Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 1828 – 1844
© TÜBİTAK
doi:10.55730/1300-0098.3235

Nonlinear evolution equations related to Kac-Moody algebras $A_r^{(1)}$: spectral aspects

Vladimir S. GERDJIKOV[1,2,*]

[1]Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, Bulgaria

[2]Institute for Advanced Physical Studies, 111 Tsarigradsko chaussee, Sofia, Bulgaria

Received: 04.01.2022 • **Accepted/Published Online:** 23.02.2022 • **Final Version:** 20.06.2022

Abstract: We analyze three types of integrable nonlinear evolution equations (NLEE) related to the Kac-Moody algebras $A_r^{(1)}$. These are $\mathbb{Z}_h$-reduced derivative NLS equations (DNLS), multicomponent mKdV equations and 2-dimensional Toda field theories (2dTFT). We outline the basic tools of this analysis: i) the gradings of the simple Lie algebras using their Coxeter automorphisms; ii) the construction of the relevant Lax representations; and iii) the spectral properties of the Lax operators and their reduction to Riemann–Hilbert problems. We also formulate the minimal set of scattering data which allow one to recover the asymptotics of the fundamental analytic solutions to L and its potential.

Key words: Integrable nonlinear evolution equations, graded simple Lie algebras, Kac-Moody algebras, Riemann–Hilbert problems

1. Introduction

The general theory of the nonlinear evolution equations (NLEE) allowing Lax representation is well developed [1, 5, 8, 11, 13, 20, 34, 35]. This paper is an extension of the old report [9] and more recent papers [10, 15, 16, 18] and deals with three types of NLEE.

The first one is as generalizations of the derivative NLS equations [9], see also [26, 28]:

$$i\frac{\partial \psi_k}{\partial t} + \gamma \frac{\partial}{\partial x}\left(\operatorname{cotan} \frac{\pi k}{h} \cdot \psi_{k,x} + i \sum_{p=1}^{h-1} \psi_p \psi_{k-p} \right) = 0, \qquad k = 1, 2, \dots, h-1, \tag{1.1}$$

where γ is a constant and the index $k-p$ should be understood modulus h, $\psi_0 = \psi_h = 0$. The system (1.1) allows also the involutions:

$$\begin{aligned} &\text{a)} \qquad \psi_k = -\psi_k^*, \qquad && \gamma = -\gamma^*, \\ &\text{b)} \qquad \psi_k = \psi_{h-k}^*, \qquad && \gamma = \gamma^*. \end{aligned} \tag{1.2}$$

*Correspondence: vgerdjikov@math.bas.bg

2010 *AMS Mathematics Subject Classification:* 23584

This work is licensed under a Creative Commons Attribution 4.0 International License.

The second type of such equations is known as the mKdV type equations [7, 15, 16, 18]. They are multicomponent generalizations of the famous mKdV equation. Examples of such systems will be given below in the text. Here we will demonstrate the Hamiltonian of one of them:

$$\begin{aligned} H = \frac{1}{\alpha}\int_{-\infty}^{\infty} dx \Bigg(-4\left(\frac{\partial q_1}{\partial x}\right)\left(\frac{\partial q_5}{\partial x}\right) + \frac{1}{4}\left(\frac{\partial q_3}{\partial x}\right)^2 \\ + \sqrt{3}q_3\left(q_1\frac{\partial q_2}{\partial x} + 3q_2\frac{\partial q_1}{\partial x} - 3q_4\frac{\partial q_5}{\partial x} - q_5\frac{\partial q_4}{\partial x} + 2q_5^2\frac{\partial q_2}{\partial x} - 2q_1^2\frac{\partial q_4}{\partial x}\right) - \frac{1}{4}q_3^4 \\ + 3(q_1q_2 + q_4q_5)^2 + 2q_3(q_1^3 + q_5^3 + 3q_2^2q_5 + 3q_1q_4^2) + 3q_3^2(q_1q_5 + q_2q_4)\Bigg). \end{aligned} \tag{1.3}$$

The corresponding NLEE takes the form:

$$\frac{\partial q_j}{\partial t} = \frac{\partial}{\partial x}\frac{\delta H}{\delta q_j(x,t)}. \tag{1.4}$$

The third type of NLEE contains the famous 2-dimensional Toda field theories discovered by [28]. For the $sl(h)$ algebras they take the form:

$$\frac{\partial^2 \vec{q}_j}{\partial x \partial t} = \sum_{j=0}^{h-1} \alpha_j e^{-2(\alpha,\vec{q})}, \qquad j = 1, \dots, h-1. \tag{1.5}$$

where $\vec{q}(x,t)$ is an $h-1$-component vector functions; the vectors $\alpha_j = e_j - e_{j+1}$, $j = 1, \dots, h-1$ are the simple roots of $sl(h)$ while $\alpha_0 = -e_1 + e_h$ is the minimal root of $sl(h)$.

Each of these types of equations possesses Lax representation, i.e. for each of them there exist a pair of first order matrix linear operators $L(\lambda)$ and $M(\lambda)$ depending on the spectral parameter λ and such, that the NLEE appears as the commutativity condition $[L, M] = 0$. The potentials of L and M depend on the variables x and t and take values in a simple Lie algebra $\mathfrak{g}$. Below for simplicity we choose $\mathfrak{g} \simeq sl(r+1)$. The Lax pairs generating these NLEE are special in the sense, that each one possesses $\mathbb{Z}_h$ Mikhailov reduction group [28].

In fact to each Lax pair one can relate a Kac-Moody algebra. For the first two types of NLEE we described above this construction is provided in [7]. We will demonstrate that the Lax operator $\tilde{L}(\lambda)$ for the third type of these NLEE can be obtained from $L(\lambda)$ by a gauge transformation. One of our aims is to construct this transformation explicitly. Next we will show that the spectral problem L and $\tilde{L}$ can be reduced to the same Riemann–Hilbert problem.

Section 2 contains preliminaries about the Kac-Moody algebras necessary to construct the Lax representations for the NLEE described above. We assume that the readers are familiar with the theory of simple Lie algebras, their root systems and Weyl reflections [25]. In particular we provide a convenient basis of $sl(h)$ which is compatible with the $\mathbb{Z}_h$-reduction. In Section 3 we derive the constraints on the Lax operator L which lead to additional reductions like (1.2) and their consequences for the scattering matrix and scattering data of L. We also give several particular cases of the DNLS equations and their reductions. In Section 4 we establish the gauge transformation that makes L and $\tilde{L}$ equivalent. In Section 5 we outline the construction of the fundamental analytic solutions (FAS) of L, their asymptotics for $x \to \pm\infty$ and their symmetry properties. We demonstrate that the FAS are directly related to the solutions of a Riemann–Hilbert problem (RHP) on a set of lines intersecting at the origin and closing angles π/h. Thus instead of solving the inverse scattering problem for the operator L we can treat the RHP and use the dressing Zakharov–Shabat method [30, 36, 37] for constructing the soliton solutions [28, 34]. We also derive the simplest integrals of motion of the DNLS equations. Section 6 contains discussion and conclusion.

2. Preliminaries

2.1. Kac-Moody algebras

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$. Let us introduce related infinite-dimensional algebra:

$$\mathfrak{g}[\lambda, \lambda^{-1}] = \left\{ \sum_{i=n}^{m} v_i \lambda^i : v_i \in \mathfrak{g}, n, m \in \mathbb{Z} \right\}, \tag{2.1}$$

with generic element:

$$f[\lambda] = \left\{ \sum_{i=0}^{m} f_i \lambda^i : f_i \in \mathfrak{g}, m \in \mathbb{Z} \right\}. \tag{2.2}$$

There is a natural Lie algebraic structure on $\mathfrak{g}[\lambda, \lambda^{-1}]$. Let φ be an automorphism of $\mathfrak{g}$ of order h and let:

$$L(\mathfrak{g}, \varphi) = \left\{ f \in \mathfrak{g}[\lambda, \lambda^{-1}] : \varphi(f(\lambda)) = f\,[\lambda\omega] \right\}, \qquad \omega = \exp\left(\frac{2\pi i}{h} \right). \tag{2.3}$$

$L(\mathfrak{g}, \varphi)$ is a Lie subalgebra of $\mathfrak{g}[\lambda, \lambda^{-1}]$. If $\mathfrak{g}$ is simple and if h is the Coxeter number of $\mathfrak{g}$ then $L(\mathfrak{g}, \varphi)$ with an appropriate central extensions, is called a Kac-Moody algebra. It is obvious that Kac-Moody algebras are graded algebras.

