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Computing forgotten topological index of zero-divisor graphs of commutative rings

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Abstract: The main objective of this paper is to calculate the forgotten topological index of the zero-divisor graph of $\mathbb{Z}_n$. Let p, q and r be distinct prime numbers. We calculate the forgotten topological index of the ring $\Gamma(\mathbb{Z}_n)$ where $n = p^\alpha, pq, p^2q, p^2q^2, pqr$. Also, we study the forgotten topological index of the product of rings of integers modulo n. We construct a polynomial algorithm to compute the forgotten topological index of $\Gamma(\mathbb{Z}_n)$.

Key words: Topological indices, forgotten topological index, zero-divisor graph, vertex degree, algorithm

1. Introduction and preliminaries

Zero-divisor graphs of commutative rings entered the area of algebraic combinatorics by the work of I. Beck [9]. His definition of zero-divisor graph has a vertex set on R and any two elements $x, y \in R$ are adjacent whenever $xy = 0$. This definition of a zero-divisor graph of a commutative ring was later modified by Anderson and Livingston in [5]. After the introduction of zero-divisor graphs, different types of graphs related to commutative rings emerged such as annihilating-ideal graphs, comaximal graphs, total graphs [1, 4, 6, 35, 37, 39–41, 43]. In recent years, many works have been done on topological indices of graphs. Indeed, these works include zero-divisor graphs of commutative rings. There are different types of topological indices such as distance-based and degree-based. One of the most well-studied distance-based index which is called Wiener index was studied for zero-divisor graphs in [7, 38]. Let $G = (V(G), E(G))$ be a simple and undirected graph. The Wiener index of a graph G is defined as:

$$\mathrm{W}(G) = \sum_{u,v \in V(G)} d(u,v)$$

where $d(u,v)$ stands for the distance between the vertices u and v of G.

The first and the second Zagreb indices which are degree-based indices were introduced in [23] and more details about these indices can be found in [24]. These Zagreb indices are shown by M_1 and M_2 and defined as:

$$M_1 = M_1(G) = \sum_{u \in V(G)} {d_u}^2 \text{ and } M_2 = M_2(G) = \sum_{uv \in E(G)} d_u d_v$$

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2010 *AMS Mathematics Subject Classification:* 05C25, 05C09, 13A70, 68R10 respectively. Many researchers have studied some recent results on these indices [10, 25, 28, 44]. Moreover, the Zagreb indices of zero-divisor graphs of $\Gamma(\mathbb{Z}_n)$ were calculated in [38].

Recently, the Sombor index was introduced by Gutman in [26] and defined as:

$$\mathrm{SO}(G) = \sum_{uv \in E(G)} \sqrt{{d_u}^2 + {d_v}^2}$$

where d_u and d_v is degree of vertices u and v, respectively. This novel topological index is degree-based and has attracted authors and many research have been done about this index [11, 22, 27, 34, 45].

Moreover, the topological indices such as Wiener index, PI index, Zagreb index, and eccentric index measure the characters of drug molecules. Recently, there are many studies to determine these topological indices of special molecular graphs in chemical, nanomaterials and pharmaceutical engineering, etc. [13–20, 42].

Followed by the first and second Zagreb indices, Furtula and Gutman (2015) introduced the forgotten topological index (also called F-index) which was defined as,

$$\mathrm{FT}(G) = \sum_{u \in V(G)} {d_u}^3 = \sum_{uv \in E(G)} ({d_u}^2 + {d_v}^2)$$

where d_v is denoted as the degree of vertex v (the number of vertices adjacent to vertex v) [12]. Furtula and Gutman analyzed that predictive ability of the forgotten topological index is almost similar to that of the first Zagreb index and for the eccentric factor and entropy, and both of them obtain correlation coefficients larger than 0.95. This fact implies the reason why the forgotten topological index is useful for testing the chemical and pharmacological properties of drug molecular structures. More recently, Gao et al. presented the forgotten topological index of some significant drug molecular structures [21]. Also, many researchers continue to examine the forgotten topological index with different studies [8, 29–33, 36].

The works mentioned above led us to study the forgotten topological index of $\Gamma(\mathbb{Z}_n)$, $\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m)$, and $\Gamma(\mathbb{Z}_n \times \mathbb{Z}_m \times \mathbb{Z}_r)$. In section 2, we calculate the forgotten topological index of the graph $\Gamma(\mathbb{Z}_n)$ and give an algorithm for this calculation. Finally, in section 3, we examine the forgotten topological index of zero-divisor graphs of products of rings modulo integers n and m.

