4(\mu-h^2+2h)f_4(h)}{4h+\mu+1-\alpha}, \tag{4.13}$$

and by (4.9) we have

$$f_5(h+2)=\frac{-4(\mu-h^2-2h)f_4(h)}{-4h+\mu+1-\alpha}. \tag{4.14}$$

Let $f_4(h)\neq 0$. Applying (4.13) and (4.14) to (4.8) we have

$$\alpha^3-(3\mu+11)\alpha^2+(3\mu^2+6\mu+19)\alpha-(\mu^3-5\mu^2+3\mu+9)=0, \tag{4.15}$$

which implies that we have three solution of (4.15)

$$\alpha_1=\mu+1=(\sqrt{\mu+1}+0)^2; \tag{4.16}$$
$$\alpha_2=\mu+5+4\sqrt{\mu+1}=(\sqrt{\mu+1}+2)^2; \tag{4.17}$$
$$\alpha_3=\mu+5-4\sqrt{\mu+1}=(\sqrt{\mu+1}-2)^2. \tag{4.18}$$

This is in agreement with the values in Theorem 4.1.

In this case, we have

$$f_1(h) = \frac{4(\mu - h^2 - 2h)f_4(h+2)}{4h + \mu + 9 - \alpha}, \tag{4.19}$$

and

$$f_5(h) = \frac{-4(\mu - h^2 + 2h)f_4(h-2)}{-4h + \mu + 9 - \alpha}. \tag{4.20}$$

Similarly, solving the second system of equations, we will get the same values of $\alpha_1, \alpha_2,$ and α_3. Moreover, we have

$$f_2(h) = \frac{4f_3(h+2)}{4h + \mu + 9 - \alpha}, \tag{4.21}$$

and

$$f_6(h) = \frac{-4f_3(h-2)}{-4h + \mu + 9 - \alpha}. \tag{4.22}$$

4.2. The submodule $\widetilde{L}_0$ where c acts as a scalar $\mu + 1$

Let $\mu \neq 0$, and $\alpha = \mu + 1$ (the Casimir constant related to the weight 0). In this case, we have

$$f_1(h) = \frac{(\mu - h^2 - 2h)f_4(h+2)}{h+2}, \tag{4.23}$$

$$f_5(h) = \frac{(\mu - h^2 + 2h)f_4(h-2)}{h-2}, \tag{4.24}$$

$$f_2(h) = \frac{f_3(h+2)}{h+2}, \tag{4.25}$$

$$f_6(h) = \frac{f_3(h-2)}{h-2}. \tag{4.26}$$

Since $\mu \neq 0$ and $f_1(h), f_2(h), f_5(h), f_6(h) \in \mathbb{C}[h]$, it follows that $f_3(h)$ and $f_4(h)$ must belong to $\mathbb{C}[h].h$.

Let $\widetilde{L}_0 = L^{\mu+1}$ be the submodule of $L_{(\lambda,2)}$ consisting of all eigenvectors of c with eigenvalue $\mu + 1$. Then any $v \in \widetilde{L}_0$ must be written as

$$\begin{aligned} v = (\ &(\mu - h^2 - 2h)g(h+2), f(h+2), hf(h), \\ &hg(h), (\mu - h^2 + 2h)g(h-2), f(h-2)\ ) \end{aligned} \tag{4.27}$$

for some $f(h), g(h) \in \mathbb{C}[h]$.

Proposition 4.2 *Let $\mu \neq 0$. Then $\widetilde{L}_0$ is isomorphic to M_λ^C.*

Proof Define $\varphi : M_\lambda^C \longrightarrow \widetilde{L}_0$, where

$$\varphi(f(h)+g(h)B) = (\ (\mu-h^2-2h)g(h+2), f(h+2), hf(h), \\ hg(h), (\mu-h^2+2h)g(h-2), f(h-2)\ ). \tag{4.28}$$

It easy to show that φ is a linear bijective map, hence it is enough to show that φ is a module homomorphism. Let $f(h), g(h) \in \mathbb{C}[h]$, we have

$$\varphi(x.(f(h)+g(h)B)) = \varphi(\frac{1}{2}(\mu-h^2+2h)g(h-2)+\frac{1}{2}f(h-2)B)$$
$$= (\ \frac{1}{2}(\mu-h^2-2h)f(h), \frac{1}{2}(\mu-h^2-2h)g(h), \frac{1}{2}h(\mu-h^2+2h)g(h-2),$$
$$\frac{1}{2}hf(h-2), \frac{1}{2}(\mu-h^2+2h)f(h-4), \frac{1}{2}(\mu-h^2+6h-8)g(h-4)\ ).$$

