Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 1814 – 1827
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doi:10.55730/1300-0098.3234

$\mathbb{Z}$-graded identities on the infinite-dimensional Grassmann algebra and arithmetic tools: revisited

Dedicated to Professor Vesselin Drensky on the occasion of his seventieth anniversary

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Received: 06.11.2021 • **Accepted/Published Online:** 09.03.2022 • **Final Version:** 20.06.2022

Abstract: The main purpose of this paper is to provide a survey of results concerning the $\mathbb{Z}$-gradings on the infinite-dimensional Grassmann algebra E over a field of characteristic zero. First, we provide graded identities and central polynomials for E equipped with fine gradings on E by the semigroup $(\mathbb{Z}^*, \times)$. We also describe briefly techniques in order to illustrate some important methods to exhibit graded identities and central polynomials of E for other abelian groups. In particular, over a field of characteristic zero, so-called 2-induced gradings of full support were considered. In order to obtain these descriptions, we strongly use elementary number theory as a tool, providing an interesting connection between this area and PI-Theory.

Key words: Grassmann algebra, graded identity, graded central polynomial, full support

1. Introduction

The Grassmann algebra E of an infinite-dimensional vector space L is one of the most important algebras satisfying a polynomial identity. The celebrated papers of Kemer have it as a key ingredient [19–21]. More precisely, Kemer proved that any associative PI-algebra over a field of characteristic zero satisfies the same identities (is PI-equivalent) of the Grassmann envelope of a finite-dimensional associative superalgebra. We also recall that the polynomial identities of the Grassmann algebra were described by Latyshev [23], and later on, in a different manner, and with great details, by Krakowski and Regev [22]. The identities of E have been extensively studied also in positive characteristic. The interested reader can consult the paper [13] and its references for a description of the polynomial identities in the case where the ground field is infinite of positive characteristics. Due to these results, it is well known that the polynomial identities of the Grassmann algebra are consequences of a single identity, namely the triple commutator $[x_1, x_2, x_3]$. A concept that has a close connection to polynomial identities is the notion of central polynomials. The central polynomials for infinite-dimensional Grassmann algebra are quite interesting. In [2], Brandão et al. described a finite set of polynomials generating $C(E)$ as a T-space (vector space that is invariant under the endomorphisms of the free algebra) over a field of characteristic 0, while this is not the case when the field is infinite of characteristic $p > 2$. This is the first example of an associative algebra whose T-space of central polynomials is not finitely generated.

Assume G an abelian group. The structure of G-graded algebras motivated the study of their G-graded identities. In light of this it seems a natural and interesting problem to investigate more closely the structure

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2010 *AMS Mathematics Subject Classification:* 15A75, 16R10.

of the graded identities and graded central polynomials of E. A complete study of the graded identities of E graded by any group is still far from being understood. For example, in [7] the authors studied all homogeneous superalgebra structures defined on the Grassmann algebra, which were denoted by E_k, E_{k^*} and E_∞. Moreover, their $\mathbb{Z}_2$-graded identities were described in [4, 7, 11]. Furthermore, gradings and graded identities on E by abelian groups of order larger than 2 were also investigated in [3, 5, 6, 16, 18]. Recently, in [14], Guimarães investigated gradings and graded identities by the additive group of the real numbers, starting a study on gradings by groups that may even be uncountable. On the other hand, the literature for graded central polynomials is still restricted. We also mention two recent papers. Guimarães, Fidelis and Koshlukov [15] exhibited a basis for the T_2-space of the $\mathbb{Z}_2$-graded central polynomials of E, in each grading, and a basis for the $T_{\mathbb{Z}}$-space of the $\mathbb{Z}$-graded central polynomials for three types of $\mathbb{Z}$-gradings on E, denoted by E^∞, E^{k^*} and E^k, and they exhibited a basis for the $T_{\mathbb{Z}}$-ideal of the identities of these gradings. Moreover, Guimarães, Fidelis and Dias [16] provided a basis for the central polynomials for all $\mathbb{Z}_q$-gradings on E, when the ground field has characteristic different from 2 and q is an odd prime. It should be noted that the information about types of nonhomogeneous Grassmann superalgebras was not mentioned, we direct the interested reader to [17] for further information and references.

This paper is organized as follows. In Section 2 we give the necessary background on associative graded algebras and graded identities. The monographs [8, 12], and their references, give a good account on the results already obtained. In Section 3, we provide graded identities and central polynomials for E equipped with fine gradings by the semigroup $(\mathbb{Z}^*, \times)$. Assuming G an any abelian group, in Section 4 we construct certain type of G-grading on E whose T_G-ideal of all graded identities is close related to $T_{G/H}$-ideal of E, for some subgroup H of G, and this generalize some results found in [5]. Of course, such results extended to the central polynomials. As a consequence, in Section 5, we provide a survey of results concerning the $\mathbb{Z}$-gradings on the infinite-dimensional Grassmann algebra over a field of characteristic zero. In particular, we exhibit a new proof for obtaining a basis of graded identities and central polynomials for the structures called of 2-induced $\mathbb{Z}$-grading on E of full support.

2. Preliminaries

Hereinafter, unless specified otherwise, K denotes a field of characteristic different from 2. Let G be a set equipped with an operation "$\cdot$" that satisfies the following condition: for all g_1, $g_2 \in G$, $g_1 \cdot g_2 \in G$. Such structure is called semigroup. A grading by G (or simply a G-grading) on an associative algebra A is a vector space decomposition

$$\Gamma\colon A = \oplus_{g\in G} A_g, \tag{2.1}$$

such that $A_g A_h \subseteq A_{gh}$, for all g, $h \in G$. In this case one says that A is G-graded. The subspaces A_g are the homogeneous components of the grading and a nonzero element a of A is homogeneous if $a \in A_g$ for some $g \in G$; we denote this by $\|a\|_G = g$ (or simply $\|a\| = g$ when the group G is clear from the context). The support of the previous grading is the set $Sup(\Gamma) = \{g \in G \mid A_g \neq 0\}$. In particular, a grading on A is said to be of full support if its support is the whole of G. A vector space (subalgebra, ideal) B of A is said to be a G-homogeneous vector space (subalgebra, ideal) if $B = \oplus_{g\in G} B \cap A_g$. It is natural to use the concepts and results of classical ring theory in the G-graded environment. Thus, if A, B are G-graded algebras, a homomorphism of algebras $\varphi\colon A \to B$ is called *G-homomorphism* if $\varphi(A_g) \subseteq B_g$ for all $g \in G$. In particular, this gives the notion of an endomorphism of graded algebras, or G-endomorphism for short. Assuming G a group, if H is a normal subgroup of G, we define the quotient G/H-grading as $A = \oplus_{\overline{g} \in G/H} A_{\overline{g}}$ where $A_{\overline{g}} = \oplus_{h \in H} A_{gh}$.