Roughly speaking, the elements of a Kac-Moody algebras are formal series in λ with coefficients in some properly graded finite-dimensional simple Lie algebra. The grading in $\mathfrak{g}$ is constructed using the Coxeter automorphism C of $\mathfrak{g}$, and the Coxeter number is order h of C: $C^h \equiv \mathbb{1}$.

Let us assume that H_j, $j = 1, \dots, r$ and E_α, $\alpha \in \Delta$ are the Cartan–Weyl basis of $\mathfrak{g}$. Here $H_j \in \mathfrak{h}$ are the basis of the Cartan subalgebra, $r = \dim \mathfrak{h}$ is the rank of $\mathfrak{g}$; the Weyl generators E_α are labeled by the roots $\alpha \in \Delta$ of $\mathfrak{g}$. We will denote by α_j the simple roots of $\mathfrak{g}$. One of the realization of the Coxeter automorphism is as the maximal element of the Weyl group of $\mathfrak{g}$. For example we can choose $C_1 = S_{\alpha_1} S_{\alpha_2} \dots S_{\alpha_r}$ to be the composition of Weyl reflections with respect to the simple roots.

The Coxeter automorphism introduces grading in $\mathfrak{g}$. Indeed, since $C_1^h \equiv \mathbb{1}$ then it has h eigenvalues ω^k, $k = 0, 1, \dots, h-1$ and h eigensubspaces in $\mathfrak{g}$, i.e.:

$$\mathfrak{g} = \mathop{\oplus}_{k=0}^{h-1} \mathfrak{g}^{(k)}, \tag{2.4}$$

such that

$$C_1(X) = \omega^k X \qquad \forall X \in \mathfrak{g}^{(k)}. \tag{2.5}$$

Such grading has the special property:

$$[X,Y] \in \mathfrak{g}^{(k+m \mod h)}, \qquad \forall X \in \mathfrak{g}^{(k)}, \quad Y \in \mathfrak{g}^{(m)}. \tag{2.6}$$

With all that in mind we can provide explicit realization of $L(\mathfrak{g}, C_1)$ introducing basis in each of the subspaces $\mathfrak{g}^{(k)}$ as follows:

$$X = \sum_{k=n}^{m} X^{(k)} \lambda^k, \quad n, m \in \mathbb{Z}, \tag{2.7}$$

where $X^{(k)} \in \mathfrak{g}^{(k \text{ mod } h)}$. Each of the subspaces $\mathfrak{g}^{(k)}$ has a basis given by:

$$\mathcal{E}_\alpha^{(k)} = \sum_{s=0}^{h-1} \omega^{-sk} C_1^s(E_\alpha), \qquad \mathcal{H}_j^{(k)} = \sum_{s=0}^{h-1} \omega^{-sk} C_1^s(H_j). \tag{2.8}$$

Note that $\mathcal{H}_j^{(k)}$ is nonvanishing only if k is an exponent.

Below we will mostly use the algebra $\mathfrak{g} \simeq sl(r+1)$ which has rank r and Coxeter number $h = r+1$. The root system of $sl(r+1)$ is $\Delta \equiv \{e_k - e_j, \quad k \neq j$ where $1 \leq k, j \leq r+1$. The simple roots are $\alpha_j = e_j - e_{j+1}$, $j = 1, \dots, r$.

$$sl(h, \mathbb{C}) = \mathop{\oplus}_{k=0}^{h-1} \mathfrak{g}^{(k)}. \tag{2.9}$$

We will use a convenient basis in $\mathfrak{g}^{(k)}$, namely:

$$J_s^{(k)} = \sum_{j=1}^{h} \omega^{kj} E_{j,j+s}, \qquad C_1^{-1} J_s^{(k)} C_1 = \omega^{-k} J_s^{(k)}. \tag{2.10}$$

Obviously, $J_s^{(k)}$ satisfies the commutation relations:

$$\left[J_s^{(k)}, J_l^{(m)} \right] = \left(\omega^{ms} - \omega^{kl} \right) J_{s+l}^{(k+m)}. \tag{2.11}$$

2.2. Another realization of the Coxeter automorphisms

Each grading relevant for Kac-Moody algebras is fixed up by a Coxeter automorphism C which satisfies $C^h = \mathbb{1}$, where h is the Coxeter number of $\mathfrak{g}$. For the first one we choose C to be an element of the Cartan subgroup:

$$\tilde{C}_1 = \exp\left(\frac{2\pi i}{h} \sum_{k=1}^{r} H_{\omega_k} \right), \tag{2.12}$$

where ω_k are the fundamental weights of $\mathfrak{g}$. Using the Cartan–Weyl commutation relations one easily finds that:

$$\tilde{C}_1(H) = H, \qquad \tilde{C}_1(E_\alpha) \equiv \tilde{C}_1 E_\alpha \tilde{C}_1^{-1} = exp\left(\frac{2\pi i}{h}\sum_{k=1}^{r}\frac{2(\omega_k,\alpha)}{(\alpha,\alpha)}\right) = \omega^{\mathrm{ht}\,(\alpha)}E_\alpha, \tag{2.13}$$

where $\omega = \exp(2\pi i/h)$ and $\mathrm{ht}\,(\alpha)$ is the height of the root α. Then

$$\mathfrak{g}^{(k)} \equiv \{E_\beta, \qquad \mathrm{ht}\,(\beta) = k \mod (h)\}. \tag{2.14}$$

We have two different choices for the Coxeter automorphism C_1 and $\tilde{C}_1$ for the algebra $\mathfrak{g} \simeq \mathfrak{sl}(h)$, $h = r+1$. In what follows we assume that the reader is familiar with the basic concepts of the simple and affine Lie algebras, see for example [6, 25, 27]. Each of these choices satisfies $C_1^h = \mathbb{1}$ and $\tilde{C}_1^h = \mathbb{1}$, and each of these automorphisms induces a grading in $\mathfrak{g}$

$$\mathfrak{g} = \mathop{\oplus}_{k=0}^{r} \mathfrak{g}^{(k)}, \qquad \tilde{\mathfrak{g}} = \mathop{\oplus}_{s=0}^{r} \tilde{\mathfrak{g}}_s\,. \tag{2.15}$$

Here the linear subspaces are such that

$$C_1 X C_1^{-1} = \omega^{-k}X, \qquad \tilde{C}_1 Y \tilde{C}_1^{-1} = \omega^{-s}Y, \tag{2.16}$$

where $X \in \mathfrak{g}^{(k)}, Y \in \tilde{\mathfrak{g}}_s$ and $\omega = e^{2\pi i/(r+1)}$. Each of the gradings satisfies

$$[\mathfrak{g}^{(k)}, \mathfrak{g}^{(m)}] \in \mathfrak{g}^{(k+m)}, \qquad [\tilde{\mathfrak{g}}_s, \tilde{\mathfrak{g}}_p] \in \tilde{\mathfrak{g}}_{s+p}, \tag{2.17}$$

where $(k+m)$ and $(s+p)$ must be understood modulo $(r+1)$.

In what follows we will specify the choice of the automorphisms setting C_1, $\tilde{C}_1$ to be

$$C_1 = \sum_{p=1}^{r+1} E_{p,p+1} = J_1^{(0)}, \qquad \tilde{C}_1 = \sum_{p=1}^{r+1} \omega^{p-1}E_{p,p} = J_0^{(1)}, \tag{2.18}$$

where the $(r+1)\times(r+1)$ matrices E_{km} are defined by $(E_{km})_{sp} = \delta_{ks}\delta_{mp}$.