2. Forgotten topological index of zero-divisor graph of $\mathbb{Z}_n$

Recently, the zero-divisor graph of the ring $\mathbb{Z}_n$ is a popular research in spectral graph and chemical graph theory. Many researchers have studied in this area. Singh and Bhat have examined adjacency matrix and Wiener index of zero-divisor graph $\Gamma(\mathbb{Z}_n)$ [38]. Later, Asir and Rabikka have studied Wiener index of zero-divisor graph of $\Gamma(\mathbb{Z}_n)$ [7]. Now, we analyze the forgotten topological index of zero-divisor graph $\Gamma(\mathbb{Z}_n)$ in this section.

Theorem 2.1 *Let p be a prime number, then followings hold:*

(i) If $p = 2$, then $\mathrm{FT}(\Gamma(\mathbb{Z}_{p^2})) = 0$,

(ii) If $p > 2$, then $\mathrm{FT}(\Gamma(\mathbb{Z}_{p^2})) = 2(p-2)^2\binom{p-1}{2}$.

Proof

(i) It is clear that $\mathbb{Z}_4$ has only one nonzero zero-divisor which is 2, then proof follows.

(ii) Since $\mathbb{Z}_{p^2} \cong K_{p-1}$, the proof follows.

□

Now we are about to calculate the forgotten topological index of zero-divisor graph for powers of p greater than or equal to 3.

Theorem 2.2 *Let $p > 2$ be a prime number and $\alpha \in \mathbb{N}$ with $\alpha \geq 3$, then the forgotten topological index of $\Gamma(\mathbb{Z}_{p^\alpha})$ is*

$$\mathrm{FT}(\Gamma(\mathbb{Z}_{p^\alpha})) = p^\alpha(p-1)\left[\sum_{i=1}^{\lfloor\frac{\alpha-1}{2}\rfloor} \frac{(p^i-1)^3}{p^{i+1}} + \sum_{i=\lceil\frac{\alpha}{2}\rceil}^{\alpha-1} \frac{(p^i-2)^3}{p^{i+1}}\right].$$

Proof We can show the zero-divisors of $(\mathbb{Z}_{p^\alpha})$ as follows:

$$\begin{aligned} A_1 &= \{px \mid x = 1, \ldots, p^{\alpha-1}-1, p \nmid x\}, \\ &\cdots \\ A_i &= \{p^i x \mid x = 1, \ldots, p^{\alpha-i}-1, p \nmid x\}, \\ &\cdots \\ A_{\alpha-1} &= \{p^{\alpha-1}x \mid x = 1, \ldots, p-1, p \nmid x\}. \end{aligned}$$

The vertex set of the graph $\Gamma(\mathbb{Z}_{p^\alpha})$ equals $\bigcup_{i=1}^{\alpha-1} A_i$ with $A_i \cap A_j = \emptyset$ for $i \neq j$, where $i, j \in \{1, \ldots, \alpha-1\}$. Besides, $|A_i|$ means the number of vertices of A_i. We calculate the number of vertices of all zero-divisor sets as $|A_1| = p^{\alpha-1} - p^{\alpha-2}$, $|A_2| = p^{\alpha-2} - p^{\alpha-3}$, ..., $|A_i| = p^{\alpha-i-j}(p-1)$, ..., $|A_{\alpha-1}| = p-1$. Moreover, the degree of each vertex in these zero-divisor sets can be defined as follows

$$d_u = \begin{cases} p-1, & u \in A_1 \\ p^2-1, & u \in A_2 \\ \cdots \\ p^i-1, & i < \alpha/2 \\ p^i-2, & i \geq \alpha/2 \\ \cdots \\ p^{\alpha-2}-2, & u \in A_{\alpha-2} \\ p^{\alpha-1}-2, & u \in A_{\alpha-1} \end{cases}$$

for all $u \in A_i$. The forgotten topological index of $\Gamma(\mathbb{Z}_{p^\alpha})$ can be attained that

$$
\begin{aligned}
\mathrm{FT}(\Gamma(\mathbb{Z}_{p^\alpha})) &= \sum_{u\in V(\Gamma(\mathbb{Z}_{p^\alpha}))} {d_u}^3 \qquad (2.1)\\
&= \sum_{i=1}^{\alpha-1}\sum_{u\in A_i} {d_u}^3\\
&= \sum_{i=1}^{\lfloor\frac{\alpha-1}{2}\rfloor}\sum_{u\in A_i} {d_u}^3 + \sum_{i=\lceil\frac{\alpha}{2}\rceil}^{\alpha-1}\sum_{u\in A_i} {d_u}^3\\
&= \sum_{i=1}^{\lfloor\frac{\alpha-1}{2}\rfloor} \mid A_i \mid (p^i-1)^3 + \sum_{i=\lceil\frac{\alpha}{2}\rceil}^{\alpha-1} \mid A_i \mid (p^i-2)^3\\
&= \sum_{i=1}^{\lfloor\frac{\alpha-1}{2}\rfloor} p^{\alpha-i-1}(p-1)(p^i-1)^3 + \sum_{i=\lceil\frac{\alpha}{2}\rceil}^{\alpha-1} p^{\alpha-i-1}(p-1)(p^i-2)^3\\
&= p^\alpha(p-1)\left[\sum_{i=1}^{\lfloor\frac{\alpha-1}{2}\rfloor}\frac{(p^i-1)^3}{p^{i+1}} + \sum_{i=\lceil\frac{\alpha}{2}\rceil}^{\alpha-1}\frac{(p^i-2)^3}{p^{i+1}}\right].
\end{aligned}
$$