An important example of graded algebra is the Grassmann algebra. This algebra played a major role in Kemer's proof in [21] of Specht's conjecture. Such algebra is defined as follows: let L be a vector space with a basis $\mathcal{B} = \{e_1, e_2, \dots\}$. The infinite-dimensional Grassmann (or exterior) algebra E of L has a basis consisting of 1 and all monomials $e_{i_1} e_{i_2} \cdots e_{i_k}$, where $i_1 < i_2 < \ldots < i_k$ for any $k \geq 1$. The multiplication in E is induced by $e_i e_j = -e_j e_i$, for all i and j. It is easy to see that the set $\mathcal{B}_E = \{1, e_{i_1} e_{i_2} \cdots e_{i_k} \mid i_1 < i_2 < \ldots < i_k, k \geq 1\}$ is a basis for E. Hence $E = E_{(0)} \oplus E_{(1)}$, where $E_{(0)}$ is the subspace spanned by 1 and all monomials of even length while $E_{(1)}$ is spanned by the monomials of odd length. Clearly $E_{(0)}$ is the centre of E and $E_{(1)}$ is the "anticommuting" part of E. This decomposition gives the natural (or canonical) $\mathbb{Z}_2$-grading on E, denoted by E_{can}.

Definition 2.1 *Let $b = c_1 \cdots c_k$, where $c_j \in \mathcal{B}$. The set $\{c_1, \dots, c_k\}$ is called the support of b, and it will be denoted it by* $\mathrm{supp}(b)$.

Given b_1, $b_2 \in \mathcal{B}_E$, we have $b_1 b_2 \neq 0$ if and only if b_1 and b_2 have disjoint supports. Moreover, if b_1 or b_2 is equal to 1, we assume that b_1 and b_2 have disjoint supports. We say that a grading on the Grassmann algebra E is homogeneous if the elements in $\mathcal{B}$ are homogeneous. Below, as it was given for the first time in [7], we recall the construction of the homogeneous $\mathbb{Z}_2$-gradings on E. All homogeneous $\mathbb{Z}_2$-gradings on E are given by the following possibilities of $\mathbb{Z}_2$-degree on the basis of L:

$$\|e_i\|_k = \begin{cases} 0, & \text{if } i = 1, \dots, k \\ 1, & \text{otherwise} \end{cases} \quad ,$$

$$\|e_i\|_{k^*} = \begin{cases} 1, & \text{if } i = 1, \dots, k \\ 0, & \text{otherwise} \end{cases} \quad ,$$

$$\|e_i\|_\infty = \begin{cases} 0, & \text{for } i \text{ even} \\ 1, & \text{for i odd} \end{cases} \quad .$$

We then induce the $\mathbb{Z}_2$-grading on E by putting

$$\|e_{j_1} \cdots e_{j_n}\| = \|e_{j_1}\| + \cdots + \|e_{j_n}\| \mod 2,$$

and extending it to E by linearity. These gradings are denoted by E_k, E_{k^*} and E_∞, respectively. When $\|e_i\| = 1$ for all i, we obtain the $\mathbb{Z}_2$-grading by E_{can}, the natural (or canonical) grading on E. We draw the readers' attention that if L is homogeneous (and $G = \mathbb{Z}_2$), then one can always choose a basis of L which is homogeneous in the grading. The procedure just described is more general, and we will present it below.

Let G be a semigroup equipped with an operation "$\cdot$", consider $g_1, \dots, g_r$ in G and suppose that $g_i \cdot g_j = g_j \cdot g_i$, for all $i, j \in \{1, \dots, r\}$. We take $v_1, \dots, v_r$ that can be $v_j \in \mathbb{N}$ or $v_j = \infty$, for $1 \leq j \leq r$. Consider

$$L = L_{g_1}^{v_1} \oplus \cdots \oplus L_{g_r}^{v_r}$$

a decomposition of L in r subspaces such that $v_j = \dim L_{g_j}^{v_j}$. Then we can define a G-grading $E_{(g_1,\dots,g_r)}^{(v_1,\dots,v_r)}$ on E whose support will be denoted by $S_{(g_1,\dots,g_r)}^{(v_1,\dots,v_r)}$. Observe that the condition $g_i \cdot g_j = g_j \cdot g_i$, for all $i, j \in \{1, \dots, r\}$, is essential. In this case we say that $E_{(g_1,\ldots,g_r)}^{(v_1,\ldots,v_r)}$ is the *r-induced* G-grading. We say that g_1, ..., g_r are the *lower indexes* and v_1, ..., v_r are the *upper indexes* of the grading.

Throughout the rest of the paper, unless otherwise stated, all the gradings on E will be considered homogeneous gradings.

Let $X_G = \cup_{g\in G} X_g$ be the disjoint union of infinite countable sets of variables $X_g = \{x_1^g, x_2^g, \ldots\}$, $g \in G$. We denote by $K\langle X_G\rangle$ the free associative G-graded algebra freely generated by X_G. This algebra has a natural G-grading where the homogeneous component $(K\langle X_G\rangle)_g$ is the span of all monomials $x_{i_1}^{g_1}\cdots x_{i_m}^{g_m}$ such that $g_1\cdots g_m = g$. The elements in $K\langle X_G\rangle$ are called *G-polynomials* (or simply polynomials when the semigroup G is inferred from the context). Let $f(x_1^{g_1},\ldots,x_m^{g_m})$ be a polynomial in $K\langle X_G\rangle$. The degree of f in the variable $x_i^{g_i}$ is defined in the usual manner, thus the notion of multilinear polynomials arises naturally. In order to make the notation simpler, when the grading group is clear from the context, we shall omit the double indexes of the variables $x_i^{g_i}$.