Further we will use a convenient basis in the affine Lie algebra $A_r^{(1)}$ which is compatible with the gradings, see [9, 12] and [6, 25, 27]:

$$J_s^{(k)} = \sum_{j=0}^{r} \omega^{kj} E_{j+1,j+s+1}. \tag{2.19}$$

The elements of this basis satisfy the commutation relations

$$\left[J_s^{(k)}, J_l^{(m)}\right] = \left(\omega^{ms} - \omega^{kl}\right) J_{s+l}^{(k+m)}. \tag{2.20}$$

Besides it is easy to check that

$$C_1^{-1} J_s^{(k)} C_1 = \omega^{-k} J_s^{(k)}, \qquad \tilde{C}_1^{-1} J_s^{(k)} \tilde{C}_1 = \omega^{-s} J_s^{(k)} \tag{2.21}$$

and

$$J_s^{(k)} J_p^{(m)} = \omega^{sm} J_{s+p}^{(k+m)}, \qquad (J_s^{(k)})^{-1} = (J_s^{(k)})^\dagger. \tag{2.22}$$

Using this we can specify bases in each of the linear subspaces as follows:

$$\mathfrak{g}^{(k)} \equiv \mathrm{span}\{J_s^{(k)}, \quad s = 1, \dots, r+1\}, \qquad \tilde{\mathfrak{g}}_s \equiv \mathrm{span}\{J_s^{(k)}, \quad k = 1, \dots, r+1\}. \tag{2.23}$$

The realization of the Coxeter automorphism by C_1 corresponds to choosing it as a Weyl group element $C_1 = S_{\alpha_1} \dots S_{\alpha_r}$, where α_k are the simple roots of $\mathfrak{sl}(r+1)$ and S_α is the Weyl reflection with respect to the root α. In the other realization $\tilde{C}_1$ is an element of the Cartan subgroup of $\mathfrak{sl}(r+1)$. Then $\mathfrak{g}^{(0)} \equiv \mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{sl}(r+1)$. Both realizations are equivalent, i.e. there exists a similarity transformation which takes C_1 into $\tilde{C}_1$. In the first realization each of the linear subspaces $\mathfrak{g}^{(k)}$ (with the exception of $\mathfrak{g}^{(0)}$) has a one-dimensional section with the Cartan subalgebra, i.e.

$$\mathfrak{g}^{(s)} \cap \mathfrak{h} \equiv c_s J_0^{(s)}, \tag{2.24}$$

where c_s is an arbitrary constant.

3. Lax representations

3.1. The $\mathbb{Z}_h$ DNLS and the mKdV equations

Lax operators subject to $\mathbb{Z}_h$ reductions mean that the ordinary differential operators L and M can be written as:

$$\mathcal{L}\chi(x,t,\lambda) \equiv \left(\frac{d}{dx} + U(x,t,\lambda) \right) \chi(x,t,\lambda) = 0, \tag{3.1}$$

$$\mathcal{M}\chi(x,t,\lambda) = \left(\frac{d}{dt} + V(x,t,\lambda) \right) \chi(x,t,\lambda) = \lambda^2 \chi(x,t,\lambda) K, \tag{3.2}$$

where the potentials $U(x,t,\Lambda)$ and $V(x,t,\Lambda)$ are polynomials of λ. For simplicity we assume that:

$$U(x,t,\lambda) = U_0(x,t) + \lambda U_1, \tag{3.3}$$

$$V(x,t,\lambda) = V_0(x,t) + \lambda V_1(x,t) + \lambda^2 K. \tag{3.4}$$

Obviously the Lax pair (3.1) possesses $\mathbb{Z}_h$ and $\mathbb{D}_h$-reduction groups [28]. For the case of $\mathbb{Z}_h$-reduction this means that we impose on (3.1) and (3.2) a $\mathbb{Z}_h$-reduction by [28]:

$$U(x,t,\lambda) = C_1^{-1} U(x,t,\lambda\omega) C_1, \qquad V(x,t,\lambda) = C_1^{-1} V(x,t,\lambda\omega) C_1, \tag{3.5}$$

where C_1 is given by (2.18). Of course U_k, $V_k \in \mathfrak{g}^{(k)}$ which means that:

$$U_0(x) = \sum_{j=1}^{h-1} \psi_j(x,t) J_j^{(0)}, \qquad U_1 = -a\omega^{-1/2} J_0^{(1)}, \qquad K = -bJ_0^{(2)}. \tag{3.6}$$

Then the requirement that (3.1) and (3.2) are compatible for all values of λ allows us to express $V_0(x,t)$ and $V_1(x,t)$ in terms of $\psi_j(x,t)$ as follows:

$$\begin{aligned} V_1(x,t) &= \sum_{k=1}^{N} v_{1,k}(x,t)J_j^{(1)}, \qquad v_{1,p} = -\frac{b}{a}\omega^{(p+1)/2}\cos(p\pi/N)\psi_p(x,t), \\ V_0(x,t) &= \sum_{k=1}^{N-1} v_{0,k}(x,t)J_j^{(0)}, \qquad v_{0,p} = \gamma\left(i\,\mathrm{cotan}\,\frac{p\pi}{N}\;\psi_{p,x} - \sum_{k+s=p}^{N}\psi_k\psi_s(x,t)\right), \end{aligned} \tag{3.7}$$

where $\gamma = \frac{b\omega}{a^2}$. The λ-independent term in the Lax representation vanishes whenever the functions ψ_k satisfy the DNLS equation (1.1).

The approach to the mKdV equations is similar. In fact they are the next element of the hierarchy related to L. Their dispersion law is $\lambda^3 K_3$ where K_3 is a constant matrix belonging to $\mathfrak{g}^{(3)}\cap\mathfrak{h}$. Thus its Lax pair is:

$$\begin{aligned} L\psi &\equiv \frac{\partial\psi}{\partial x} + (Q(x,t) - \lambda J_0^{(1)})\psi(x,t,\lambda) = 0, \\ M_3\psi &\equiv \frac{\partial\psi}{\partial t} + \left(\sum_{k=0}^{2} V_k(x,t)\lambda^k - \lambda^3 K_3\right)\psi(x,t,\lambda) = \lambda^3\psi(x,t,\lambda)K_3. \end{aligned} \tag{3.8}$$

Skipping the details we write down the corresponding equations for the case $r=5$. Thus is a system of 5 equations for five functions:

$$\begin{aligned} \alpha\frac{\partial q_1}{\partial t} &= \frac{\partial}{\partial x}\left(4\frac{\partial^2 q_1}{\partial x^2} + \sqrt{3}\left(4\frac{\partial q_2}{\partial x}q_5 + 2q_3\frac{\partial q_4}{\partial x} + 3\frac{\partial q_3}{\partial x}q_4\right)\right. \\ &\qquad \left. + 6q_3(q_2^2+q_5^2) + 3q_1q_3^2 + 6q_4(q_1q_2+q_4q_5)\right), \\ \alpha\frac{\partial q_2}{\partial t} &= \frac{\partial}{\partial x}\left(\sqrt{3}\left(4\frac{\partial q_1}{\partial x}q_1 + q_5\frac{\partial q_3}{\partial x} - 2\frac{\partial q_5}{\partial x}q_3\right) + 6q_5(q_1q_2+q_4q_5) + 3q_2q_3^2 + 12q_1q_3q_4\right), \\ \alpha\frac{\partial q_3}{\partial t} &= \frac{\partial}{\partial x}\left(-\frac{1}{2}\frac{\partial^2 q_3}{\partial x^2} + \sqrt{3}\left(\frac{\partial q_2}{\partial x}q_1 + 3q_2\frac{\partial q_1}{\partial x} - \frac{\partial q_4}{\partial x}q_5 - 3\frac{\partial q_5}{\partial x}q_4\right)\right. \\ &\qquad \left. + 6(q_3(q_2q_4+q_1q_5) + q_1q_4^2 + q_2^2q_5) + 2q_1^3 - q_3^3 + 2q_5^3\right), \\ \alpha\frac{\partial q_4}{\partial t} &= \frac{\partial}{\partial x}\left(\sqrt{3}\left(2\frac{\partial q_1}{\partial x}q_3 - q_1\frac{\partial q_3}{\partial x} - 4\frac{\partial q_5}{\partial x}q_5\right) + 6q_1(q_1q_2+q_4q_5) + 3q_4q_3^2 + 12q_2q_3q_5\right), \\ \alpha\frac{\partial q_5}{\partial t} &= \frac{\partial}{\partial x}\left(4\frac{\partial^2 q_5}{\partial x^2} - \sqrt{3}\left(4\frac{\partial q_4}{\partial x}q_1 + 2q_3\frac{\partial q_2}{\partial x} + 3\frac{\partial q_3}{\partial x}q_2\right)\right. \\ &\qquad \left. + 6q_3(q_1^2+q_4^2) + 3q_5q_3^2 + 6q_2(q_1q_2+q_4q_5)\right). \end{aligned} \tag{3.9}$$

