Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 1809 – 1813
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doi:10.55730/1300-0098.3233

Symmetric polynomials in the free metabelian associative algebra of rank 2

Dedicated to the 70$^{\text{th}}$ anniversary of Vesselin Drensky

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Received: 19.01.2022 • **Accepted/Published Online:** 08.04.2022 • **Final Version:** 20.06.2022

Abstract: Let F be the free metabelian associative algebra generated by x and y over a field of characteristic zero. We call a polynomial $f \in F$ symmetric, if $f(x,y) = f(y,x)$. The set of all symmetric polynomials coincides with the algebra F^{S_2} of invariants of the symmetric group S_2. In this paper, we give the full description of the algebra F^{S_2}.

Key words: Metabelian, symmetric polynomial

1. Introduction

Let $K[X_n]$ be the algebra of polynomials in n commuting variables over a field K of characteristic zero, where $X_n = \{x_1, \dots, x_n\}$. It is well known that the algebra

$$K[X_n]^{S_n} = \{p \in K[X_n] \mid p(x_1, \dots, x_n) = p(x_{\pi(1)}, \dots, x_{\pi(n)}),\ \forall \pi \in S_n\}$$

of symmetric polynomials is generated by elementary symmetric polynomials $\sigma_1, \dots, \sigma_n$, where

$$\sigma_1 = x_1 + \cdots + x_n\ ,\ \sigma_2 = x_1x_2 + \cdots + x_1x_n + \cdots + x_{n-1}x_n\ , \dots,\ \sigma_n = x_1 \cdots x_n.$$

One may consider noncommutative or nonassociative analogues of the above result. As a pioneer, Wolf [12] in 1936 handled the problem for the algebra $K\langle X_n\rangle^{S_n}$, where $K\langle X_n\rangle$ is the free associative algebra. One may also see the work of Bergeron et al. [3] on the invariants and coinvariants of the symmetric groups in noncommuting variables. For a survey on symmetric polynomials in noncommutative variables, we suggest the paper by Boumova et al. [4]. In a recent work [1], the algebra of symmetric polynomials of the free algebra of rank three in the variety of Grassmann algebras was described. One may also see the work [8] on the symmetric polynomials of the algebra generated by two 2×2 generic traceless matrices and of its Lie subalgebra. When considering the nonassociative case, the recent papers [6, 9], [10], and [7] consider symmetric polynomials of free metabelian Lie algebras, free metabelian Leibniz algebras, and free metabelian Poisson algebras, respectively.

In the present paper, we describe the algebra F^{S_2} of symmetric polynomials in the free metabelian associative algebra F of rank two, and provide a finite generating set for F^{S_2}.

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2010 *AMS Mathematics Subject Classification:* 17B01; 17B30; 16S15.

2. Preliminaries

Let A be the free associative algebra of rank two over a field K of characteristic zero. Then, the algebra $F = A/(A')^2$ is the free metabelian associative algebra of rank two, where $A' = A[A, A]A$ stands for the commutator ideal of A generated by all elements of the form $[a, b] = ab - ba$, when $a, b \in A$. The algebra F satisfies the metabelian identity $[a, b][c, d] = 0$ (see [11]). Assume that F is freely generated by x and y. Let all commutators be left normed: $[a, b, c] = [[a, b], c]$. Then it is well known that (see [2, 5]) the commutator ideal F' of F is of a basis consisting of elements of the form

$$x^m y^n [x, y, \underbrace{x, \ldots, x}_{k}, \underbrace{y, \ldots, y}_{l}] \ , \quad m, n, k, l \geq 0.$$

However, for the needs of the paper we use another basis of the algebra F as follows.

$$\underbrace{x^m y^n}_{\text{basis of } F/F'} \quad , \quad \underbrace{x^m y^n [x, y] x^k y^l}_{\text{basis of } F'} \quad , \quad m, n, k, l \geq 0.$$

The metabelian identity implies that $xyu = yxu$ and $uxy = uyx$ for every element $u \in F'$. This yields the following construction. We consider the action of the commutative polynomial algebra $K[x_1, y_1, x_2, y_2]$ on F' defined as

$$(x_1^a y_1^b x_2^c y_2^d) u = x^a y^b u x^c y^d \ , \quad u \in F'.$$

Hence, the vector space F' is the free left $K[x_1, y_1, x_2, y_2]$-module generated by $[x, y]$ via this action.