Let A be an associative algebra equipped with a G-grading Γ as given in Eq. (2.1). An m-tuple $(a_1,\ldots,a_m)$ such that $a_i \in A_{g_i}$ for $i = 1$, ..., m, is called *f-admissible substitution* (or simply an admissible substitution). If $f(a_1,\ldots,a_m) = 0$ for every admissible substitution $(a_1,\ldots,a_m)$, we say that the polynomial f is a G-polynomial identity for the graded algebra A, or simply G-identity. In additional, we say that f is a graded central polynomial, or G-central polynomial, for A if $f(a_1,\ldots,a_m) \in Z(A)$ for every admissible substitution $(a_1,\ldots,a_m)$, where $Z(A)$ denotes the centre of A. We denote by $T_G(\Gamma)$ and $C_G(\Gamma)$ the set of all G-identities and all G-central polynomials for the algebra A equipped with grading Γ, respectively. If G is the trivial group, we recover the usual notion of ordinary polynomial identities and central polynomials, in this case, we use the notation $K\langle X\rangle$ for the free associative algebra and x_i's for the variables. As in the ordinary case, it is clear that $T_G(\Gamma)$ and $C_G(\Gamma)$ are vector subspaces of $K\langle X_G\rangle$. They are invariant under all endomorphisms of $K\langle X_G\rangle$ which respect the grading. Such vector subspaces are called *T_G-spaces*; in particular $T_G(\Gamma)$ is called *T_G-ideal*.

The intersection of T_G-ideals (respectively T_G-spaces) of $K\langle X_G\rangle$ is also a T_G-ideal (respectively a T_G-space). A subset $\mathcal{P} \subset K\langle X_G\rangle$ generates the T_G-ideal $T_G(\Gamma)$ (the T_G-space $C_G(\Gamma)$) if it equals the intersection of all T_G-ideals (T_G-spaces) in $K\langle X_G\rangle$ that contain $\mathcal{P}$, and we denote it by $\langle\mathcal{P}\rangle_{I_G}$, or sometimes, when the semigroup G is known from the context, by $\langle\mathcal{P}\rangle_I$, I for ideal, (respectively $\langle\mathcal{P}\rangle_{S_G}$, or $\langle\mathcal{P}\rangle_S$, S for space).

A polynomial $f \in K\langle X_G\rangle$ is regular if every one of its variables appears in every monomial of f, not necessarily with the same degree. It is easy to see that every T_G-ideal is generated by its regular polynomials. In addition, it is well known that in studying ordinary polynomial identities in characteristic 0, it is sufficient to consider the multilinear ones. An analogous fact holds for graded identities and graded central polynomials as well, and the proofs repeat verbatim those of the ordinary case. Thus if A is a G-graded algebra over a field F of characteristic 0 the ideal of all G-identities of A is generated as a T_G-ideal by its multilinear polynomials, and analogously for G-central polynomials. Given algebras A and B equipped with a G-grading Γ_A and Γ_B, respectively, we say that A and B are PI-equivalent whenever $T(A) = T(B)$ (in the ordinary sense). It is well known that $T_G(\Gamma_A) = T_G(\Gamma_B)$ implies that $T(A) = T(B)$. The following results are well known and their proofs can be found for example in [8, Section 5.1].

Theorem 2.2 *The polynomials $[x_1,x_2][x_1,x_3]$ and $[x_1,x_2][x_3,x_4]+[x_1,x_3][x_2,x_4]$ belong to $T(E)$. Moreover, we have $[x_1,x_2] \in C(E)$. Over a field of characteristic zero, then $T(E) = \langle[x_1,x_2,x_3]\rangle_I$ and $C(E) =$*

$\langle [x_1, x_2], [x_1, x_2][x_3, x_4] \rangle_S$.

In the next sections, we shall describe the respective T_G-space of graded central polynomials, as well as a basis for their graded identities, for such some G-structures on the Grassmann algebra.

3. A fine grading on E by semigroup $(\mathbb{Z}^*, \times)$

Fine gradings on associative algebras occurred for the first time in the paper [1]. Such gradings are those where each homogeneous component is of dimension ≤ 1. Note that any n-induced G-grading on E is not fine, for all $n \in \mathbb{N}$. In this section, a fine grading on the Grassmann algebra by multiplicative semigroup $G = (\mathbb{Z}^*, \times)$ will be constructed. Furthermore, a basis of its graded identities, as well as central polynomials, will be provide.

Let $\mathfrak{P}$ be the set of prime integers and let L be an infinite-dimensional vector space with basis $\{e_p \mid p \in \mathfrak{P}\} \cup \{e_1\}$. We consider the Grassmann algebra $E = E(L)$, and define the $(\mathbb{Z}^*, \times)$-grading on E defined as follows:

$$\|e_1\| = -1 \qquad \textit{and} \qquad \|e_p\| = p,$$

for all $p \in \mathfrak{P}$. Denoting by $S_{\mathfrak{P}}$ the support of this grading, for each $r \in S_{\mathfrak{P}}$ there exist $p_1 < \ldots < p_j$ elements in $\mathfrak{P}$ such that if $r > 0$, then

$$r = \|e_{p_1} \cdots e_{p_j}\| = \|e_{p_1}\| \times \cdots \times \|e_{p_j}\| = p_1 \times \cdots \times p_j,$$

and if $r < 0$, then

$$r = \|e_1 e_{p_1} \cdots e_{p_j}\| = \|e_1\| \times \|e_{p_1}\| \times \cdots \times \|e_{p_j}\| = -(p_1 \times \cdots \times p_j),$$

and from fundamental theorem of arithmetics, it follows that $\mathrm{dim A_r} \leq 1$. This is an example of a fine grading on E. We call the reader's attention to information that a similar construction by group $(\mathbb{R}^*, \times)$ can be seen in [14, Subection 6.2]. At that opportunity, Guimarães's construction considers that $S_{\mathfrak{P}}$ is a subset of $\mathbb{N}$.