The corresponding Hamiltonian was given in (1.3) in the Introduction.

3.2. The 2-dimensional Toda field theories

The importance of the reduction group discovered by Mikhailov [28] is well known. Its first achievement was the discovery of the integrability of the 2-dimensional Toda field theories (2dTFT) which allow generalizations to any simple Lie algebra [28, 29]. If we choose a reduction group isomorphic to the $\mathbb{Z}_h$ we naturally obtain Lax pair on a Kac-Moody algebra. Here we have the freedom to choose the simple Lie algebra $\mathfrak{g}$, but we also have to comply with the relativistic invariance of TFT. In two-dimensional space-time this results in a symmetry of the Lax pair under the change $\lambda \to \lambda^{-1}$.

The first Lax pair for the 2dTFT has the form:

$$
\begin{aligned}
\tilde{L}\tilde{\psi} &\equiv \frac{\partial \tilde{\psi}}{\partial x} + (\vec{q}_x - \lambda \mathcal{J}(x,t))\tilde{\psi}(x,t,\lambda) = 0, \\
\tilde{M}\tilde{\psi} &\equiv \frac{\partial \tilde{\psi}}{\partial t} - (\vec{q}_t - \frac{1}{\lambda}\mathcal{K}(x,t))\tilde{\psi}(x,t,\lambda) = 0, \\
\vec{q} &= \sum_{s=1}^{r} q_s(x,t) H_s, \quad \mathcal{J}(x,t) = \sum_{\alpha\in\delta} f_\alpha(x,t) E_\alpha, \quad \mathcal{K}(x,t) = \sum_{\alpha\in\delta} g_\alpha(x,t) E_{-\alpha}.
\end{aligned}
\tag{3.10}
$$

Here $\vec{q}$, $\mathcal{J}(x,t)$ and $\mathcal{K}(x,t)$ are elements of a $\mathbb{Z}_h$-graded simple Lie algebra $\mathfrak{g}$, δ is the set of admissible roots of $\mathfrak{g}$. In other words, δ contains all simple roots α_j, $j = 1, \dots, r$ and the minimal one α_0. We should mention here that the grading is achieved by using Coxeter automorphism $\tilde{C}_1$ which belongs to the Cartan subgroup of $\mathfrak{g}$.

There are several obvious differences with respect to the construction of the previous subsection. As we mentioned above, the Lax pair $\tilde{L}(\lambda^{-1})$, $\tilde{M}(\lambda^{-1})$ produce precisely the same NLEE as the original pair $\tilde{L}(\lambda)$ and $\tilde{M}(\lambda)$. Another difference is that the Coxeter automorphism $\tilde{C}$ is realized as an element of the Cartan subgroup, see Eq. (2.13). As a result the structure of the subspaces $\tilde{\mathfrak{g}}^{(k)}$ is different. In particular $\tilde{\mathfrak{g}}^{(0)} \simeq \mathfrak{h}$, i.e. the λ-independent terms in both $\tilde{L}$ and $\tilde{M}$ are diagonal matrices. In addition the terms proportional to λ in $\tilde{L}$ and to λ^{-1} in $\tilde{M}$ cannot be chosen as diagonal matrices, and become functions of x and t. These facts seem to make the 2dTFT more difficult to solve than the $\mathbb{Z}_h$ DNLS and mKdV equations.

On the other hand the fact that we are using Coxeter automorphisms allows us to treat the problem adequately. Indeed, now the subspaces $\tilde{\mathfrak{g}}^{(k)}$ for $k \neq 0$ are spanned by the Weyl generators E_α corresponding to the roots of height $\mathrm{ht}\,(\alpha) = k \mod h$. In particular $\mathcal{J}(x,t)$ is linear combination of all admissible roots.

Following [28] we choose $\mathfrak{g} \simeq sl(r+1)$ and write down the compatibility condition of the pair $\tilde{L}$ and $\tilde{M}$. The terms proportional to λ and λ^{-1} in the equation $[\tilde{L}, \tilde{M}] = 0$ give simple differential equations:

$$
\begin{aligned}
&\lambda && \frac{\partial \mathcal{J}}{\partial t} + [\vec{q}_t, \mathcal{J}(x,t)] = 0, \\
&\lambda^{-1} && \frac{\partial \mathcal{K}}{\partial x} - [\vec{q}_x, \mathcal{K}(x,t)] = 0, \\
&\lambda^{0} && \qquad \frac{\partial^2 \vec{q}}{\partial x \partial t} = [\mathcal{J}, \mathcal{K}].
\end{aligned}
\tag{3.11}
$$

In order to solve them we need to use the commutation relations of the Cartan–Weyl basis, see [25]. Here with some abuse of notations we have used the duality between the Cartan subalgebra $\mathfrak{h}$ and the r-dimensional Euclidean space, i.e. by $\vec{q}$ we mean an element of the Cartan subalgebra which is dual to the vector $\vec{q} \in \mathbb{E}^r$.

$$
[\vec{q}, E_{\pm\alpha}] = \pm(\alpha, \vec{q}) E_{\pm\alpha}, \qquad [E_\alpha, E_{-\alpha}] = H_\alpha. \tag{3.12}
$$

The solution to the first two equations in (3.11) are given by:

$$
f_\alpha(x,t) = \exp(-(\alpha, \vec{q}(x,t))), \qquad g_\alpha(x,t) = \exp(-(\alpha, \vec{q}(x,t))), \tag{3.13}
$$

and the 2dTFT equations take the form:

$$
\frac{\partial \vec{q}}{\partial x \partial t} = \sum_{\alpha \in \delta_0} e^{-2(\alpha, \vec{q})} H_\alpha, \tag{3.14}
$$

which (after identifying H_α with α) coincide with (1.5)

3.3. Additional involutions and examples

Along with the $\mathbb{Z}_N$-reduction (3.5), we can introduce one of the following involutions ($\mathbb{Z}_2$-reductions):

$$
\begin{aligned}
&\text{a)} && K_0^{-1} U^\dagger(x,t,-\lambda^*) K_0 = -U(x,t,\lambda), \\
&\text{b)} && K_0^{-1} U^*(x,t,\lambda^*) K_0 = U(x,t,\lambda), \\
&\text{c)} && U^T(x,t,-\lambda) = -U(x,t,\lambda),
\end{aligned}
\tag{3.15}
$$

where

$$
K_0 = \sum_{k=1}^{h} E_{k,h-k+1}.
$$

An immediate consequences of Eq. (3.15) are the constraints on the potentials:

$$
\begin{aligned}
&\text{a)} && K_0^{-1} U_0^\dagger(x,t) K_0 = -U_0(x,t), && K_0^{-1} U_1^\dagger K_0 = U_1, \\
&\text{b)} && K_0^{-1} U_0^*(x,t) K_0 = U_0(x,t), && K_0^{-1} U_1^* K_0 = U_1, \\
&\text{c)} && U_0^T(x,t) = -U_0(x,t), && U_1 = U_1,
\end{aligned}
\tag{3.16}
$$

and the constraints on the fundamental solutions:

$$
\begin{aligned}
&\text{a)} \qquad K_0^{-1}\chi^\dagger(x,t,-\lambda^*)K_0 = \chi^{-1}(x,t,\lambda), \\
&\text{b)} \qquad K_0^{-1}\chi^*(x,t,\lambda)K_0 = \chi(x,t,\lambda), \\
&\text{c)} \qquad \chi^T(x,t,\lambda) = \chi^{-1}(x,t,\lambda),
\end{aligned}
\tag{3.17}
$$