□

Example 2.3 *For the graph $\Gamma(\mathbb{Z}_{243})$, we have $p^\alpha = 3^5$. Then, $\mathrm{FT}(\Gamma(\mathbb{Z}_{243})) = 1{,}089{,}476$, and the set of zero-divisors can be written as follows:*

$$
\begin{aligned}
A_1 = \{&3,6,12,15,21,24,30,33,39,42,48,51,57,60,66,69,75,78,84,87,93,96,102,\\
&105,111,114,120,123,129,132,138,141,147,150,156,159,165,168,174,177,\\
&183,186,192,195,201,204,210,213,219,222,228,231,237,240\},\\
A_2 = \{&9,18,36,45,63,72,90,99,117,126,144,153,171,180,198,207,225,234\},\\
A_3 = \{&27,54,108,135,189,216\},\\
A_4 = \{&81,162\}.
\end{aligned}
$$

Additionally, these sets create the graph which can be shown in Figure 1, where $A_1 = \{3x \mid x = 1,\dots,p^{\alpha-1}-1, p \nmid x\}$, $A_2 = \{9x \mid x = 1,\dots,p^{\alpha-2}-1, p \nmid x\}$, $A_3 = \{27x \mid x = 1,\dots,p^{\alpha-3}-1, p \nmid x\}$, and $A_4 = \{81x \mid x = 1,\dots,p^{\alpha-4}-1, p \nmid x\}$.

In the next theorem, we give the forgotten index of a zero-divisor graph $\Gamma(\mathbb{Z}_{pq})$ for distinct primes p and q.

Theorem 2.4 *Let p and q be prime numbers with $p \neq q$. Then the forgotten index of the graph $\Gamma(\mathbb{Z}_{pq})$ is*

$$\mathrm{FT}(\Gamma(\mathbb{Z}_{pq})) = (p-1)(q-1)\Big[(q-1)^2 + (p-1)^2\Big].$$

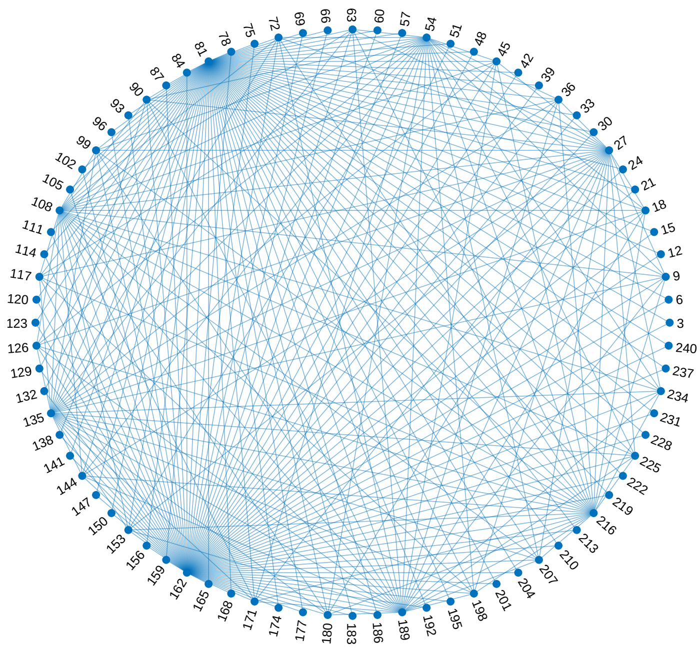

Figure 1. The graph $\Gamma(\mathbb{Z}_{243})$

Proof The graph $\Gamma(\mathbb{Z}_{pq})$ is a complete bipartite graph. The bipartitions of $\Gamma(\mathbb{Z}_{pq})$ are $A_1 = \{px \mid x = 1, 2, \ldots, q-1\}$ and $A_2 = \{qx \mid x = 1, 2, \ldots, p-1\}$. Since $|A_1| = \varphi(\frac{pq}{p}) = q-1$ and $|A_2| = \varphi(\frac{pq}{q}) = p-1$, then the size of this graph is $(p-1)(q-1)$. It follows that

$$\begin{aligned}
\mathrm{FT}(\Gamma(\mathbb{Z}_{pq})) &= \sum_{i=1}^{2} \sum_{u \in A_i} {d_u}^3 \\
&= |A_1|(q-1)^3 + |A_2|(p-1)^3 \\
&= (p-1)(q-1)^3 + (q-1)(p-1)^3 \\
&= (p-1)(q-1)\Big[(p-1)^2 + (q-1)^2\Big].
\end{aligned}$$