Denote by E_{fine} the fine G-grading on E with support $S_{\mathfrak{P}}$ exactly as before. Let us describe the graded identities and central polynomial for $E_{\mathbb{Z}}$. Assume that for each $g \in S_{\mathfrak{P}}$, there is $b_g \in \mathcal{B}_E$ such that $A_g = \mathrm{sp}_K\{b_g\}$. Of course that $b_1 = 1_E$, and we say that every elements g and h in $S_{\mathfrak{P}}$ are "related", denoted by $g \sim h$, if $\mathrm{supp}(b_g) \cap \mathrm{supp}(b_h) \neq \emptyset$, and $g \not\sim h$, otherwise. Moreover, we set $S_{\mathfrak{P}} = S_e \cup S_o$, where

$$S_e = \{g \in S_{\mathfrak{P}} \mid |\mathrm{supp}(b_g)| \text{ is even}\}$$

and

$$S_o = \{g \in S_{\mathfrak{P}} \mid |\mathrm{supp}(b_g)| \text{ is odd}\}.$$

Theorem 3.1 *Let G be the semigroup $(\mathbb{Z}^*, \times)$. Over an arbitrary field of characteristic different from 2, we have that:*

(a) The T_G-ideal of the graded identities of E_{fine} is generated by polynomials

(i) x^g, if $g \in \mathbb{Z}^* \setminus S_{\mathfrak{P}}$;

(ii) $x_1^{g_1} x_2^{g_2}$, if $g_1 \sim g_2$;

(iii) $[x_1^{g_1}, x_2^{g_2}]$, if either g_1 or g_2 lies in S_e and $g_1 \not\sim g_2$;

(iv) $x_1^{g_1}x_2^{g_2} + x_2^{g_2}x_1^{g_1}$, *if* g_1 *and* g_2 *lies in* S_o *and* $g_1 \not\sim g_2$.

(b) The T_G*-space of the graded central polynomials of* E_{fine} *is generated by polynomials*

(i) x^g, *if* $g \in S_e$;

(ii) $x_1^{h_1} f x_2^{h_2}$, *where* $f \in T_G(E_{fine})$ *and* $x_1^{h_1}$, $x_2^{h_2} \in X_G \cup \{1\}$.

Proof Let I be the T_G-ideal generated by the G-identities from (i)–(iv). It is clear that $I \subseteq T_G(E_{fine})$. Let us $f = f(x_1^1, \ldots, x_s^1, x_1^{g_1}, \ldots, x_n^{g_n}) \in T_G(E_{fine})$. Here the variables of G-degree 1 may not occur in f. As before mentioned, assume that f is a regular polynomial. The identities (i) and (ii) implies that the variables in f belong to $S_{\mathfrak{P}}$ and if x is a variable in f with G-degree different from 1, then f is linear in x. Thus, the G-identity f can be reduced to $f = \sum_i \alpha_i M_i$, where $\alpha_i \in K$ and each one of the monomial

$$M_i \equiv (x_1^1)^{m_1^i} \cdots (x_s^1)^{m_s^i} x_1^{g_1} \cdots x_l^{g_l} x_{l+1}^{g_{l+1}} \cdots x_n^{g_n} \pmod{I}$$

in which $m_1^i, \ldots, m_s^i$ are nonnegative integer, g_1, ..., g_l lie in $S_e^* = S_e \setminus \{1\}$, g_{l+1}, ..., g_n are in S_o and $g_i \not\sim g_j$, for all i, $j \in \{1, \ldots, n\}$. In addition, for each variable x_k^1, $1 \leq k \leq s$, we put $r_k = \min\{m_i^k\}$. By the regularity of f we have $r_k > 0$. Applying identity (iii), if needed, there exists an element $f' = f'(x_1^1, \ldots, x_s^1, x_1^{g_1}, \ldots, x_n^{g_n})$, where its variables is contained in the set of variables of f, such that

$$f \equiv (x_1^1)^{r_1} \cdots (x_s^1)^{r_s} f' \pmod{I}. \tag{3.1}$$

Note that the degree of the polynomial f' is lower than the degree of f, or f is multihomogeneous. It is easy to see that $f' \in T_G(E_{fine})$, if the first case happens the result follows by induction, i.e. $f' \in I$. Otherwise $f \equiv \alpha M_i$, and we are done. This implies that $T_G(E_{fine}) \subseteq I$, and, therefore, we have the first statement.

For the second item, we denote by V the T_G-space generated by (i) and (ii). Of course $V \subseteq C_G(E_{fine})$. Now we take a regular polynomial $h \in C_G(E_{fine}) \setminus T_G(E_{fine})$. If $h \in T_G(E_{fine})$, then we have the result by item (a). Repeating step by step what was done in (3.1), we may write

$$h \equiv (x_1^1)^{r_1} \cdots (x_s^1)^{r_s} h' \pmod{V},$$

for some G-monomial h' multilinear in $K\langle X_G \rangle$. It is clear that $\|h'\|_{can} = \|h\|_{can} \equiv 0 \pmod{2\mathbb{Z}}$ and, therefore, h is a consequence of x^g, for some $g \in S_e$. Therefore, the result follows. □

4. G-graded identities on E

From now on G will denote an abelian group. In order to study the graded central polynomials and graded identities of an n-induced $\mathbb{Z}$-grading on E, we will need to investigate the relations between the graded identities of E with respect to G-gradings and to its quotient G/H-gradings, for some special subgroup H of G. For this, it is reasonable to consider the following definition.

Definition 4.1 *Let* G *be an abelian group and* $H \leq G$. *We say that a* G*-grading on* E *is an* H*-central* G*-grading, if*

(i) such grading is homogeneous and of full support, and

(ii) for each $h \in H$, the homogeneous component E_h of the grading has infinitely many monomials of even lengths with pairwise disjoint supports.

When $G = \mathbb{Z}$ and $H = d\mathbb{Z}$, we simply say that such grading is d-central.

We also have to consider the homomorphism between free graded algebras

$$\pi_H \colon K\langle X_G \rangle \to K\langle X_{G/H} \rangle$$

given by $\pi_H(x_i^g) = x_i^{\overline{g}}$, where x_i^g denotes a variable of G-degree g while $x_i^{\overline{g}}$ is a variable of G/H-degree $\overline{g}$.

Lemma 4.2 *(a) If I is a $T_{G/H}$-ideal of $K\langle X_{G/H}\rangle$, then the set $J = \{f \in K\langle X_G\rangle \mid \pi_H(f) \in I\}$ is a T_G-ideal of $K\langle X_G\rangle$.*

(b) Let $f \in K\langle X_G\rangle$ and let A be a G-graded algebra. If $\pi_H(f) \in T_{G/H}(A)$, then $f \in T_G(A)$. In addition, If $\pi_H(f) \in C_{G/H}(A)$, then $f \in C_G(A)$.