Also,

$$x.\varphi(f(h)+g(h)B) = x.(\ (\ (\mu-h^2-2h)g(h+2), f(h+2), hf(h),$$
$$hg(h), (\mu-h^2+2h)g(h-2), f(h-2)\ )\ )$$

$$= (\ \frac{1}{2}(\mu-h^2-2h)f(h), \frac{1}{2}(\mu-h^2-2h)g(h), \frac{1}{2}h(\mu-h^2+2h)g(h-2),$$
$$\frac{1}{2}hf(h-2), \frac{1}{2}(\mu-h^2+2h)f(h-4), \frac{1}{2}(\mu-h^2+6h-8)g(h-4)\ )$$
$$=\varphi(x.(f(h)+g(h)B)).$$

Now

$$\varphi(y.(f(h)+g(h)B)) = \varphi(\frac{1}{2}(\mu-h^2-2h)g(h+2)+\frac{1}{2}f(h+2)B)$$

$$= (\ \frac{1}{2}(\mu-h^2-2h)f(h+4), \frac{1}{2}(\mu-h^2-6h-8)g(h+4),$$
$$\frac{1}{2}h(\mu-h^2-2h)g(h+2), \frac{1}{2}hf(h+2),$$
$$\frac{1}{2}(\mu-h^2+2h)f(h), \frac{1}{2}(\mu-h^2+2h)g(h)\ ).$$

$$y.\varphi(f(h)+g(h)B) = y(\ (\ (\mu-h^2-2h)g(h+2), f(h+2), hf(h),$$
$$hg(h), (\mu-h^2+2h)g(h-2), f(h-2)\ )\ )$$

$$= (\ \frac{1}{2}(\mu-h^2-2h)f(h+4), \frac{1}{2}(\mu-h^2-6h-8)g(h+4), \frac{1}{2}h(\mu-h^2-2h)g(h+2),$$
$$\frac{1}{2}hf(h+2), \frac{1}{2}(\mu-h^2+2h)f(h), \frac{1}{2}(\mu-h^2+2h)g(h)\ )$$
$$=\varphi(y.(f(h)+g(h)B)).$$

Again,

$$\varphi(h.(f(h)+g(h)B)) = \varphi(hf(h)+hg(h)B)$$

$$\begin{aligned}
=&(\ (\mu-h^2-2h)(h+2)g(h+2),(h+2)f(h+2),h^2f(h),\\
&h^2g(h),(\mu-h^2+2h)(h-2)g(h-2),(h-2)f(h-2)\ )\\
=&h.\varphi(f(h)+g(h)B).
\end{aligned}$$

Hence $M^C_\lambda \cong \widetilde{L}_0$. □

4.3. The submodule $\widetilde{L}_2$ where c acts as a scalar $\mu+5+4\sqrt{\mu+1}$

Let $\alpha=\mu+5+4\sqrt{\mu+1}$ (the Casimir constant related to the weight 2). Note that

$$(\mu-h^2+2h) = -(h-1-\sqrt{\mu+1})(h-1+\sqrt{\mu+1}),$$

and

$$(\mu-h^2-2h) = -(h+1-\sqrt{\mu+1})(h+1+\sqrt{\mu+1}).$$

Hence, in this case we have

$$f_1(h) = -\,(h+1+\sqrt{\mu+1})f_4(h+2), \tag{4.29}$$

$$f_5(h) = -\,(h-1-\sqrt{\mu+1})f_4(h-2), \tag{4.30}$$

$$f_2(h) = \frac{f_3(h+2)}{h+1-\sqrt{\mu+1}}, \tag{4.31}$$

$$f_6(h) = \frac{f_3(h-2)}{h-1+\sqrt{\mu+1}}. \tag{4.32}$$

Since $\mu\neq 0$ and $f_2(h), f_6(h)\in\mathbb{C}[h]$, it follows that $f_3(h)$ must belongs to $\mathbb{C}[h].(h-1-\sqrt{\mu+1})(h+1+\sqrt{\mu+1})$.

Let $\widetilde{L}_2 = L^{\mu+5+4\sqrt{\mu+1}}$ be the submodule of $L_{(\lambda,2)}$ consisting of all eigenvectors of c with eigenvalue $\mu+5+4\sqrt{\mu+1}$. Then for any $v\in\widetilde{L}_2$, v must be written as

$$\begin{aligned}
v = (\ &-(h+1+\sqrt{\mu+1})g(h+2),(h+3+\sqrt{\mu+1})f(h+2),\\
&(h-1-\sqrt{\mu+1})(h+1+\sqrt{\mu+1})f(h),g(h),\\
&-(h-1-\sqrt{\mu+1})g(h-2),(h-3-\sqrt{\mu+1})f(h-2)\ )
\end{aligned} \tag{4.33}$$

for some $f(h), g(h)\in\mathbb{C}[h]$.