Recall that every element of the set

$$F^{S_2} = \{f(x, y) \in F \mid f(x, y) = f(y, x)\}$$

is called a symmetric polynomial of the free associative algebra F. Note that F^{S_2} coincides with the algebra of invariants of the symmetric group S_2. In the next section, we give a generating set for the algebra F^{S_2}.

3. Main results

The next lemma describes the forms of symmetric polynomials in the left $K[x_1, y_1, x_2, y_2]$-module F'.

Lemma 3.1 *Let $p \in K[x_1, y_1, x_2, y_2]$. Then the followings are equivalent.*

(1) $p(x_1, y_1, x_2, y_2)[x, y] \in (F')^{S_2}$.

(2) $p(x_1, y_1, x_2, y_2) = -p(y_1, x_1, y_2, x_2)$.

(3) $p(x_1, y_1, x_2, y_2) = (x_1 - y_1) p_1(x_1, y_1, x_2, y_2) + (x_2 - y_2) p_2(x_1, y_1, x_2, y_2)$, *for some $p_1, p_2 \in K[x_1, y_1, x_2, y_2]^{S_2}$.*

Proof (1) $\Rightarrow$ (2) Let $\tau_{12} \in S_2$ be the transposition exchanging x and y. Then,

$$\begin{aligned} p(x_1, y_1, x_2, y_2)[x, y] &= \tau_{12}\Big(p(x_1, y_1, x_2, y_2)[x, y]\Big) \\ &= p(y_1, x_1, y_2, x_2)[y, x] \\ &= -p(y_1, x_1, y_2, x_2)[x, y]. \end{aligned}$$

Hence, $\big(p(x_1,y_1,x_2,y_2)+p(y_1,x_1,y_2,x_2)\big)[x,y]=0$. Therefore, $p(x_1,y_1,x_2,y_2)+p(y_1,x_1,y_2,x_2)=0$ in the free left $K[x_1,y_1,x_2,y_2]$-module F' generated by the single element $[x,y]$.

$(2)\Rightarrow(1)$ and $(3)\Rightarrow(2)$ are clear.

$(2)\Rightarrow(3)$ We may assume that $p(x_1,y_1,x_2,y_2)=(x_1-y_1)p_1(x_1,y_1,x_2,y_2)+q(y_1,x_2,y_2)$, where q does not depend on x_1. Then,

$$\begin{aligned} p(y_1,x_1,y_2,x_2)&=(y_1-x_1)p_1(y_1,x_1,y_2,x_2)+q(x_1,y_2,x_2)\\ &=-p(x_1,y_1,x_2,y_2)\\ &=(y_1-x_1)p_1(x_1,y_1,x_2,y_2)-q(y_1,x_2,y_2). \end{aligned}$$

Substituting $x_1=y_1$, we get that $q(y_1,y_2,x_2)=-q(y_1,x_2,y_2)$. Hence, one may express that

$$q(y_1,x_2,y_2)=(x_2-y_2)q_1(y_1,x_2,y_2)+q_2(y_1,y_2),$$

where q_2 does not depend on x_1,x_2. Then

$$\begin{aligned} q(y_1,y_2,x_2)&=(y_2-x_2)q_1(y_1,y_2,x_2)+q_2(y_1,x_2)\\ &=-q(y_1,x_2,y_2)\\ &=(y_2-x_2)q_1(y_1,x_2,y_2)-q_2(y_1,y_2). \end{aligned}$$

Now $x_2=y_2$ yields that $2q_2(y_1,y_2)=0$, and hence $q_2=0$. □

Remark 3.2 *Note that the algebra* $K[x_1,y_1,x_2,y_2]^{S_2}$ *is generated by* x_1+y_1, x_2+y_2, x_1y_1, x_2y_2 *and* $x_1y_2+x_2y_1$ *(see [6]). In addition, the following holds.*