Proof The first statement is obvious. The second statement follows word by word, with small modifications, the idea of [5, Lemma 3.1]. □

Notice that an H-central G-grading on E has the strong property that each homogeneous component of even degree contains a "significant part" of the centre of E, which allows us to obtain its graded polynomial identities and graded central polynomials. The importance of this definition is given by the following proposition and such result is an adaptation of [5, Proposition 3.3] under the hypothesis the abelian group G is not necessarily finite. Thus, it is possible to invert the implication of the previous result. It will be important for our goals in this paper.

Proposition 4.3 *Let G be an abelian group and $H \leq G$. Assume $\Gamma\colon E = \bigoplus_{g\in G} A_g$ an H-central G-grading on E, and denote by Γ' its induced G/H-grading. Moreover, let $f \in K\langle X_G\rangle$ be a multilinear polynomial. Then $f \in T_G(\Gamma)$ if and only if $\pi_H(f) \in T_{G/H}(\Gamma')$.*

Proof We just need to prove one part. Let us $f(x_1^{g_1}, \ldots, x_n^{g_n}) \in T_G(\Gamma)$ such that $f' = \pi_H(f)$. Consider an arbitrary f'-admissible substitution

$$x_j^{\overline{g}} \longmapsto \sum_{h\in H} a_j^{g+h},$$

where each a_j^g is an element of $\mathcal{B}_E$ and a_j^g means that a_j^g is homogeneous of G-degree g. Since f is multilinear, for all j, we may assume that the substitution is of the form

$$x_j^{\overline{g}} \longmapsto a_j^{g+h_j},$$

for some $h_j \in H$.

For each $h \in H$, the component A_h has infinitely many monomials of even length with pairwise disjoint supports. Therefore, for each index j, there exists $b_j^{-h_j}$ of even length with G-degree $-h_j$ such that $a_j^{g+h_j}$ and $b_j^{-h_j}$ have pairwise disjoint supports. We can choose the monomials $b_j^{-h_j}$ with pairwise disjoint support. So

$$c_j^g = a_j^{g+h_j} b_j^{-h_j}$$

is a homogeneous element of degree g in the G-grading Γ.

Now we consider the G-graded substitution given by

$$x_j^g \longmapsto c_j^g.$$

Due to $f \in T_{\mathbb{Z}}(\Gamma)$ and each $b_j^{-h_j} \in Z(E)$ we obtain

$$0 = f(c_1^{g_1}, \ldots, c_n^{g_n}) = f(a_1^{g_1+h_1} b_1^{-h_1}, \ldots, a_n^{g_n+h_n} b_n^{-h_n}) = (\prod_{j=1}^{r} b_j^{-h_j}) f'(a_1^{\overline{g_1}}, \ldots, a_n^{\overline{g_n}}).$$

Since the monomials b_j's have pairwise disjoint supports, we obtain $\pi(f) = f'$ lies in $T_{G/H}(\Gamma')$. $\square$

Corollary 4.4 *Assume the conditions of Proposition 4.3. We then have $f \in C_G(\Gamma)$ if and only if $\pi_H(f) \in C_{G/H}(\Gamma')$.*

Proof As in the previous proof, it is enough to prove one part. Suppose that $\pi_H(f)$ is not in $C_{G/H}(\Gamma')$, then there exists an admissible substitution $(a_1, \ldots, a_n)$ such that $\pi_H(f)(a_1, \ldots, a_n)$ is not in $Z(E) = E_{(0)}$. In other words, there is a homogeneous element $a \in E$ of G-degree g such that $[\pi_H(f)(a_1, \ldots, a_n), a^{\bar{g}}] \neq 0$. Thus, the polynomial $\pi_H(k') = [\pi_H(f), x^{\bar{g}}]$ is not in $T_{G/H}(\Gamma')$, for a new variable $x^{\bar{g}}$. It follows from the previous proposition that $k' = [f, x^g]$ is not in $T_G(\Gamma)$, i.e. f is not graded central polynomial for Γ and we are done. $\square$

The previous result is a key ingredient in the proof of the main theorem of the next section. To finish this section, we will give a sufficient condition for a subgroup of $\mathbb{Z}$ to be $\mathbb{Z}_d$-central, for some $d \geq 1$.

Proposition 4.5 *Let $A = \bigoplus_{r \in \mathbb{Z}} A_r$ be an arbitrary $\mathbb{Z}$-grading on E (not necessarily homogeneous). Suppose that there exist integers $m < 0 < n$ such that both $L \cap A_m$ and $L \cap A_n$ contain infinitely many elements of the basis of L. Writing $d = \gcd(m, n)$, we have that:*

(a) If m/d and n/d have distinct parities, then A is a d-central $\mathbb{Z}$-grading.

(b) If m/d and n/d have same parity, then A is a $2d$-central $\mathbb{Z}$-grading.

Proof Consider B the subalgebra of A generated by all e_i in $(L \cap A_m) \cup (L \cap A_n)$. Notice that B is a graded subalgebra of A and $\mathbb{Z}$-isomorphic to some 2-induced $\mathbb{Z}$-grading $E_{(m,n)}^{(\infty,\infty)}$.

Claim 1: $S_{(m,n)}^{(\infty,\infty)} = d\mathbb{Z}$, where $\gcd(m, n) = d$. In fact, we consider $\gcd(m/d, n/d) = 1$. We have that $\overline{(m/d)}$ is a generator of the group $\mathbb{Z}_{n/d}$. Since $\mathbb{Z}_{n/d}$ is finite, there exist positive integers α, α' such that $\alpha\overline{(m/d)} = \bar{1}$ and $\alpha'\overline{(m/d)} = -\bar{1}$ in $\mathbb{Z}_{n/d}$. Therefore, there exist integers β, β' such that

$$1 - \alpha(m/d) = \beta(n/d),$$

$$-1 - \alpha'(m/d) = \beta'(n/d).$$

As $m < 0$ we have $1 - \alpha(m/d) \geq 0$ and $-1 - \alpha'(m/d) \geq 0$, it follows that β and β' are nonnegative integers. Hence

$$d = \alpha m + \beta n,$$

$$-d = \alpha' m + \beta' n.$$

This implies that $d\mathbb{Z} \subseteq S_{(m,n)}^{(\infty,\infty)}$. On the other hand, if $k \in S_{(m,n)}^{(\infty,\infty)}$, there exist nonnegative integers r and s such that $k = rm + sn$. It follows from the definition of greatest common divisor that d divides k, and hence we have the claim.