Proposition 4.3 *Let $\mu\neq 0$. Then $\widetilde{L}_2$ is isomorphic to $M^C_{\lambda+2}$.*

Proof Note that in $M^C_{\lambda+2}$, the value

$$\begin{aligned}\mu' =&(\lambda+2)^2+2(\lambda+2)\\ =&\lambda^2+6\lambda+8\\ =&\mu+4\sqrt{\mu+1}+4.\end{aligned}$$

Define $\varphi : M^C_{\lambda+2} \longrightarrow \widetilde{L}_2$, where

$$\begin{aligned}\varphi(f(h)+g(h)B) = (&-(h+1+\sqrt{\mu+1})f(h+2),\\ &-(h+3+\sqrt{\mu+1})g(h+2),\\ &-(h-1-\sqrt{\mu+1})(h+1+\sqrt{\mu+1})g(h), f(h),\\ &-(h-1-\sqrt{\mu+1})f(h-2),\\ &-(h-3-\sqrt{\mu+1})g(h-2)\ ).\end{aligned}$$

It easy to show that φ is a linear bijective map, hence it is enough to show that φ is a module homomorphism. Let $f(h), g(h) \in \mathbb{C}[h]$. Then

$$\varphi(x.(f(h)+g(h)B)) = \varphi(\frac{1}{2}(\mu'-h^2+2h)g(h-2)+\frac{1}{2}f(h-2)B)$$

$$\begin{aligned}= (&-\frac{1}{2}(h+1+\sqrt{\mu+1})(\mu'-h^2-2h)g(h),\\ &-\frac{1}{2}(h+3+\sqrt{\mu+1})f(h),\\ &-\frac{1}{2}(h-1-\sqrt{\mu+1})(h+1+\sqrt{\mu+1})f(h-2),\\ &\frac{1}{2}(\mu'-h^2+2h)g(h-2),\\ &-\frac{1}{2}(h-1-\sqrt{\mu+1})(\mu'-h^2+6h-8)g(h-4),\\ &-\frac{1}{2}(h-3-\sqrt{\mu+1})f(h-4)\ ),\end{aligned}$$

and

$$\begin{aligned}x.\varphi(f(h)+g(h)B) = x.(&-(h+1+\sqrt{\mu+1})f(h+2),\\ &-(h+3+\sqrt{\mu+1})g(h+2),\\ &-(h-1-\sqrt{\mu+1})(h+1+\sqrt{\mu+1})g(h),\\ &f(h), -(h-1-\sqrt{\mu+1})f(h-2),\\ &-(h-3-\sqrt{\mu+1})g(h-2)\ )\end{aligned}$$

$$\begin{aligned}
= (\ &-\frac{1}{2}(h+1+\sqrt{\mu+1})(\mu-h^2-2h+4+4\sqrt{\mu+1})g(h),\\
&-\frac{1}{2}(h+3+\sqrt{\mu+1})f(h),\\
&-\frac{1}{2}(h-1-\sqrt{\mu+1})(h+1+\sqrt{\mu+1})f(h-2),\\
&-\frac{1}{2}(h-3-\sqrt{\mu+1})(h+1+\sqrt{\mu+1})g(h-2),\\
&-\frac{1}{2}(h-5-\sqrt{\mu+1})(\mu-h^2+2h)g(h-4),\\
&-\frac{1}{2}(h-3-\sqrt{\mu+1})f(h-4)\ ).
\end{aligned}$$

Note that, in the first components,

$$\mu' - h^2 - 2h = (\mu + 4\sqrt{\mu+1} + 4 - h^2 - 2h).$$

Also,

$$\begin{aligned}
\mu' - h^2 + 2h &= -\ (h-1-\sqrt{\mu'+1})(h-1+\sqrt{\mu'+1})\\
&= -\ (h-3-\sqrt{\mu+1})(h+1+\sqrt{\mu+1}).
\end{aligned}$$

Moreover,

$$\begin{aligned}
(\mu' - h^2 + 6h - 8) &= (\mu + 4\sqrt{\mu+1} - h^2 + 6h - 4)\\
&= -\ (h-1+\sqrt{\mu+1})(h-5-\sqrt{\mu+1})
\end{aligned}$$

which make the fifth components in both equations equal. Hence $\varphi(x.(f(h+g(h)B))) = x.\varphi(f(h)+g(h)B)$. Now

$$\begin{aligned}
\varphi(y.(f(h)+g(h)B)) &= \varphi(\frac{1}{2}(\mu' - h^2 - 2h)g(h+2) + \frac{1}{2}f(h+2)B)\\
= (\ &-\frac{1}{2}(h+1+\sqrt{\mu+1})(\mu'-h^2-6h-8)g(h+4),\\
&-\frac{1}{2}(h+3+\sqrt{\mu+1})f(h+4),\\
&-\frac{1}{2}(h-1-\sqrt{\mu+1})(h+1+\sqrt{\mu+1})f(h+2),\\
&\frac{1}{2}(\mu'-h^2-2h)g(h+2),\\
&-\frac{1}{2}(h-1-\sqrt{\mu+1})(\mu'-h^2+2h)g(h),\\
&-\frac{1}{2}(h-3-\sqrt{\mu+1})f(h)\ ),
\end{aligned}$$