□

Example 2.5 *For the zero-divisor graph of* $\mathbb{Z}_{119}$*, we attain* $n = pq$ *where* $p = 7$ *and* $q = 17$*. Then,* $\mathrm{FT}(\Gamma(\mathbb{Z}_{119})) = 28,032$*, and the set of zero-divisors as follows:*

$$A_1 = \{7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, 84, 91, 98, 105, 112\},$$

$$A_2 = \{17, 34, 51, 68, 85, 102\}.$$

These sets give rise to the graph depicted in Figure 2, where $A_1 = \{7x \mid x = 1, 2, \ldots, q-1\}$ *and* $A_2 = \{17x \mid x = 1, 2, \ldots, p-1\}$*.*

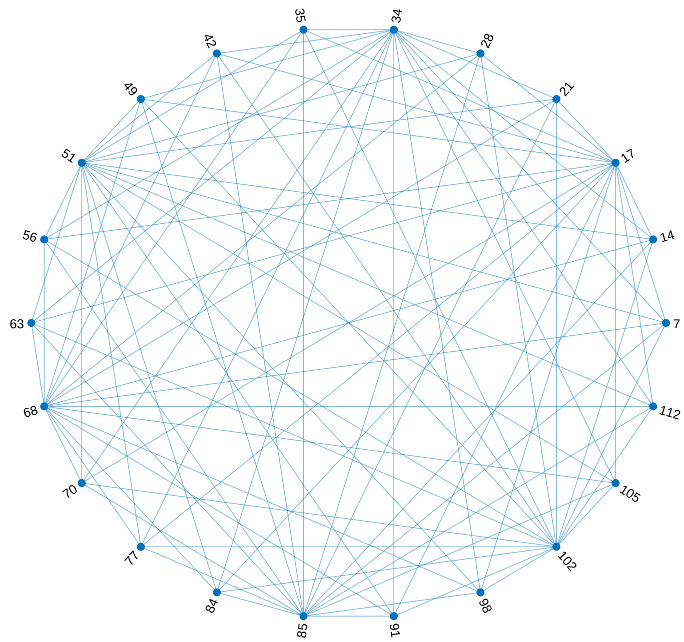

Figure 2. The graph $\Gamma(\mathbb{Z}_{119})$

Theorem 2.6 *Let $\Gamma(\mathbb{Z}_{p^2q})$ be a zero-divisor graph and p and q be distinct prime numbers. If $n = p^2q$, then the forgotten index of $\Gamma(\mathbb{Z}_{p^2q})$ is*

$$\mathrm{FT}(\Gamma(\mathbb{Z}_{p^2q})) = (p-1)(q-1)\Big[(p-1)^3 + p(q-1)^2 + (p-1)^2(p+1)^3 + \frac{(pq-2)^3}{q-1}\Big].$$

Proof Since the proper divisors of $n = p^2q$ are p, p^2, q and pq, then the vertex set can be partitioned as $V(\Gamma(\mathbb{Z}_n)) = A_1 \cup A_2 \cup A_3 \cup A_4$ and $A_i \cap A_j = \emptyset$ for $i \neq j$, where $i, j \in \{1, \ldots, 4\}$ and

$$A_1 = \{px \mid x = 1, 2, \ldots, pq-1, p \nmid x, q \nmid x, \},$$
$$A_2 = \{qx \mid x = 1, 2, \ldots, p^2-1, p \nmid x\},$$
$$A_3 = \{p^2x \mid x = 1, 2, \ldots, q-1\},$$
$$A_4 = \{pqx \mid x = 1, 2, \ldots, p-1\}.$$

We can calculate the number of vertices of all zero-divisor sets as $|A_1| = (p-1)(q-1)$, $|A_2| = p(p-1)$, $|A_3| = (q-1)$, and $|A_4| = (p-1)$. Also, the degree of each vertex in these zero-divisor sets can be determined as

$$d_u = \begin{cases} |A_4|, & u \in A_1 \\ |A_3|, & u \in A_2 \\ |A_2| + |A_4|, & u \in A_3 \\ |A_1| + |A_3| + |A_4| - 1, & u \in A_4 \end{cases}.$$