More specifically from Eq. (3.16) there follows:

$$
\begin{aligned}
&\text{a)} \qquad \psi_j^*(x,t) = -\psi_j(x,t), && j = 1,\dots,h-1. \\
&\text{b)} \qquad \psi_j^*(x,t) = \psi_{h-j}(x,t), && j = 1,\dots,h-1, \\
&\text{c)} \qquad \psi_j(x,t) = -\psi_{h-j}(x,t), && j = 1,\dots,h-1.
\end{aligned}
\tag{3.18}
$$

The involutions a) and b) lead directly to the constraints in Eq. (1.2). All three involutions are valid reductions for the L operator. However, the third involution is not applicable to the DNLS equations and thus the constraint (3.18c) is not compatible with Eq. (1.1). The reason for that is that the second operator M is not compatible with it. In fact the reduction (3.16c) is compatible only with M-operators whose highest order term in λ is of odd power. In other words, this involutions is good only for NLEE that have odd dispersion laws.

Let us write down examples of DNLS systems of equations. The involution (1.2a) reduces Eq. (1.1) to a system of equations for h real-valued functions $u_k = i\psi_k$, $\gamma = i\gamma_0$:

$$
\frac{\partial u_k}{\partial t} + \gamma_0 \frac{\partial}{\partial x}\left(\operatorname{cotan}\frac{\pi k}{h}\cdot u_{k,x} - \sum_{p=1}^{h-1} u_p u_{k-p} \right) = 0, \qquad k = 1,2,\dots,h-1. \tag{3.19}
$$

The other two examples are obtained with involution (1.2b). If $h=5$ the involution (1.2b) leads to: $\psi_0 = \psi_5 = 0$, $\psi_1 = \psi_4^*$, $\psi_2 = \psi_3^*$, i.e. we have only two independent complex-valued fields and

$$
\begin{aligned}
i\frac{\partial \psi_1}{\partial t} + \gamma \operatorname{cotan}\frac{\pi}{5}\frac{\partial^2 \psi_1}{\partial x^2} + i\gamma \frac{\partial}{\partial x}\left(2\psi_2\psi_1^* + (\psi_2^*)^2\right) = 0, \\
i\frac{\partial \psi_2}{\partial t} + \gamma \operatorname{cotan}\frac{2\pi}{5}\frac{\partial^2 \psi_2}{\partial x^2} + i\gamma \frac{\partial}{\partial x}\left(2\psi_1^*\psi_2^* + (\psi_1)^2\right) = 0.
\end{aligned}
\tag{3.20}
$$

For $h=6$ and $\psi_1 = \psi_5^*$, $\psi_2 = \psi_4^*$, $\psi_3 = \psi_3^*$, so we have a system for two complex-valued fields ψ_1 and ψ_2 and the real field ψ_3:

$$
\begin{aligned}
i\frac{\partial \psi_1}{\partial t} + \gamma \operatorname{cotan}\frac{\pi}{6}\frac{\partial^2 \psi_1}{\partial x^2} + 2i\gamma \frac{\partial}{\partial x}\left(\psi_1^*\psi_2 + \psi_2^*\psi_3\right) = 0, \\
i\frac{\partial \psi_2}{\partial t} + \gamma \operatorname{cotan}\frac{2\pi}{6}\frac{\partial^2 \psi_2}{\partial x^2} + i\gamma \frac{\partial}{\partial x}\left(\psi_1^2 + 2\psi_1^*\psi_3 + (\psi_2^*)^2\right) = 0, \\
\frac{\partial \psi_3}{\partial t} + 2\gamma \frac{\partial}{\partial x}\left(\psi_1\psi_2 + \psi_1^*\psi_2^*\right) = 0,
\end{aligned}
\tag{3.21}
$$

4. The interrelations between L and $\tilde{L}$

Our aim is to compare the two approaches and demonstrate that they are related by gauge transformation. As a result the two hierarchies of NLEE: the one containing 2dTFT and the other containing the mKdV are shown to be equivalent. We also demonstrate that the spectral problems of both Lax operators L and $\tilde{L}$ can be reduced to the same Riemann–Hilbert problems. In addition the minimal sets of scattering data for both Lax operators are equivalent.

Let us somewhat simplify the notations in the two Lax operators

$$\begin{aligned} L\psi &\equiv \frac{\partial \psi}{\partial x} + (Q(x,t) - \lambda J)\psi(x,t,\lambda) = 0, \\ \tilde{L}\tilde{\psi} &\equiv \frac{\partial \tilde{\psi}}{\partial x} + (\vec{q}_x - \lambda \mathcal{J}(x,t))\tilde{\psi}(x,t,\lambda) = 0, \end{aligned} \tag{4.1}$$

where $J = \operatorname{diag}(1, \omega, \omega^2, \dots, \omega^r)$ and $\omega = \exp(2\pi i/h)$.

Following [7] in writing down $\mathcal{J}(x,t)$ we will use the Chevalie basis of $A_r^{(1)}$ algebra. It is given by:

$$\begin{aligned} E_{\alpha_0} &= E_{r+1,1}\lambda, & E_{\alpha_j} &= E_{j,j+1}\lambda, & j &= 1, \dots, r \\ E_{-\alpha_0} &= E_{1,r+1}\lambda^{-1}, & E_{-\alpha_j} &= E_{j+1,j}\lambda^{-1}, & j &= 1, \dots, r \\ H_{\alpha_0} &= E_{r+1,r+1} - E_{1,1}, & H_{\alpha_j} &= E_{j,j} - E_{j+1,j+1}, & j &= 1, \dots, r. \end{aligned} \tag{4.2}$$

Thus we have the explicit matrix form of $\mathcal{J}(x,t)$.

The next step is to analyze the direct and the inverse scattering problems for L and $\tilde{L}$. These problems have been solved for the operator L some time ago [3, 4, 21, 22, 24]. Important factor for it was in the fact that J is constant and diagonal matrix.

As for the operator $\tilde{L}$ we first check the characteristic equation for $\mathcal{J}(x,t)$ with the result:

$$\det(\mathcal{J}(x,t) - z\mathbb{1}) = z^h - 1, \qquad h = r + 1. \tag{4.3}$$

Therefore the eigenvalues of $\mathcal{J}(x,t)$ are ω^p, $p = 0, 1, \dots, h-1$, i.e. they are independent of both t and x. Thus there exists a matrix $w(x,t)$ which diagonalize $\mathcal{J}(x,t)$:

$$\mathcal{J}(x,t) = w^{-1} J w(x,t), \tag{4.4}$$

where J is the same as in the operator L. It is only natural to apply gauge transformation to $\tilde{L}$ with $w(x,t)$. In addition we will need to multiply $\tilde{L}$ by i and have to replace λ by $i\lambda$. Thus it is easy to find that these transformations map $\tilde{L}$ into L where $Q(x,t) = -w_x w^{-1} + w\vec{q}_x w^{-1}$. As a result we have shown that the operator $\tilde{L}$ is gauge equivalent to L, so they have the same spectral properties.