$$
\begin{aligned}
y.\varphi(f(h)+g(h)B) = y\,(&-(h+1+\sqrt{\mu+1})f(h+2),\\
&-(h+3+\sqrt{\mu+1})g(h+2),\\
&-(h-1-\sqrt{\mu+1})(h+1+\sqrt{\mu+1})g(h),\\
&\ f(h), -(h-1-\sqrt{\mu+1})f(h-2),\\
&-(h-3-\sqrt{\mu+1})g(h-2)\ )\\
=(\ &-\frac{1}{2}(h+5+\sqrt{\mu+1})(\mu-h^2-2h)g(h+4),\\
&-\frac{1}{2}(h+3+\sqrt{\mu+1})f(h+4),\\
&-\frac{1}{2}(h-1-\sqrt{\mu+1})(h+1+\sqrt{\mu+1})f(h+2),\\
&-\frac{1}{2}(h+3+\sqrt{\mu+1})(h-1\sqrt{\mu+1})g(h+2),\\
&-\frac{1}{2}(h-1-\sqrt{\mu+1})(\mu-h^2+2h+4+4\sqrt{\mu+1})g(h),\\
&-\frac{1}{2}(h-3-\sqrt{\mu+1})f(h)\ ).
\end{aligned}
$$

Again, when we return to the value of $\mu' = \mu + 4\sqrt{\mu+1} + 4$, we find that

$$\varphi(y.(f(h+g(h)B))) = y.\varphi(f(h)+g(h)B).$$

Finally,

$$
\begin{aligned}
\varphi(h.(f(h)+g(h)B)) =&\varphi(hf(h)+hg(h)B)\\
=&(\ -(h+1+\sqrt{\mu+1})(h+2)f(h+2),\\
&-(h+3+\sqrt{\mu+1})(h+2)g(h+2),\\
&-h(h-1-\sqrt{\mu+1})(h+1+\sqrt{\mu+1})g(h),\\
&\ hf(h), -(h-1-\sqrt{\mu+1})(h-2)f(h-2),\\
&-(h-3-\sqrt{\mu+1})(h-2)g(h-2)\ )\\
=&h.\varphi(f(h)+g(h)B).
\end{aligned}
$$

Hence $M^C_{\lambda+2} \cong \widetilde{L}_2$. □

4.4. The submodule $\widetilde{L}_{-2}$ where c acts as a scalar $\mu+5-4\sqrt{\mu+1}$

Let $\mu \neq 0$ and let $\alpha = \mu + 5 - 4\sqrt{\mu+1}$ (the Casimir constant related the the weight -2). Since

$$(\mu - h^2 + 2h) = -(h-1-\sqrt{\mu+1})(h-1+\sqrt{\mu+1}),$$

and

$$(\mu - h^2 - 2h) = -(h + 1 - \sqrt{\mu + 1})(h + 1 + \sqrt{\mu + 1}),$$

it follows that,

$$f_1(h) = -(h + 1 - \sqrt{\mu + 1})f_4(h + 2), \tag{4.34}$$

$$f_5(h) = -(h - 1 + \sqrt{\mu + 1})f_4(h - 2), \tag{4.35}$$

$$f_2(h) = \frac{f_3(h + 2)}{h + 1 + \sqrt{\mu + 1}}, \tag{4.36}$$

$$f_6(h) = \frac{f_3(h - 2)}{h - 1 - \sqrt{\mu + 1}}. \tag{4.37}$$

Since $f_2(h), f_6(h) \in \mathbb{C}[h]$, it follows that

$$f_3(h) \in \mathbb{C}[h].(h - 1 + \sqrt{\mu + 1})(h + 1 - \sqrt{\mu + 1}).$$

Consider that $\widetilde{L}_{-2} = L^{\mu+5-4\sqrt{\mu+1}}$ is the submodule of $L_{(\lambda,2)}$ consisting of all eigenvectors of c with eigenvalue $\mu + 5 - 4\sqrt{\mu + 1}$. Then for any $v \in \widetilde{L}_{-2}$, v must be written as

$$\begin{aligned} v = (\ &-(h + 1 - \sqrt{\mu + 1})g(h + 2), (h + 3 - \sqrt{\mu + 1})f(h + 2), \\ &(h - 1 + \sqrt{\mu + 1})(h + 1 - \sqrt{\mu + 1})f(h), g(h), \\ &-(h - 1 + \sqrt{\mu + 1})g(h - 2), (h - 3 + \sqrt{\mu + 1})f(h - 2)\ ) \end{aligned} \tag{4.38}$$

for some $f(h), g(h) \in \mathbb{C}[h]$.