Then, the forgotten topological index of $\Gamma(\mathbb{Z}_{p^2q})$ can be attained that

$$\begin{aligned}
\mathrm{FT}(\Gamma(\mathbb{Z}_{p^2q})) &= \sum_{u\in V(\Gamma(\mathbb{Z}_{p^2q}))} {d_u}^3 \\
&= \sum_{i=1}^{4}\sum_{u\in A_i} {d_u}^3 \\
&= |A_1||A_4|^3 + |A_2||A_3|^3 + |A_3|(|A_2|+|A_4|)^3 + |A_4|(|A_1|+|A_3|+|A_4|-1)^3 \\
&= (p-1)(q-1)(p-1)^3 + p(p-1)(q-1)^3 + (q-1)(p.(p-1)+(p-1))^3 \\
&\quad + (p-1)\Big[(p-1)(q-1)+(q-1)+(p-1)-1\Big]^3 \\
&= (p-1)^4(q-1) + p(p-1)(q-1)^3 + (q-1)(p^2-1)^3 + (p-1)(pq-2)^3 \\
&= (p-1)\Big[(p-1)^3(q-1) + p(q-1)^3 + (p-1)^2(p+1)^3(q-1) + (pq-2)^3\Big] \\
&= (p-1)(q-1)\Big[(p-1)^3 + p(q-1)^2 + (p-1)^2(p+1)^3 + \frac{(pq-2)^3}{q-1}\Big].
\end{aligned}$$

□

Example 2.7 *For the graph $\Gamma(\mathbb{Z}_{175})$, we have $n=p^2q$ where $p=5$ and $q=7$. Then, $\mathrm{FT}(\Gamma(\mathbb{Z}_{175}))=232,548$, and the set of zero-divisors can be written as follows:*

$$\begin{aligned}
A_1 &= \{5,10,15,20,30,40,45,55,60,65,80,85,90,95,110,115,120,130,135,145,\\
&\qquad 155,160,165,170\},\\
A_2 &= \{7,14,21,28,42,49,56,63,77,84,91,98,112,119,126,133,147,154,161,168\},\\
A_3 &= \{25,50,75,100,125,150\},\\
A_4 &= \{35,70,105,140\}.
\end{aligned}$$

Moreover, these sets give rise to the graph which can be shown in Figure 3, where $A_1=\{5x \mid x=1,2,\ldots,pq-1, p\nmid x, q\nmid x,\}$, $A_2=\{7x \mid x=1,2,\ldots,p^2-1, p\nmid x\}$, $A_3=\{25x \mid x=1,2,\ldots,q-1\}$, $A_4=\{35x \mid x=1,2,\ldots,p-1\}$.

Theorem 2.8 *Let $\Gamma(\mathbb{Z}_{p^2q^2})$ be a zero-divisor graph and p and q be distinct prime numbers. If $n=p^2q^2$, then the forgotten index of $\Gamma(\mathbb{Z}_{p^2q^2})$ is*

$$\begin{aligned}
\mathrm{FT}(\Gamma(\mathbb{Z}_{p^2q^2})) = (p-1)(q-1)\Big[&(p-1)^3q + p(q-1)^3 + (p-1)^2(p+1)^3q \\
&+ p(q-1)^2(q+1)^3 + (pq-2)^3 + \frac{(p^2q-2)^3}{p-1} + \frac{(pq^2-2)^3}{q-1}\Big].
\end{aligned}$$

Proof Since the proper divisors of $n=p^2q^2$ are p,q,p^2,q^2,pq,p^2q and pq^2, then the vertex set can be

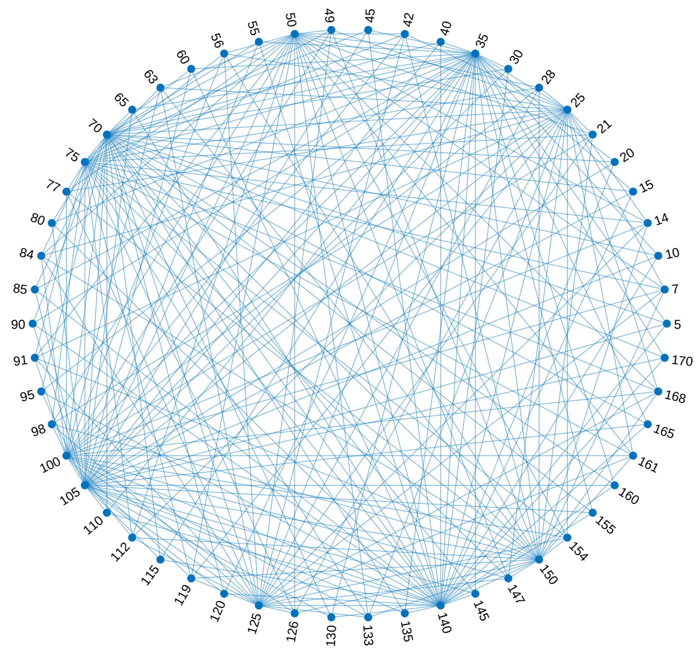

Figure 3. The graph $\Gamma(\mathbb{Z}_{175})$

partitioned into seven subsets as $V(\Gamma(\mathbb{Z}_n)) = \bigcup_{i=1}^{7} A_i$ and $A_i \cap A_j = \emptyset$ for $i \neq j$, where $i, j \in \{1, \ldots, 7\}$ and