Claim 2: If $w \in A_r$, where $r \in Sup(B)$, then there exist infinitely many monomials with pairwise disjoint supports in A_r whose length is equal to the length of w. Indeed, we write

$$w = e_{i_1} \cdots e_{i_l} e_{j_1} \cdots e_{j_k},$$

where e_{i_1}, ..., e_{i_l} are in $L \cap A_m$ while e_{j_1}, ..., e_{j_k} are in $L \cap A_n$. In this case for any choice of l elements in $L \cap A_m$ and k in $L \cap A_n$, we can construct a monomial in A_r with the same length of w. This proves the Claim 2.

By Claim 1, $Sup(B) = d\mathbb{Z}$, where $d = \gcd(m,n)$. If m/d and n/d have distinct parities we have that $n/d + (-m/d)$ is odd. This implies that

$$0 = (n/d) \times m + (-m/d) \times n,$$

i.e. then A_0 has infinitely many monomials of length $n/d + (-m/d)$. As $A_0 A_0 \subseteq A_0$, A_0 has infinitely many monomials of length $2(n + (-m))$ (even number). The first statement follows. For the second statement, we observe that $n/d + (-m/d)$ is even. In this case we can ensure that there are infinitely many elements of even length in B_0. In this case, consider $w \in B_d$, similar for B_{-d}. If $\|w\|_{can} = 0$, then the result follows for each rd, with $r > 0$, just apply the claim 2 and observe that

$$\underbrace{B_d \cdots B_d}_{\times r} \subseteq B_{rd}.$$

Similarly for $r < 0$. Otherwise, we take another monomial $w' \in B_d$. In this case, $ww' \in B_{2d}$ is of even length and the result follows as it was done before. Hence we have the proof of the proposition, since $Sup(B) \subseteq Sup(A)$. $\square$

5. Graded identities and central polynomials for 2-induced $\mathbb{Z}$-gradings on E

Although the results exposed in this section are already known, we intent to give a survey of results which point out relations between Arithmetic tools and PI-Theory. For this, we will give new proofs for these results in order to illustrate the use of the results of the previous section. We will start this section, by understanding what happens in the support of a structure n-induced $\mathbb{Z}$-grading on E.

Let $S_{(r_1,\ldots,r_n)}^{(v_1,\ldots,v_n)} = d\mathbb{Z}$ and assume that $r_1 < r_2 < \ldots < r_n$ (this can always be achieved by permuting simultaneously the lower and upper indices). If $r_1 > 0$, then it is clear that $S_{(r_1,\ldots,r_n)}^{(v_1,\ldots,v_n)} \subset \mathbb{N}_0$. If $r_n < 0$, we conclude that $S_{(r_1,\ldots,r_n)}^{(v_1,\ldots,v_n)} \subset \{-k \mid k \in \mathbb{N}_0\}$. Hence it follows that $r_1 < 0 < r_n$. We fix the unique integer $i \in \{1, 2, \ldots, n\}$ such that

$$r_1 < r_2 < \ldots < r_i < 0 < r_{i+1} < \ldots < r_n.$$

If $v_j = \dim L_{r_j}^{v_j} < \infty$, for $j = i+1$, ..., n, then $S_{(r_1,\ldots,r_n)}^{(v_1,\ldots,v_n)} \subset \{a \in \mathbb{Z} \mid a \geq r_1 v_1 + r_2 v_2 + \cdots + r_i v_i\}$. A similar conclusion holds if $v_j = \dim L_{r_j}^{v_j} < \infty$, for $j = 1$, 2, ..., i. We summarize these comments in the following proposition.

Proposition 5.1 *Let $E^{(v_1,\dots,v_n)}_{(r_1,\dots,r_n)}$ be an n-induced $\mathbb{Z}$-grading on E. If $S^{(v_1,\dots,v_n)}_{(r_1,\dots,r_n)}$ is a subgroup of $\mathbb{Z}$, then there exist i, $j \in \{1,\dots,n\}$ such that $r_i < 0 < r_j$, and $v_i = v_j = \infty$.*

A question that naturally arises is whether the reciprocal of this last result is true. The general case is an open problem so far, we will probably need additional conditions in order to solve this problem. In [3, 10], the authors gave a criterion for the support of a 2 and 3-induced $\mathbb{Z}$-grading on E to be a subgroup of $\mathbb{Z}$. More precisely the following theorem was proved.

Theorem 5.2 *$S^{(\infty,\infty)}_{(r_1,r_2)} = d\mathbb{Z}$ if and only if $r_1 < 0 < r_2$ and $\gcd(r_1, r_2) = d$.*

The proof of the latter result can be seen in Claim 1 of Proposition 4.5. Due to these comments we can formulate the following result.

Proposition 5.3 *Let $E^{(v_1,\dots,v_n)}_{(r_1,\dots,r_n)}$ be an n-induced $\mathbb{Z}$-grading on E. If there exist i, $j \in \{1,\dots,n\}$ such that $r_i < 0 < r_j$, $\gcd(r_i, r_j) = 1$, and $v_i = v_j = \infty$, then $E^{(v_1,\dots,v_n)}_{(r_1,\dots,r_n)}$ is of full support.*

We are ready to make use of these comments to obtain graded identities and central polynomial for some particular cases of n-induced $\mathbb{Z}$-grading on E. As a consequence of Propositions 4.3, 4.5, 5.1 and Corollary 4.4, we have the following.