Proposition 4.4 *$\widetilde{L}_{-2}$ is isomorphic to $M^C_{\lambda-2}$.*

Proof Note that in $M^C_{\lambda-2}$, the value

$$\begin{aligned} \mu' =&(\lambda - 2)^2 + 2(\lambda - 2) \\ =&\lambda^2 - 2\lambda = \mu - 4\sqrt{\mu + 1} + 4. \end{aligned}$$

Define $\varphi : M^C_{\lambda-2} \longrightarrow \widetilde{L}_{-2}$, where

$$\begin{aligned} \varphi(f(h) + g(h)B) = (\ &-(h + 1 - \sqrt{\mu + 1})f(h + 2), \\ &-(h + 3 - \sqrt{\mu + 1})g(h + 2), \\ &-(h - 1 + \sqrt{\mu + 1})(h + 1 - \sqrt{\mu + 1})g(h), f(h), \\ &-(h - 1 + \sqrt{\mu + 1})f(h - 2), \\ &-(h - 3 + \sqrt{\mu + 1})g(h - 2)\ ). \end{aligned}$$

It is easy to show that φ is a linear bijective map, hence it is enough to show that φ is a module homomorphism. Let $f(h), g(h) \in \mathbb{C}[h]$, we have that

$$
\begin{aligned}
\varphi(x.(f(h)+g(h)B)) =& \varphi(\frac{1}{2}(\mu'-h^2+2h)g(h-2)+\frac{1}{2}f(h-2)B) \\
&= \Big(-\frac{1}{2}(h+1-\sqrt{\mu+1})(\mu'-h^2-2h)g(h), \\
&\quad -\frac{1}{2}(h+3-\sqrt{\mu+1})f(h), \\
&\quad -\frac{1}{2}(h-1+\sqrt{\mu+1})(h+1-\sqrt{\mu+1})f(h-2), \\
&\quad \frac{1}{2}(\mu'-h^2+2h)g(h-2), \\
&\quad -\frac{1}{2}(h-1+\sqrt{\mu+1})(\mu'-h^2+6h-8)g(h-4), \\
&\quad -\frac{1}{2}(h-3+\sqrt{\mu+1})f(h-4)\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
x.\varphi(f(h)+g(h)B) &= x.\Big(-(h+1-\sqrt{\mu+1})f(h+2), \\
&\quad -(h+3-\sqrt{\mu+1})g(h+2), \\
&\quad -(h-1+\sqrt{\mu+1})(h+1-\sqrt{\mu+1})g(h), \\
&\quad f(h), -(h-1+\sqrt{\mu+1})f(h-2), \\
&\quad -(h-3+\sqrt{\mu+1})g(h-2)\Big) \\
&= \Big(-\frac{1}{2}(h+1-\sqrt{\mu+1})(\mu-h^2-2h+4-4\sqrt{\mu+1})g(h), \\
&\quad -\frac{1}{2}(h+3-\sqrt{\mu+1})f(h), \\
&\quad -\frac{1}{2}(h-1+\sqrt{\mu+1})(h+1-\sqrt{\mu+1})f(h-2), \\
&\quad -\frac{1}{2}(h-3+\sqrt{\mu+1})(h+1-\sqrt{\mu+1})g(h-2), \\
&\quad -\frac{1}{2}(h-5+\sqrt{\mu+1})(\mu-h^2+2h)g(h-4), \\
&\quad -\frac{1}{2}(h-3+\sqrt{\mu+1})f(h-4)\Big).
\end{aligned}
$$

Note that, in the first components,

$$
\mu'-h^2-2h = (\mu-4\sqrt{\mu+1}+4-h^2-2h).
$$

Also,

$$\begin{aligned}\mu' - h^2 + 2h =& -(h-1+\sqrt{\mu'+1})(h-1-\sqrt{\mu'+1})\\ =& -(h-3+\sqrt{\mu+1})(h+1-\sqrt{\mu+1}).\end{aligned}$$