$$A_1 = \{px \mid x = 1, 2, \ldots, pq^2 - 1, p \nmid x, q \nmid x\},$$

$$A_2 = \{qx \mid x = 1, 2, \ldots, p^2q - 1, p \nmid x, q \nmid x\},$$

$$A_3 = \{p^2x \mid x = 1, 2, \ldots, q^2 - 1, q \nmid x\},$$

$$A_4 = \{q^2x \mid x = 1, 2, \ldots, p^2 - 1, p \nmid x\},$$

$$A_5 = \{pqx \mid x = 1, 2, \ldots, pq - 1, p \nmid x, q \nmid x\},$$

$$A_6 = \{p^2qx \mid x = 1, 2, \ldots, q - 1\},$$

$$A_7 = \{pq^2x \mid x = 1, 2, \ldots, p - 1\}.$$

We can calculate the number of vertices of all zero-divisor sets as $|A_1| = (p-1)q(q-1)$, $|A_2| = p(p-1)(q-1)$, $|A_3| = q(q-1)$, $|A_4| = p(p-1)$, $|A_5| = (p-1)(q-1)$, $|A_6| = (q-1)$ and $|A_7| = (p-1)$. Also, the degree of each vertex in these zero-divisor sets can be determined as

$$d_u = \begin{cases} |A_7|, & u \in A_1 \\ |A_6|, & u \in A_2 \\ |A_4| + |A_7|, & u \in A_3 \\ |A_3| + |A_6|, & u \in A_4 \\ |A_5| + |A_6| + |A_7| - 1, & u \in A_5 \\ |A_2| + |A_4| + |A_5| + |A_6| + |A_7| - 1, & u \in A_6 \\ |A_1| + |A_3| + |A_5| + |A_6| + |A_7| - 1, & u \in A_7 \end{cases}.$$

Then, the forgotten index of $\Gamma(\mathbb{Z}_{p^2q^2})$ can be attained that

$$\begin{aligned}
\mathrm{FT}(\Gamma(\mathbb{Z}_{p^2q^2})) &= \sum_{u\in V(\Gamma(\mathbb{Z}_{p^2q^2}))} {d_u}^3 \\
&= \sum_{i=1}^{7}\sum_{u\in A_i} {d_u}^3 \\
&= |A_1||A_7|^3 + |A_2||A_6|^3 + |A_3|(|A_4| + |A_7|)^3 + |A_4|(|A_3| + |A_6|)^3 \\
&+ |A_5|(|A_5| + |A_6| + |A_7| - 1)^3 \\
&+ |A_6|(|A_2| + |A_4| + |A_5| + |A_6| + |A_7| - 1)^3 \\
&+ |A_7|(|A_1| + |A_3| + |A_5| + |A_6| + |A_7| - 1)^3 \\
&= (p-1)q(q-1)(p-1)^3 + p(p-1)(q-1)(q-1)^3 \\
&+ q(q-1)(p-1)^3(p+1)^3 + p(p-1)(q-1)^3(q+1)^3 \\
&+ (p-1)(q-1)\Big[(p-1)(q-1) + (q-1) + (p-1) - 1\Big]^3 \\
&+ (q-1)\Big[p(p-1)(q-1) + p(p-1) + (p-1)(q-1) + (q-1) + (p-1) - 1\Big]^3 \\
&+ (p-1)\Big[(p-1)q(q-1) + q(q-1) + (p-1)(q-1) + (q-1) + (p-1) - 1\Big]^3 \\
&= (p-1)^4q(q-1) + p(p-1)(q-1)^4 + (p-1)^3(p+1)^3q(q-1) \\
&+ p(p-1)(q-1)^3(q+1)^3 + (p-1)(q-1)(pq-2)^3 \\
&+ (q-1)(p^2q-2)^3 + (p-1)(pq^2-2)^3 \\
&= (p-1)(q-1)\Big[(p-1)^3q + p(q-1)^3 + (p-1)^2(p+1)^3q \\
&+ p(q-1)^2(q+1)^3 + (pq-2)^3 + \frac{(p^2q-2)^3}{p-1} + \frac{(pq^2-2)^3}{q-1}\Big].
\end{aligned}$$

□

Example 2.9 *For the graph* $\Gamma(\mathbb{Z}_{225})$*, we have* $n = p^2q^2$ *where* $p = 3$ *and* $q = 5$*. Then,* $\mathrm{FT}(\Gamma(\mathbb{Z}_{225})) =$