5. The spectral properties of the Lax operators with $\mathbb{Z}_h$-reduction

5.1. The FAS of the Lax operators L

Here we just outline the procedure of constructing the FAS of L [3, 4, 21, 22, 24]. First we have to determine the regions of analyticity. For smooth potentials $U_0(x)$ that fall off fast enough for $x \to \pm\infty$

these regions are the $2h$ sectors Ω_ν separated by the rays l_ν on which $\mathrm{Re}\,\lambda(a_j - a_k) = 0$, where by a_j here and below we mean $a_j = -U_{1,jj} = -\omega^{j-1/2}$. The rays l_ν are given by:

$$l_\nu\colon\ \arg(\lambda) = \frac{\pi(\nu-1)}{h}, \qquad \nu = 1, \dots, 2h, \tag{5.1}$$

and close angles equal to π/h. Here without restrictions we have put $a = 1$; indeed, we can always change $\lambda \to \lambda' = a\lambda$.

The next step is to construct the set of integral equations equivalent to (3.1) whose solution will be analytic in Ω_ν. To this end we associate with each sector Ω_ν the relations (orderings) $\underset{\nu}{>}$ and $\underset{\nu}{<}$ by:

$$\begin{array}{l} i \underset{\nu}{>} j \\ i \underset{\nu}{<} j \end{array} \quad \text{if} \quad \begin{array}{ll} \mathrm{Re}\,\lambda(a_i - a_j) < 0 & \text{for } \lambda \in \Omega_\nu, \\ \mathrm{Re}\,\lambda(a_i - a_j) > 0 & \text{for } \lambda \in \Omega_\nu. \end{array} \tag{5.2}$$

Then the solution of the system

$$\begin{aligned} \xi^\nu_{ij}(x,\lambda) &= \delta_{ij} + i\int_{-\infty}^x dy e^{-\lambda(a_i-a_j)(x-y)} \sum_{p=1}^h U_{0;ip}(y)\xi^\nu_{pj}(y,\lambda), \qquad i \underset{\nu}{>} j; \\ \xi^\nu_{ij}(x,\lambda) &= i\int_{\infty}^x dy e^{-\lambda(a_i-a_j)(x-y)} \sum_{p=1}^h U_{0;ip}(y)\xi^\nu_{pj}(y,\lambda), \qquad i \underset{\nu}{<} j; \end{aligned} \tag{5.3}$$

will be the FAS of L in the sector Ω_ν. The asymptotics of $\xi^\nu(x,\lambda)$ and $\xi^{\nu-1}(x,\lambda)$ along the ray l_ν can be written in the form [10, 21]:

$$\begin{aligned} \lim_{x\to-\infty} e^{-\lambda U_1 x}\xi^\nu(x,\lambda e^{i0})e^{\lambda U_1 x} &= S^+_\nu(\lambda), && \lambda \in l_\nu, \\ \lim_{x\to-\infty} e^{-\lambda U_1 x}\xi^{\nu-1}(x,\lambda e^{-i0})e^{\lambda U_1 x} &= S^-_\nu(\lambda), && \lambda \in l_\nu, \\ \lim_{x\to\infty} e^{-\lambda U_1 x}\xi^\nu(x,\lambda e^{i0})e^{\lambda U_1 x} &= T^-_\nu D^+_\nu(\lambda), && \lambda \in l_\nu, \\ \lim_{x\to\infty} e^{-\lambda U_1 x}\xi^{\nu-1}(x,\lambda e^{-i0})e^{\lambda U_1 x} &= T^+_\nu D^-_\nu(\lambda), && \lambda \in l_\nu, \end{aligned} \tag{5.4}$$

where the matrices S^+_ν, T^+_ν (resp. S^-_ν, T^-_ν) are upper-triangular (resp. lower-triangular) with respect to the ν-ordering. They provide the Gauss decomposition of the scattering matrix with respect to the ν-ordering, i.e.:

$$T_\nu(\lambda) = T^-_\nu(\lambda)D^+_\nu(\lambda)\hat{S}^+_\nu(\lambda) = T^+_\nu(\lambda)D^-_\nu(\lambda)\hat{S}^-_\nu(\lambda), \qquad \lambda \in l_\nu. \tag{5.5}$$

More careful analysis shows [21] that in fact $T_\nu(\lambda)$ belongs to a subgroup $\mathcal{G}_\nu$ of $SL(N,\mathbb{C})$. Indeed, with each ray l_ν one can relate a subalgebra $\mathfrak{g}_\nu \subset sl(N,\mathbb{C})$. Each such $sl(2)$-subalgebra can be specified by a pair of indices (k,s) and is generated by:

$$h^{(k,s)} = E_{kk} - E_{ss}, \qquad e^{(k,s)} = E_{ks}, \qquad f^{(k,s)} = E_{sk}, \qquad k \underset{\nu}{<} s. \tag{5.6}$$

Then the scattering matrix $T_\nu(\lambda)$ will be a product of mutually commuting matrices $T_\nu^{(k,s)}$ of the form:

$$T_\nu^{(k,s)} = \mathbb{1} + (a^+_{\nu;ks}(\lambda) - 1)E_{kk} + (a^-_{\nu;ks}(\lambda) - 1)E_{ss} - b^-_{\nu;ks}(\lambda)E_{ks} + b^+_{\nu;ks}(\lambda)E_{sk}, \tag{5.7}$$

where $k \underset{\nu}{<} s$, with only 4 nontrivial matrix elements, just like the ZS (or AKNS) system. The $\mathbb{Z}_n$-symmetry imposes the following constraints on the FAS and on the scattering matrix and its factors:

$$\begin{aligned} C_0\xi^\nu(x,\lambda\omega)C_0^{-1} &= \xi^{\nu-2}(x,\lambda), \qquad & C_0T_\nu(\lambda\omega)C_0^{-1} &= T_{\nu-2}(\lambda), \\ C_0S^\pm_\nu(\lambda\omega)C_0^{-1} &= S^\pm_{\nu-2}(\lambda), \qquad & C_0D^\pm_\nu(\lambda\omega)C_0^{-1} &= D^\pm_{\nu-2}(\lambda, \end{aligned} \tag{5.8}$$

where the index $\nu - 2$ should be taken modulo $2N$. Consequently we can view as independent only the data on two of the rays, e.g., on l_1 and $l_{2N} \equiv l_0$; all the rest will be recovered using (5.8).

If in addition we impose the $\mathbb{Z}_2$-symmetry (3.17a), then we will have also:

$$\begin{aligned} &\text{a)} \qquad K_0^{-1}(\xi^\nu(x,-\lambda^*))^\dagger K_0 = (\xi^{N+1-\nu}(x,\lambda))^{-1}, \\ &\qquad\qquad K_0^{-1}(S^\pm_\nu(-\lambda^*))K_0 = (S^\mp_{N+1-\nu}(\lambda))^{-1}, \\ &\text{b)} \qquad K_0^{-1}(\xi^\nu(x,\lambda^*))^* K_0 = (\xi^\nu(x,\lambda))^{-1}, \\ &\qquad\qquad K_0^{-1}(S^\pm_\nu(\lambda^*))K_0 = (S^\mp_{N+1-\nu}(\lambda))^{-1}, \end{aligned} \tag{5.9}$$

and analogous relations for $T^\pm_\nu(\lambda)$ and $D^\pm_\nu(\lambda)$. One can prove also that $D^+_\nu(\lambda)$ (resp. $D^-_\nu(\lambda)$) allows analytic extension for $\lambda \in \Omega_\nu$ (resp. for $\lambda \in \Omega_{\nu-1}$. Another important fact is that $D^+_\nu(\lambda) = D^-_{\nu+1}(\lambda)$ for all $\lambda \in \Omega_\nu$ [21].