Moreover,

$$\begin{aligned}(\mu' - h^2 + 6h - 8) =&(\mu - 4\sqrt{\mu+1} - h^2 + 6h - 4)\\ =& -(h-1-\sqrt{\mu+1})(h-5+\sqrt{\mu+1})\end{aligned}$$

which make the fifth components in both equations equal. Hence

$$\varphi(x.(f(h+g(h)B))) = x.\varphi(f(h)+g(h)B).$$

Now

$$\begin{aligned}\varphi(y.(f(h)+g(h)B)) = \varphi(\frac{1}{2}(\mu-h^2-2h)g(h+2) + \frac{1}{2}f(h+2))\\ = (-\frac{1}{2}(h+1-\sqrt{\mu+1})(\mu'-h^2-6h-8)g(h+4),\\ -\frac{1}{2}(h+3-\sqrt{\mu+1})f(h+4),\\ -\frac{1}{2}(h-1+\sqrt{\mu+1})(h+1-\sqrt{\mu+1})f(h+2),\\ \frac{1}{2}(\mu'-h^2-2h)g(h+2),\\ -\frac{1}{2}(h-1+\sqrt{\mu+1})(\mu'-h^2+2h)g(h),\\ -\frac{1}{2}(h-3+\sqrt{\mu+1})f(h)\ ),\end{aligned}$$

$$\begin{aligned}y.\varphi(f(h)+g(h)B) = y\,(-(h+1-\sqrt{\mu+1})f(h+2), -(h+3-\sqrt{\mu+1})g(h+2),\\ -(h-1+\sqrt{\mu+1})(h+1-\sqrt{\mu+1})g(h), f(h),\\ -(h-1+\sqrt{\mu+1})f(h-2), -(h-3+\sqrt{\mu+1})g(h-2)\ )\\ = (-\frac{1}{2}(h+5-\sqrt{\mu+1})(\mu-h^2-2h)g(h+4),\\ -\frac{1}{2}(h+3-\sqrt{\mu+1})f(h+4),\\ -\frac{1}{2}(h-1+\sqrt{\mu+1})(h+1-\sqrt{\mu+1})f(h+2),\\ -\frac{1}{2}(h+3-\sqrt{\mu+1})(h-1+\sqrt{\mu+1})g(h+2),\\ -\frac{1}{2}(h-1+\sqrt{\mu+1})(\mu-h^2+2h+4-4\sqrt{\mu+1})g(h),\\ -\frac{1}{2}(h-3-\sqrt{\mu+1})f(h)\ ).\end{aligned}$$

Again, when we return to the value of $\mu' = \mu - 4\sqrt{\mu+1} + 4$, we find that

$$\varphi(y.(f(h+g(h)B))) = y.\varphi(f(h)+g(h)B).$$

Finally,

$$\begin{aligned}\varphi(h.(f(h)+g(h)B)) =& \varphi(hf(h)+hg(h)B)\\ =& (\ -(h+1-\sqrt{\mu+1})(h+2)f(h+2),\\ & -(h+3-\sqrt{\mu+1})(h+2)g(h+2),\\ & -h(h-1+\sqrt{\mu+1})(h+1-\sqrt{\mu+1})g(h),\\ & hf(h), -(h-1+\sqrt{\mu+1})(h-2)f(h-2),\\ & -(h-3+\sqrt{\mu+1})(h-2)g(h-2)\ )\\ =& h.\varphi(f(h)+g(h)B).\end{aligned}$$

Hence $M^C_{\lambda-2} \cong \widetilde{L}_{-2}$. □

Note another important result due to Kostant [11, Corollary 5.5]):

Lemma 4.5 *Let c be the Casimir element of $U(\mathfrak{sl}_2(\mathbb{C}))$, and let M be an $\mathfrak{sl}_2(\mathbb{C})$-module where c acts as the scalar ρ. Assume that the values of the Casimir constant*

$$\left\{\alpha_i := (\sqrt{\rho}+2i)^2 \mid i = -n, -n+1, \ldots, n-1, n\right\}$$

are all distinct. Define

$$P_i = \{z \in M \otimes V(n) \mid \ c.z = \alpha_i z\},$$

such that if $0 \neq P_i$, P_i is the maximal submodule of $M \otimes V(n)$ in which c acts as the scalar α_i. Then

$$M \otimes V(n) = \bigoplus_{i=-n}^{n} P_i.$$

□

Note that the values of the Casimir constant are not distinct in the cases when $\mu = -1, 0$.

The following theorem summarizes all results we had so far in this section.

Theorem 4.6 *Let $\mu \in \mathbb{C}\setminus\{-1,0\}$. Then*

$$L_{(\lambda,2)} = \widetilde{L}_2 \oplus \widetilde{L}_0 \oplus \widetilde{L}_{-2} \cong M^C_{\lambda+2} \oplus M^C_{\lambda} \oplus M^C_{\lambda-2}.$$