$1,208,678$, *and the set of zero-divisors can be written as follows.*

$$A_1 = \{3,6,12,21,24,33,39,42,48,51,57,66,69,78,84,87,93,96,102,111,114,$$
$$123,129,132,138,141,147,156,159,168,174,177,183,186,192,201,204,$$
$$213,219,222\},$$
$$A_2 = \{5,10,20,35,40,55,65,70,80,85,95,110,115,130,140,145,155,160,170,$$
$$185,190,205,215,220\},$$
$$A_3 = \{9,18,27,36,54,63,72,81,99,108,117,126,144,153,162,171,189,198,$$
$$207,216\},$$
$$A_4 = \{25,50,100,125,175,200\},$$
$$A_5 = \{15,30,60,105,120,165,195,210\},$$
$$A_6 = \{45,90,135,180\},$$
$$A_7 = \{75,150\}.$$

These sets build the graph $\Gamma(\mathbb{Z}_{225})$, *where* $A_1 = \{3x \mid x = 1,2,\ldots,pq^2-1, p \nmid x, q \nmid x\}$, $A_2 = \{5x \mid x = 1,2,\ldots,p^2q-1, p \nmid x, q \nmid x\}$, $A_3 = \{9x \mid x = 1,2,\ldots,q^2-1, q \nmid x\}$, $A_4 = \{25x \mid x = 1,2,\ldots,p^2-1, p \nmid x\}$, $A_5 = \{15x \mid x = 1,2,\ldots,pq-1, p \nmid x, q \nmid x\}$, $A_6 = \{45x \mid x = 1,2,\ldots,q-1\}$, *and* $A_7 = \{75x \mid x = 1,2,\ldots,p-1\}$.

In the next theorem, the relation of the forgotten topological index of $\Gamma(\mathbb{Z}_{pqr})$ is represented.

Theorem 2.10 *Let* $\Gamma(\mathbb{Z}_{pqr})$ *be a zero-divisor graph and* p,q *and* r *be distinct prime numbers. Then, the forgotten topological index of* $\Gamma(\mathbb{Z}_{pqr})$ *is*

$$\mathrm{FT}(\Gamma(\mathbb{Z}_{pqr})) = (p-1)(q-1)(r-1)\Bigg[(p-1)^2 + (q-1)^2 + (r-1)^2 + \frac{(pq-1)^3}{(p-1)(q-1)} + \frac{(pr-1)^3}{(p-1)(r-1)} + \frac{(qr-1)^3}{(q-1)(r-1)}\Bigg].$$

Proof Since the proper divisors of $n = pqr$ are p,q,r,pq,pr and qr, then the vertex set can be partitioned as $V(\Gamma(\mathbb{Z}_n)) = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$ and $A_i \cap A_j = \emptyset$ for $i \neq j$, where $i,j \in \{1,\ldots,6\}$ and

$$A_1 = \{px \mid x = 1,2,\ldots,qr-1, q \nmid x, r \nmid x\},$$
$$A_2 = \{qx \mid x = 1,2,\ldots,pr-1, p \nmid x, r \nmid x\},$$
$$A_3 = \{rx \mid x = 1,2,\ldots,pq-1, p \nmid x, q \nmid x\},$$
$$A_4 = \{pqx \mid x = 1,2,\ldots,r-1\},$$
$$A_5 = \{prx \mid x = 1,2,\ldots,q-1\},$$
$$A_6 = \{qrx \mid x = 1,2,\ldots,p-1\}.$$

The number of vertices of all zero-divisor sets can be calculated as $|A_1| = (q-1)(r-1)$, $|A_2| = (p-1)(r-1)$, $|A_3| = (p-1)(q-1)$, $|A_4| = (r-1)$, $|A_5| = (q-1)$, and $|A_6| = (p-1)$. Besides, the degree of each vertex in these zero-divisor sets can be determined as

$$d_u = \begin{cases} |A_6|, & u \in A_1 \\ |A_5|, & u \in A_2 \\ |A_4|, & u \in A_3 \\ |A_3| + |A_5| + |A_6|, & u \in A_4 \\ |A_2| + |A_4| + |A_6|, & u \in A_5 \\ |A_1| + |A_4| + |A_5|, & u \in A_6 \end{cases}.$$