5.2. The inverse scattering problem and the Riemann–Hilbert problem

The next important step is the possibility to reduce the solution of the ISP for the generalized Zakharov–Shabat system to a (local) RHP. More precisely, we have:

$$\begin{aligned} &\xi^\nu(x,t,\lambda) = \xi^{\nu-1}(x,t,\lambda)G_\nu(x,t,\lambda), && \lambda \in l_\nu, \\ &G_\nu(x,t,\lambda) = e^{\lambda U_1 x - \lambda^2 V_2 t}G_{0,\nu}(\lambda)e^{-\lambda U_1 x + \lambda^2 V_2 t}, \quad && G_{0,\nu}(\lambda) = \hat{S}^-_\nu S^+_\nu(\lambda)\Big|_{t=0}. \end{aligned} \tag{5.10}$$

The collection of all relations (5.10) for $\nu = 1, 2, \dots, 2h$ together with

$$\lim_{\lambda\to\infty} \xi^\nu(x,t,\lambda) = \mathbb{1}, \tag{5.11}$$

can be viewed as a local RHP posed on the collection of rays $\Sigma \equiv \{l_\nu\}_{\nu=1}^{2h}$ with canonical normalization. Rather straightforwardly we can prove that if $\xi^\nu(x,\lambda)$ is a solution of the RHP (5.10), (5.11) then $\chi^\nu(x,\lambda) = \xi^\nu(x,\lambda)e^{-\lambda U_1 x}$ is a FAS of L with potential

$$U_0(x,t) = \lim_{\lambda\to\infty} \lambda\left(U_1 - \xi^\nu(x,t,\lambda)U_1\hat{\xi}^\nu(x,t,\lambda)\right). \tag{5.12}$$

The analyticity properties of $D_k^{\pm}(\lambda)$ allow one to reconstruct them from the sewing function $G(\lambda)$ (5.10) and from the locations of their simple zeroes and poles through

$$\ln \mathcal{D}_k(\lambda) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\mu}{\mu-\lambda} \ln \mu_{G_0,k} + \sum_{j=1}^{N}\sum_{s=0}^{N-1} \ln \frac{\lambda-\lambda_{j,s}^+}{\lambda-\lambda_{j,s}^-}, \tag{5.13}$$

where $\mu_{G_0,k}$ is the principal upper minor of $G_0(t,\lambda)$ of order k. The zeroes and poles $\lambda_{j,s}^{\pm}$ of $\mathcal{D}$ are in fact discrete eigenvalues of L, which due to the reductions come in multiplets:

$$\begin{aligned} &\text{a)} \qquad \lambda_{j,s}^+ = \lambda_j^+ \omega^{s-1}, \quad \lambda_{j,s}^- = -(\lambda_{j,s}^+)^*, \quad \lambda_j^+ \in \Omega_1, \\ &\text{b)} \qquad \lambda_{j,s}^+ = \lambda_j^+ \omega^{s-1}, \quad \lambda_{j,s}^- = (\lambda_{j,s}^+)^*, \quad \lambda_j^+ \in \Omega_1. \end{aligned} \tag{5.14}$$

Consider first case a). For odd N and $\lambda_j^+ \in \Omega_1$ it is easy to check that all eigenvalues are inside sectors $\Omega_{2\nu-1}$ with odd indices; if $\lambda_j^+ \in \Omega_0$ then all eigenvalues are inside sectors $\Omega_{2\nu}$ with even indices. Thus for each generic choice of λ_j^+ we have a multiplet of $2N$ discrete eigenvalues. However, if we choose $\arg \lambda_j^+ = \pi/(2h)$, then the set of $\lambda_{j,s}^-$ coincides with the set of $\lambda_{j,s}^+$ and we have smaller multiplets with N discrete eigenvalues each. Thus one may conclude that for odd N the DNLS equations have two types of one-soliton solutions corresponding to the two different types of multiplets. For even N the situation is different. All multiplets containing λ_j^+ have exactly $2N$ discrete eigenvalues, one in each of the $2N$ sectors Ω_ν. So for odd N only one type of one-soliton solutions exists.

In the case b) we will have multiplets of $2N$ eigenvalues, one for each of the sectors Ω_ν both for N even and N odd.

More detailed analysis shows that $\mathcal{D}_k^+(\lambda)$ (resp. $\mathcal{D}_k^-(\lambda)$) are related to the principle upper (resp. lower) minors of order k of the scattering matrix $T(t,\lambda)$ by:

$$\mathcal{D}_k(\lambda) = \begin{cases} \ln m_k^+(\lambda), & \lambda \in \mathbb{C}_+ \\ -\ln m_{n-k}^-(\lambda), & \lambda \in \mathbb{C}_-. \end{cases} \tag{5.15}$$

One can view $\mathcal{D}_k(\lambda)$ as generating functionals of the conserved quantities for the related NLEE. Using the fact that $\ln m_{\nu,k}^+(\lambda)$ allows asymptotic expansions

$$\ln m_{\nu,k}^+(\lambda) = \sum_{s=1}^{\infty} M_{\nu,k}^{(s)} \lambda^{-s}. \tag{5.16}$$

We are able to calculate the local integrals of motion for the DNLS equations. We illustrate it by the two first integrals of motion of the $\mathbb{Z}_n$-NLS equation:

$$M_{1,1}^{(1)} = \frac{1}{2\omega}\int_{-\infty}^{\infty} dx \sum_{p=1}^{n} \psi_p \psi_{n-p}(x,t), \tag{5.17}$$

$$M_{1,1}^{(2)} = \frac{1}{2\omega^2}\int_{-\infty}^{\infty} dx \left\{ \sum_{p=1}^{n} i \operatorname{cotan}\left(\frac{\pi p}{n}\right)\left(\frac{d\psi_p}{dx}\psi_{n-p} - \psi_p \frac{d\psi_{n-p}}{dx}\right) - \frac{2}{3}\sum_{p+k+l=n} \psi_p\psi_k\psi_l(x,t) \right\}. \tag{5.18}$$

5.3. The minimal sets of scattering data

As a consequence of the above considerations we conclude that the following Lemma holds true:

Lemma 5.1 *Let us assume that $\xi^{\nu}(x,t,\lambda)$ is a regular solution of the RHP (5.10). Then Each of the minimal sets of scattering data:*

$$\begin{aligned} \mathcal{T}_1 &= \{S_\nu^+(\lambda), S_\nu^-(\lambda), \quad \nu = 0,1\}, \\ \mathcal{T}_2 &= \{T_\nu^+(\lambda), T_\nu^-(\lambda), \quad \nu = 0,1\}, \end{aligned} \tag{5.19}$$

determines uniquely: i) the sewing functions $G_\nu(x,t,\lambda)$ of the RHP; ii) the solution of the RHP; and iii) the potential of the Lax operator $U_0(x,t)$.

Proof i) Given $\mathcal{T}_1$ (respectively $\mathcal{T}_2$) and using (5.8) we recover all functions $S_\nu^{\pm}(\lambda,t)$ (respectively $T_\nu^{\pm}(\lambda,t)$) for all $\nu = 0,\dots,2h-1$. It remains to use (5.9) to recover all sewing functions $G_\nu(x,t,\lambda)$.
ii) It is well known that the RHP has unique regular solution.
iii) The potential of the Lax operator is determined from Eq. (5.12). □

6. Conclusion

We outlined the inverse scattering method for three types of NLEE related to the Kac-Moody algebras of the class $A_r^{(1)}$. Some of the above results have also been extended to other Kac-Moody algebras, such as $D_4^{(s)}$, $s = 1,2,3$ and $A_5^{(2)}$ [12, 14–19, 21, 22]. For many others there are still open problems.

The extension of the dressing Zakharov–Shabat method [37] to the above classes of Lax operators is also an open problem. One of the difficulties is due to the fact that the $\mathbb{Z}_h$ reductions requires dressing factors with $2h$ pole singularities. This makes the relevant linear algebraic equations rather involved.

The ideas of [1, 13, 20] about the interpretation of the inverse scattering method as a generalized Fourier transform hold true also for the $\mathbb{Z}_h$ reduces Lax operators [22–24, 31, 32]. This may allow one to derive the action-angle variables for these classes of NLEE.

Another important aspect of the theory concerns the geometric interpretation of the recursion operators and the relevant hierarchies of Hamiltonian structures [20, 33].

Acknowledgment

The author acknowledges the useful suggestions of anonymous referee and the financial support of Bulgarian National science foundations, contract KP-06N42-2.