□

4.5. Particular cases

If $\mu = 0$, then all the previous calculations of $\widetilde{L}_0, \widetilde{L}_2$, and $\widetilde{L}_{-2}$ still work. The submodule $\widetilde{L}_2$ is still the maximal submodule of $L_{(\lambda,2)}$ on which c acts as the scalar $\mu + 5 + 4\sqrt{\mu+1} = 9$, but the difference is that $\widetilde{L}_0$, and $\widetilde{L}_{-2}$ are not maximal with respect to the action of c. Our goal now is to find the maximal submodule on which c acts as the scalar 1. In this case we have

$$f_1(h) = -hf_4(h+2), \tag{4.39}$$

$$f_5(h) = -hf_4(h-2), \tag{4.40}$$

$$f_2(h) = \frac{f_3(h+2)}{h+2}, \tag{4.41}$$

$$f_6(h) = \frac{f_3(h-2)}{h-2}. \tag{4.42}$$

Since $f_5(h), f_6(h) \in \mathbb{C}[h]$, it follows that $f_3(h)$ must belong to $\mathbb{C}[h].h$. Let $\widehat{L^1}$ be the submodule of $L_{(\lambda,2)}$ consisting of all eigenvectors of c with eigenvalue 1. Then for any $v \in \widehat{L^1}$, v must be written as

$$\begin{aligned} v = (\ &-hg(h+2), f(h+2), hf(h), \\ &g(h), -hg(h-2), f(h-2)\ ) \end{aligned} \tag{4.43}$$

for some $f(h), g(h) \in \mathbb{C}[h]$. It is clear that $\widetilde{L}_0$ and $\widetilde{L}_{-2}$ are submodules of the module $\widehat{L^1}$.

The second particular case is when $\mu = -1$ ($\lambda = -1$). All of the previous calculations for $\widetilde{L}_0$, $\widetilde{L}_2$, and $\widetilde{L}_{-2}$ work also in this case. The submodule $\widetilde{L}_0$ is still the maximal submodule of $L_{(\lambda,2)}$ on which c acts as the scalar $\mu + 1 = 0$. The modules $\widetilde{L}_2$, and $\widetilde{L}_{-2}$ have the same Casimir constant equal to 4. Now we will find the maximal submodule on which c acts as the scalar 4. We have

$$f_1(h) = -(h+1)f_4(h+2), \tag{4.44}$$

$$f_5(h) = -(h-1)f_4(h-2), \tag{4.45}$$

$$f_2(h) = \frac{f_3(h+2)}{h+1}, \tag{4.46}$$

and

$$f_6(h) = \frac{f_3(h-2)}{h-1}. \tag{4.47}$$

Since $f_2(h), f_6(h) \in \mathbb{C}[h]$, also $f_3(h)$ must belong to $\mathbb{C}[h].(h-1)(h+1)$. Let $\widehat{L^4}$ be the subset of $L_{(\lambda,2)}$ of all elements such that c acts as the scalar 1. Then for any $v \in \widehat{L^4}$, v must be written as

$$\begin{aligned} v = (\ &-(h+1)g(h+2), (h+3+)f(h+2), \\ &(h-1-)(h+1+\sqrt{\mu+1})f(h), g(h), \\ &-(h-1)g(h-2), (h-3)f(h-2)\ ) \end{aligned} \tag{4.48}$$

for some $f(h), g(h) \in \mathbb{C}[h]$. which means that $\widehat{L^4} = \widetilde{L}_2 = \widetilde{L}_{-2}$.

Indeed, in the case when $\mu = -1$, it is easy to find an element $v \in L_{(\lambda,2)}$ such that $c(c-4).v \neq 0$ (for example let $v = 1 \otimes h$), which implies that $L_{(\lambda,2)}$ cannot be the direct sum of $\widetilde{L}_0$ and $\widetilde{L}_2$.

The detailed structure of this module is as follows. Consider the submodule $U = c.L_{(\lambda,2)}$. Then U has the Casimir constant 4. Now $(c-4).U \neq 0$. Hence $W = c(c-4).L_{(\lambda,2)}$ is a nonzero submodule of U which also has the Casimir constant 4, that is, $(c-4).W = 0$. Indeed, the submodule W is just the submodule $\tilde{L}_2$(or $\tilde{L}_{-2}$), which is a simple submodule. Now consider the submodule $T = (c-4).L_{(\lambda,2)}$. Then T admits two of the Casimir constants, 0 and 4. The submodules $S = (c-4)^2.L_{(\lambda,2)}$ and W are simple submodules of T, which are isomorphic to $\tilde{L}_0, \tilde{L}_2$ respectively. Moreover, $L_{(\lambda,2)} = U \oplus S$.

Remark 4.7 *This example shows that if we have Casimir constants which are not distinct (that is, the hypotheses of Kostant's Lemma 4.5 are not satisfied), then the module may decompose as a decomposition of root spaces but not as a decomposition of eigenspaces, see the enclosed picture.*

The lattice of submodules of $L_{(\lambda,2)}$ when $\mu = -1$ is shown in Figure **??**.