$$(x_1y_2+x_2y_1)^2+A(x_1y_2+x_2y_1)+B=0,$$

where $A=-(x_1+y_1)(x_2+y_2)$ *and* $B=x_1y_1\big((x_2+y_2)^2-2x_2y_2\big)+x_2y_2\big((x_1+y_1)^2-2x_1y_1\big)$. *Hence,* A *and* B *depend on* x_1+y_1, x_2+y_2, x_1y_1, x_2y_2. *Therefore, every polynomial* $p\in K[x_1,y_1,x_2,y_2]^{S_2}$ *is of the form*

$$p=q(x_1+y_1,x_2+y_2,x_1y_1,x_2y_2)+r(x_1+y_1,x_2+y_2,x_1y_1,x_2y_2)(x_1y_2+x_2y_1)$$

for some $q,r\in K[x_1+y_1,x_2+y_2,x_1y_1,x_2y_2]$.

In the next theorem, we give generators of the algebra F^{S_2} of symmetric polynomials.

Theorem 3.3 *Let* $\mathrm{char}K\neq 2$. *Then* F^{S_2} *is generated by*

$$\begin{aligned} &x+y,\quad xy+yx,\quad u_1=x[x,y]-y[x,y],\quad u_2=[x,y]x-[x,y]y,\\ &u_3=xu_1y+yu_1x,\quad u_4=xu_2y+yu_2x. \end{aligned}$$

Proof Initially, it follows from $\mathrm{char}K\neq 2$ that $x+y$ and $xy+yx$ generate $K[x,y]^{S_2}\cong(F/F')^{S_2}$. They act on F' as the polynomials

$$x_1+y_1,\ x_2+y_2,\ 2x_1y_1,\ 2x_2y_2\in K[x_1,y_1,x_2,y_2].$$

If we show that $(F')^{S_2}$ is generated by u_1, u_2, u_3, u_4 as a $K[x_1+y_1, x_2+y_2, x_1y_1, x_2y_2]$-module, then the proof will be completed. We prove the theorem in two steps.

Step 1 Let $f = f_1 + f_2 \in F^{S_2}$, where $f_1 \in K[x,y]^{S_2}$ and $f_2 \in (F')^{S_2}$. Then $f_1 = p(x+y, 2xy)$ for some $p \in K[v_1, v_2]$. Thus, $f_1 - p(x+y, 2xy) \equiv 0 \pmod{(F')^{S_2}}$. This implies that $f_1 - p(x+y, 2xy) \in (F')^{S_2}$, and for some $q \in K[x_1, y_1, x_2, y_2]$ we have

$$f - p(x+y, 2xy) = q(x_1, y_1, x_2, y_2)[x, y],$$

i.e. can be presented.

Step 2 Now let $p(x_1, y_1, x_2, y_2)[x, y] \in (F')^{S_2}$. Then

$$p(x_1, y_1, x_2, y_2) = (x_1 - y_1)p_1(x_1, y_1, x_2, y_2) + (x_2 - y_2)p_2(x_1, y_1, x_2, y_2),$$

for some $p_1, p_2 \in K[x_1, y_1, x_2, y_2]^{S_2}$ by Lemma 3.1. Then, we have that

$$p_i = q_i(x_1+y_1, x_2+y_2, x_1y_1, x_2y_2) + r_i(x_1+y_1, x_2+y_2, x_1y_1, x_2y_2)(x_1y_2 + x_2y_1)\ , \quad i = 1, 2,$$

for some explicitly given q_1, q_2, r_1, r_2 depending on x_1+y_1, x_2+y_2, x_1y_1, x_2y_2 by Remark 3.2. This implies that

$$\begin{aligned} p(x_1, y_1, x_2, y_2)[x, y] =& \Big((x_1 - y_1)\big(q_1 + r_1(x_1y_2 + x_2y_1)\big) + (x_2 - y_2)\big(q_2 + r_2(x_1y_2 + x_2y_1)\big)\Big)[x, y] \\ =& q_1\big(\underbrace{x[x, y] - y[x, y]}_{u_1}\big) + r_1\big(xu_1y + yu_1x\big) + q_2\big(\underbrace{[x, y]x - [x, y]y}_{u_2}\big) + r_2\big(xu_2y + yu_2x\big). \end{aligned}$$

The action of q_i and r_i is a linear combination of composition of multiplications from both sides by $x+y$ and $\frac{xy+yx}{2}$. □

Acknowledgement

The author is grateful to anonymous referees for careful reading and useful suggestions.

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