Theorem 5.4 *Let $E^{(v_1,\dots,v_n)}_{(r_1,\dots,r_n)}$ be an n-induced $\mathbb{Z}$-grading on E such that its support is a subgroup of $\mathbb{Z}$. Assume the conditions of Proposition 5.1 and $d = \gcd(r_i, r_j)$. Then the following properties hold:*

1. *If r_i/d and r_j/d have distinct parities then for any multilinear polynomial $f(x_1^{g_1},\dots,x_n^{g_n}) \in K\langle X_{\mathbb{Z}}\rangle$ one has*

$$f \in T_{\mathbb{Z}}(E^{(v_1,\dots,v_n)}_{(r_1,\dots,r_n)}) \text{ if and only if } \pi_{d\mathbb{Z}}(f) \in T_{\mathbb{Z}_d}(\Gamma'_d),$$

   where $\Gamma'_d : E = \oplus_{\overline{g}\in\mathbb{Z}_d} A_{\overline{g}}$ is the induced $\mathbb{Z}_d$-grading of $E^{(v_1,\dots,v_n)}_{(r_1,\dots,r_n)}$;

2. *If r_i/d and r_j/d have the same parity, then for any multilinear polynomial $f(x_1^{g_1},\dots,x_n^{g_n}) \in K\langle X_{\mathbb{Z}}\rangle$ one has*

$$f \in T_{\mathbb{Z}}(\Gamma) \text{ if and only if } \pi_{2d\mathbb{Z}}(f) \in T_{\mathbb{Z}_{2d}}(\Gamma'_{2d}),$$

   where $\Gamma'_{2d} : E = \oplus_{\overline{g}\in\mathbb{Z}_{2d}} A_{\overline{g}}$ is the induced $\mathbb{Z}_{2d}$-grading of $E^{(v_1,\dots,v_n)}_{(r_1,\dots,r_n)}$;

3. *The results (1) and (2) remain valid for graded central polynomials;*

4. *In the quotient grading of Γ', the vector space $L^{\bar{0}}$ is infinite dimensional.*

Notice that the methods used by us so far suggest that an adequate usage of arithmetic tools could contribute to the study of gradings and graded identities on the Grassmann algebra. Recall that if d is a prime number, one can apply our methods to the problems studied in [6, 7, 16]. In other words, the description of all graded identities to n-induced $\mathbb{Z}$-gradings on E whose support is a subgroup of $\mathbb{Z}$ is "reduced" to: 1) the problem of the description of graded identities on E by a finite cyclic group, and 2) the usage of techniques derived from number theory.

Theorem 5.5 *Let $E^{(\infty,\infty)}_{(r_1,r_2)}$ be a $\mathbb{Z}$-grading on E of full support. Over a field of characteristic zero, the following statements hold:*

(1) If $r_1 \times r_2$ is even, then $T_{\mathbb{Z}}(E^{(\infty,\infty)}_{(r_1,r_2)})$ is generated, as a $T_{\mathbb{Z}}$-ideal, by the polynomials

$$[u_1, u_2, u_3];$$

(2) If $r_1 \times r_2$ is odd, then $T_{\mathbb{Z}}(E^{(\infty,\infty)}_{(r_1,r_2)})$ is generated, as a $T_{\mathbb{Z}}$-ideal, by the polynomials

(i) $[x_1, x_2]$, if either $\|x_1\|$ or $\|x_2\|$ is even and

(ii) $x_1x_2 + x_2x_1$, if both $\|x_1\|$ and $\|x_2\|$ are odd.

Here each $u_i \in X_{\mathbb{Z}}$.

Proof Proposition 4.5 implies that each $\mathbb{Z}$-graded algebra $E^{(\infty,\infty)}_{(r_1,r_2)}$ is 2-central $\mathbb{Z}$-grading. If $r_1 \times r_2$ is even, we then have E_∞ is the $\mathbb{Z}_2$-grading induced. If $r_1 \times r_2$ is odd, then E_{can} is the $\mathbb{Z}_2$-grading induced. Now we apply Theorem 5.4 and the main results of [7]. □

In the next result we describe a basis for the $T_{\mathbb{Z}}$-space of graded central polynomials for such structures. Its proof follows mutatis mutandis from the previous theorem with the main results of [15] instead of [7].

Theorem 5.6 *Let $E^{(\infty,\infty)}_{(r_1,r_2)} = \oplus_{r\in\mathbb{Z}} A_r$ be a $\mathbb{Z}$-grading on E of full support and let K be a field of characteristic zero, we then have:*

(1) If $r_1 \times r_2$ is even, then $C_{\mathbb{Z}}(E^{(\infty,\infty)}_{(r_1,r_2)})$ is generated, as a $T_{\mathbb{Z}}$-space, by the polynomials

$$[u_1, u_2] \qquad \text{and} \qquad [u_1, u_2][u_3, u_4];$$

(2) If $r_1 \times r_2$ is odd, then $C_{\mathbb{Z}}(E^{(\infty,\infty)}_{(r_1,r_2)})$ is generated, as a $T_{\mathbb{Z}}$-space, by the polynomials

(i) x, where $\|x\|$ is even;

(ii) $u_1 f u_2$, where $f \in T_G(E^{(\infty,\infty)}_{(r_1,r_2)})$.

Here each $u_i \in X_{\mathbb{Z}} \cup \{1\}$.

The n-induced grading by an abelian group is a subject of recent research. The interested reader can consult [3, 9, 10, 14–16, 18] for further details and references concerning these remarkable theorems. We further recall that research about their graded identities and central polynomials over infinite fields have been recently obtained and can be consulted in [9]. However, a complete description of even these types of structures is still far from being achieved. And, as mentioned earlier, such a problem depends on the development of the theory of graded identities on E by any finite cyclic group.

Given $d \in \mathbb{Z}$, we consider the homomorphisms $\Phi_d\colon F\langle X_{\mathbb{Z}}\rangle \to F\langle X_{d\mathbb{Z}}\rangle$ and $\Psi_d\colon F\langle X_{d\mathbb{Z}}\rangle \to F\langle X_{\mathbb{Z}}\rangle$ given by $\Phi_d(x_i^n) = x_i^{dn}$ $\Psi_d(x_i^{dn}) = x_i^{dn}$, respectively. Note that the latter homomorphism is not inclusion, because even though algebras seem to have the same variables they are in distinct free algebras. To complete the description for an n-induced $\mathbb{Z}$-grading on E, we will consider the following result.