References

[1] Ablowitz MS, Kaup DJ, Newell AC, Segur H. The inverse scattering transform – Fourier analysis for nonlinear problems. Studies in Applied Mathematics 1974; 53 (4): 249-315.

[2] Adler M. On a trace functional for formal pseudo-differential operators and the symplectic structure of the Korteweg-de Vries equations. Inventiones Mathematicae 1979; 50: 219-248.

[3] Beals R, Coifman RR. Scattering and inverse scattering for first order systems. Communications in Pure and Applied Mathematics 1984; 37: 39-90.

[4] Beals R, Coifman RR. Inverse scattering and evolution equations. Communications in Pure and Applied Mathematics 1985; 38 (1): 29-42.

[5] Calogero F, Degasperis A. Spectral transform and solitons. Vol. I. North Holland, Amsterdam, 1982.

[6] Carter R. Lie algebras of finite and affine type (Cambridge University Press, Cambridge, 2005).

[7] Drinfel'd VV, Sokolov VG. Lie algebras and equations of Korteweg-de Vries type. Itogi Nauki i Techniki, Seriya Sovremennye Problemy Matematiki (Noveishie Dostizheniya), 1984; 24 (1): 81-180.

[8] Faddeev LD, Takhtadjan LA. Hamiltonian methods in the theory of solitons. (Springer Verlag, Berlin, 1987).

[9] Gerdjikov VS. Z_N –reductions and new integrable versions of derivative nonlinear Schrödinger equations. In: Nonlinear evolution equations: integrability and spectral methods, Ed. Fordy AP, Degasperis A, Lakshmanan M. Manchester University Press, (1981), p. 367-372.

[10] Gerdjikov VS. Algebraic and analytic aspects of N-wave type equations. Contemporary Mathematics 2002; 301 (1): 35-68. **nlin.SI/0206014**.

[11] Gerdjikov VS. Generalised Fourier transforms for the soliton equations. Gauge covariant formulation. Inverse Problems 1986; 2 (1): 51-74.

[12] Gerdjikov VS. Derivative nonlinear Schrödinger equations with $\mathbb{Z}_N$ and $\mathbb{D}_N$ reductions. Romanian Journal of Physics 2013; 58 (5-6): 573-582.

[13] Gerdjikov VS, Kulish PP. The generating operator for the $n \times n$ linear system. Physica D, 1981; 3D (3): 549-564.

[14] Gerdjikov VS, Mladenov DM, Stefanov AA, Varbev SK. MKdV-type of equations related to $B_2^{(1)}$ and $A_4^{(2)}$ algebra. Springer Proceedings in Physics, Nonlinear Mathematical Physics and Natural Hazards, eds. B. Aneva, M. Kouteva-Guentcheva, 2014; 163: 59-69. (Springer).

[15] Gerdjikov VS, Mladenov DM, Stefanov AA, Varbev SK. Integrable equations and recursion operators related to the affine Lie algebras $A_r^{(1)}$. Journal of Mathematical Physics 2015; 56 (5): 052702. doi: 10.1063/1.4919672. **ArXiv: 1411.0273v1 [nlin-SI]**

[16] Gerdjikov VS, Mladenov DM, Stefanov AA, Varbev SK. On mKdV equations related to the affine Kac-Moody algebra $A_5^{(2)}$. Journal of Geometry and Symmmetry in Physics 2015; 39 (1): 17–31. doi:10.7546/jgsp-39-2015-17-31 **ArXive: 1512.01475 nlin:SI**.

[17] Gerdjikov VS, Mladenov DM, Stefanov AA, Varbev SK. MKdV-type of equations related to $\mathfrak{sl}(N, \mathbb{C})$ algebra. In Mathematics in Industry, Ed. A. Slavova. Cambridge Scholar Publishing, 2015, pp. 333-341.

[18] Gerdjikov VS, Mladenov DM, Stefanov AA, Varbev SK. On the MKdV type equations related to $A_5^{(1)}$ and $A_5^{(2)}$ Kac-Moody algebras. Theoretical and Mathematical Physics, 2021; 207 (2): 604-625.

[19] Gerdjikov VS, Mladenov DM, Stefanov AA, Varbev SK. MKdV-type of equations related to $\mathfrak{sl}(N, \mathbb{C})$ algebra. Mathematics in Industry, ed. A. Slavova, Cambridge Scholar Publishing, 2014; 335-344.

[20] Gerdjikov VS, Vilasi G, Yanovski AB. Integrable Hamiltonian hierarchies. Spectral and geometric methods. Lecture Notes in Physics 2008; 748 Springer Verlag, Berlin, Heidelberg, New York. ISBN: 978-3-540-77054-1.

[21] Gerdjikov VS, Yanovski AB. Completeness of the eigenfunctions for the Caudrey–Beals–Coifman system. Journal of Mathematical Physics 1994; 35 (7): 3687-3725.

[22] Gerdjikov VS, Yanovski AB. CBC systems with Mikhailov reductions by Coxeter automorphism. I. Spectral theory of the recursion operators. Studies in Applied Mathematics 2015; 134 (2): 145-180. doi: 10.1111/sapm.12065.

[23] Gerdjikov VS, Yanovski AB. On soliton equations with $\mathbb{Z}_h$ and $\mathbb{D}_h$ reductions: conservation laws and generating operators. Journal of Geometry and Symmmetry in Physics 2013; 31 (1): 57-92.

[24] Gerdjikov VS, Yanovski AB, Completeness of the eigenfunctions for the Caudrey-Beals-Coifman system. Journal of Mathematical Physics 1994; 35 (7): 3687-3725.

[25] Helgasson S. Differential geometry, Lie groups and symmetric spaces. (Academic Press, New York, 1978).

[26] Kaup DJ, Newell AC. Soliton equations, singular dispersion relations and moving eigenvalues. Advances in Mathematics 1979; 31 (1): 67-100.

[27] Kac V. "Infinite-dimensional Lie algebras" (Cambridge University Press, Cambridge, 1994).

[28] Mikhailov AV. The reduction problem and the inverse scattering problem. Physica D, 1981; 3D (1-2): 73-117.

[29] Mikhailov AV, Olshanetsky MA, Perelomov AM. Two-dimensional generalized Toda lattice. Communications of Mathematical Physics 1981; 79 (4): 473-488.

[30] Mikhailov AV, Zakharov VE. On the integrability of classical spinor models in two-dimensional space-time. Communications of Mathematical Physics 1980; 74 (1): 21-40.

[31] Valchev T. On the Kaup-Kupershmidt equation. Completeness relations for the squared solutions. Eds. I. Mladenov, M. de Leon, Softex, Sofia 2008; 308-319.

[32] Yanovski AB. Recursion operators and expansions over adjoint solutions for the Caudrey-Beals-Coifman system with $\mathbb{Z}_p$ reductions of Mikhailov type. Journal of Geometry and Symmmetry in Physics 2013; 30 (1): 105-119.

[33] Yanovski AB, Vilasi G. Geometric theory of the recursion operators for the generalized Zakharov-Shabat system in pole gauge on the algebra $sl(n,\mathbb{C})$: with and without reductions. SIGMA 2012; 8 (1): 87-110.

[34] Zakharov VE, Manakov SV, Novikov SP, Pitaevskii LI. Theory of solitons: the inverse scattering method. (Plenum, N.Y.: Consultants Bureau, 1984).

[35] Zakharov VE, Manakov SV. The theory of resonant interaction of wave packets in nonlinear media. Soviet Physics JETP 1975; 69 (5): 842-850.

[36] Zakharov VE, Mikhailov AV. Relativistically invariant two-dimensional models of field theory which are integrable by means of the inverse scattering problem method. Soviet Physics JETP 1978; 47 (6): 1017-1027.

[37] Zakharov VE, Shabat AB. Integration of nonlinear equations of mathematical physics by the method of inverse scattering. II. Functional Analysis and Applications 1979; 13 (3): 166-174.