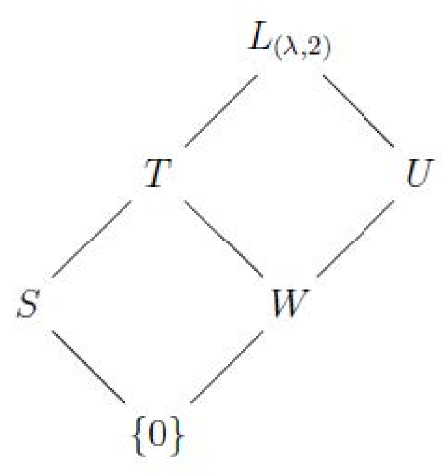

Figure. Submodules of $L_{(\lambda,2)}$ when $\mu = -1$

5. Tensor products of M_λ^C and $V(2n)$

In this part, we will give a general result about the tensor products of the module M_λ^C and a simple finite dimensional module whose highest weight is even. First we quote a classical result (see, for example, the proof in [12, Theorem 1.39]).

Theorem 5.1 *Consider simple finite dimensional modules $V(n)$ and $V(m)$, where $m \leq n$. Then*

$$V(n) \otimes V(m) \cong \bigoplus_{i=0}^{m} V(n+m-2i).$$

□

Now let us define the module

$$L_{(\lambda,2n)} = M_\lambda^C \otimes V(2n). \tag{5.1}$$

Our next theorem is a generalization of Corollary 4.6 to the module $L_{(\lambda,n)}$. The proof of this theorem will follow the steps of [12, Theorem 3.81] and [13, Corollary 18].

Theorem 5.2 *Let* $\mu \in \mathbb{C} \setminus \{-1, 0\}$. *Then*

$$L_{(\lambda,2n)} \cong \bigoplus_{i=0}^{2n} M_{\lambda+2n-2i}^C.$$

Proof Consider $L = L_{(\lambda,2n)}$ with $\lambda \in \mathbb{C} \setminus \{-2, -1, 0\}$. We will prove our theorem using induction on n. For $n = 1$ the result follows directly from Theorem 4.6.

For $n \geq 2$, using Theorem 5.1, we have

$$V(2) \otimes V(2n-2) \cong V(2n) \oplus V(2n-2) \oplus V(2n-4). \tag{5.2}$$

Now using (5.2), we have

$$\begin{aligned} M_\lambda^C \otimes V(2) \otimes V(2n-2) \cong & M_\lambda^C \otimes (V(2n) \oplus V(2n-2) \oplus V(2n-4)) \qquad (5.3)\\ \cong & M_\lambda^C \otimes V(2n) \oplus M_\lambda^C \otimes V(2n-2) \\ & \oplus M_\lambda^C \otimes V(2n-4) \\ \text{(by induction step)} \cong & M_\lambda^C \otimes V(2n) \oplus \bigoplus_{i=0}^{2n-2} M_{\lambda+2n-2-2i}^C \\ & \oplus \bigoplus_{i=0}^{2n-4} M_{\lambda+2n-4-2i}^C \\ = & M_\lambda^C \otimes V(2n) \oplus M_{\lambda+2n-2}^C \oplus \cdots \oplus M_{\lambda-2n+2}^C \\ & \oplus M_{\lambda+2n-4}^C \oplus M_{\lambda+2n-6}^C \oplus \cdots \\ & \oplus M_{\lambda-2n+6}^C \oplus M_{\lambda-2n+4}^C. \end{aligned}$$

Now using Theorem 4.6, we have

$$\begin{aligned} M_\lambda^C \otimes V(2) \otimes V(2n-2) \cong & (M_{\lambda+2}^C \oplus M_\lambda^C \oplus M_{\lambda-2}^C) \otimes V(2n-2) \qquad (5.4)\\ \cong & M_{\lambda+2}^C \otimes V(2n-2) \oplus M_\lambda^C \otimes V(2n-2) \\ & \oplus M_{\lambda-2}^C \otimes V(2n-2) \end{aligned}$$

$$
\begin{aligned}
\text{(by induction step)} &\cong \bigoplus_{i=0}^{2n-2} M^C_{\lambda+2n-2i} \oplus \bigoplus_{i=0}^{2n-2} M^C_{\lambda+2n-2-2i} \\
&\oplus \bigoplus_{i=0}^{2n-2} M^C_{\lambda+2n-4-2i} \\
&= M^C_{\lambda+2n-2} \oplus \cdots \oplus M^C_{\lambda-2n+6} \oplus M^C_{\lambda-2n+4} \\
&\oplus M^C_{\lambda+2n-2} \oplus \cdots \oplus M^C_{\lambda-2n+4} \oplus M^C_{\lambda-2n+2} \\
&\oplus M^C_{\lambda+2n-4} \oplus \cdots M^C_{\lambda-2n+2} \oplus M^C_{\lambda-2n}.
\end{aligned}
$$

By Theorem 2.5, M^C_λ has a finite length (hence both Artinian and Noetherian). Then, Krull–Schmidt theorem applies (see e.g., [10]). To complete the proof, it remains to compare (5.3) and (5.4). □

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