Theorem 5.7 *[3, Theorem 3.7] Let m and n be integers, with $m < 0 < n$ and $d = \gcd(m,n)$ and let K be a field of characteristic zero. If S is a basis for the $T_{\mathbb{Z}}$-ideal $T_{\mathbb{Z}}(E^{(\infty,\infty)}_{(\frac{m}{d},\frac{n}{d})})$, then the $T_{\mathbb{Z}}$-ideal $T_{\mathbb{Z}}(E^{(\infty,\infty)}_{(m,n)})$ is generated by the set $S' \cup N$, where*

$$S' = \{\Psi_d(\Phi_d(f)) \mid f \in S\}$$

and

$$N = \{x \in K\langle X_{\mathbb{Z}}\rangle \mid \|x\| \notin d\mathbb{Z}\}.$$

The previous result tells us that in order to determine the $\mathbb{Z}$-graded identities, as well as $\mathbb{Z}$-graded central polynomials, for a graded algebra of type $E^{(\infty,\infty)}_{(m,n)}$, when $m < 0 < n$, it is sufficient to consider the case in which such structure is of full support and it was done in Theorems 5.5 and 5.6.

The next result will establish the relationship between the 2-induced and n-induced $\mathbb{Z}$-grading on E.

Theorem 5.8 *Let $E^{(v_1,\dots,v_n)}_{(r_1,\dots,r_n)} = \bigoplus_{r\in\mathbb{Z}} A_r$ be an n-induced $\mathbb{Z}$-grading on E. Suppose that there exist integers $r_i < 0 < r_j$, with distinct parities such that both $L \cap A_{r_i}$ and $L \cap A_{r_j}$ have infinitely many generators of L. Over a field of characteristic zero, the $T_{\mathbb{Z}}$-ideal $T_{\mathbb{Z}}(A)$ is generated by the polynomials*

$$[x_1, x_2, x_3],$$

for every choice of degree $\|x_1\|$, $\|x_2\|$, $\|x_3\|$, together with their graded monomial identities. In particular, if $v_k \leq \infty$, for every $k \in \{1,\dots,n\} \setminus \{i,j\}$, then the $\mathbb{Z}$-graded monomial identities have length up to $v+1$, where $v = v_1 + \cdots + v_{i-1} + v_{i+1} + \cdots + v_{j-1} + +v_{j+1} + \cdots + v_n$.

Proof Let I will be the $T_{\mathbb{Z}}$-ideal generated by the $\mathbb{Z}$-identities from $[x_1,x_2,x_3]$, for every choice of degree $\|x_1\|$, $\|x_2\|$, $\|x_3\|$. It is clear that $I \subseteq T_{\mathbb{Z}}(E^{(v_1,\dots,v_n)}_{(r_1,\dots,r_n)})$. We define B as being the subalgebra of A generated by the generators of L that are in $(L \cap A_{r_i}) \cup (L \cap A_{r_j})$. In this case B is $\mathbb{Z}$-isomorphic to $E^{(\infty,\infty)}_{(r_i,r_j)}$, which is not necessarily full support. By Claim 1 in the proof of Proposition 4.5, it follows that $Sup(B) = d\mathbb{Z}$, where $d = \gcd(r_i,r_j)$. According to Theorem 5.7 the ideal $T_{\mathbb{Z}}(B)$ is generated by the polynomials $\Psi_d(\Phi_d(f))$, where $f \in T_{\mathbb{Z}}(E^{(\infty,\infty)}_{(\frac{r_i}{d},\frac{r_j}{d})})$, together with

$$N = \{x \in F\langle X_{\mathbb{Z}}\rangle \mid \|x\| \notin d\mathbb{Z}\}.$$

Notice that $E^{(\infty,\infty)}_{(\frac{r_i}{d},\frac{r_j}{d})}$ has full support with r_i and r_j have distinct parities. This implies that $E^{(\infty,\infty)}_{(\frac{r_i}{d},\frac{r_j}{d})}$ induces the grading E_∞. Hence $T_{\mathbb{Z}}(E^{(\infty,\infty)}_{(\frac{r_i}{d},\frac{r_j}{d})}) = I$. Thus, the inclusions

$$I \subseteq T_{\mathbb{Z}}(E^{(v_1,\dots,v_n)}_{(r_1,\dots,r_n)}) \subseteq T_{\mathbb{Z}}(B),$$

are hold, and the ideal $T_{\mathbb{Z}}(B)$ differs from I by the graded monomial identities of degree 1. Hence, the above inclusions are equalities, module monomial identities. As each variable in the corresponding monomial identity must contain variables of degree does not multiple of d. Therefore the result follows since in the evaluation it has at least one element of L of a $\mathbb{Z}$-degree different from the multiple of d. $\square$

We will end this paper with interesting examples of $\mathbb{Z}$-gradings on E different from r-induced.

Example 5.9 *Let L be an infinite-dimensional vector space with basis $\{e_n \mid n \in \mathbb{N}\}$. We get the Grassmann algebra $E = E(L)$, and we consider the attribution of degree on the generators of L given by $\|e_n\| = n$. This provides a $\mathbb{Z}$-grading on E, denoted by E_{part}. Pay attention that for all $r \in Sup(E_{part})$, there exist $0 < i_1 < \ldots < i_j$ in $\mathbb{Z}$ such that*

$$r = \|e_{i_1} \cdots e_{i_j}\| = i_1 + \cdots + i_j.$$

Notice that the homogeneous component, in this grading, of degree n has dimension that equals the number of partitions of n. Therefore, E_{part} cannot be r-induced.

Example 5.10 *Assume the notations of the previous example. For any fixed natural number d, we consider the attribution of degree on the generators of L given by*

$$\|e_n\| = \begin{cases} dk, & \text{if } n = 2k \\ -dk, & \text{if } n = 2k-1. \end{cases}$$

This provides a $\mathbb{Z}$-grading on E whose support is the subgroup $d\mathbb{Z}$ of $\mathbb{Z}$. Moreover, such grading is not r-induced, since there exist infinitely many nontrivial components of the vector space L. Therefore, there are another $\mathbb{Z}$-grading on E different from r-induced whose support is a subgroup of $\mathbb{Z}$.

Acknowledgment

The author is grateful to A. Guimarães and P. Koshlukov for the formulation of the problem, for helpful suggestions and discussions to this article.

The author acknowledges the support by grant 2019/12498-0, São Paulo Research Foundation (FAPESP).

Finally, the author thanks the anonymous referee for her/his valuable comments that improved the exposition of the paper.

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