Hence, we can obtain the forgotten index of $\Gamma(\mathbb{Z}_{pqr})$ as

$$\begin{aligned}
\mathrm{FT}(\Gamma(\mathbb{Z}_{pqr})) &= \sum_{u \in V(\Gamma(\mathbb{Z}_{p^2q}))} {d_u}^3 \\
&= \sum_{i=1}^{6} \sum_{u \in A_i} {d_u}^3 \\
&= |A_1||A_6|^3 + |A_2||A_5|^3 + |A_3||A_4|^3 + |A_4|(|A_3| + |A_5| + |A_6|)^3 \\
&+ |A_5|(|A_2| + |A_4| + |A_6|)^3 + |A_6|(|A_1| + |A_4| + |A_5|)^3 \\
&= (q-1)(r-1)(p-1)^3 + (p-1)(r-1)(q-1)^3 + (p-1)(q-1)(r-1)^3 \\
&+ (r-1)\big((p-1)(q-1) + (q-1) + (p-1)\big)^3 \\
&+ (q-1)\big((p-1)(r-1) + (p-1) + (q-1)\big)^3 \\
&+ (p-1)\big((q-1)(r-1) + (q-1) + (r-1)\big)^3 \\
&= (p-1)^3(q-1)(r-1) + (p-1)(q-1)^3(r-1) + (p-1)(q-1)(r-1)^3 \\
&+ (pq-1)^3(r-1) + (pr-1)^3(q-1) + (qr-1)(p-1) \\
&= (p-1)(q-1)(r-1)\bigg[(p-1)^2 + (q-1)^2 + (r-1)^2 \\
&+ \frac{(pq-1)^3}{(p-1)(q-1)} + \frac{(pr-1)^3}{(p-1)(r-1)} + \frac{(qr-1)^3}{(q-1)(r-1)}\bigg].
\end{aligned}$$

□

Example 2.11 *For the graph* $\Gamma(\mathbb{Z}_{165})$, *we have* $n = pqr$ *where* $p = 3$, $q = 5$ *and* $r = 11$. *Then,*

$\mathrm{FT}(\Gamma(\mathbb{Z}_{165})) = 483,040$, *and the set of zero-divisors can be written as follows:*

$$A_1 = \{3,6,9,12,18,21,24,27,36,39,42,48,51,54,57,63,69,72,78,81,84,87,93,$$
$$96,102,108,111,114,117,123,126,129,138,141,144,147,153,156,159,162\},$$
$$A_2 = \{5,10,20,25,35,40,50,65,70,80,85,95,100,115,125,130,140,145,155,160\},$$
$$A_3 = \{11,22,44,77,88,121,143,154\},$$
$$A_4 = \{15,30,45,60,75,90,105,120,135,150\},$$
$$A_5 = \{33,66,99,132\},$$
$$A_6 = \{55,110\}.$$

Additionally, these sets give rise to the graph which can be shown in Figure 4, where $A_1 = \{3x \mid x = 1,2,\ldots,qr-1, q \nmid x, r \nmid x\}$, $A_2 = \{5x \mid x = 1,2,\ldots,pr-1, p \nmid x, r \nmid x\}$, $A_3 = \{11x \mid x = 1,2,\ldots,pq-1, p \nmid x, q \nmid x\}$, $A_4 = \{15x \mid x = 1,2,\ldots,r-1\}$, $A_5 = \{33x \mid x = 1,2,\ldots,q-1\}$, $A_6 = \{55x \mid x = 1,2,\ldots,p-1\}$.

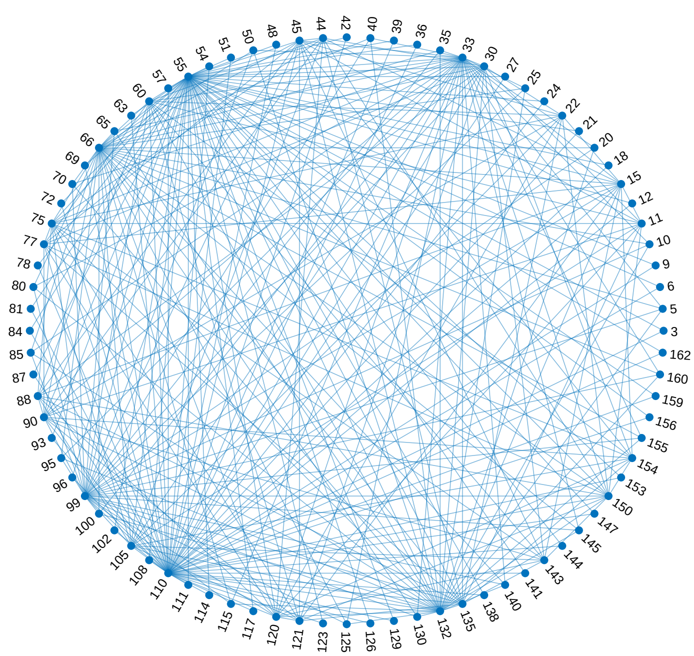

Figure 4. The graph $\Gamma(\mathbb{Z}_{165})$

2.1. Matlab code for determining the forgotten topological index of $\Gamma(\mathbb{Z}_n)$

In this subsection, we give an algorithm for calculating the forgotten topological index of $\Gamma(\mathbb{Z}_n)$ when entering an integer n.

```
n=input($"enter n for Z_n:"$)
Vert=strings(1,n-2);
Adj=zeros(n-2);
Deg=zeros(1,n-2);
for i=2:n-1
```

```
6 Vert(i-1)=int2str(i);
7 for j=2:n-1
8 if (i==j), continue, end
9 if mod(i*j,n)==0
10 Adj(i-1,j-1)=1;
11 Deg(i-1)=Deg(i-1)+1;
12 end
13 end
14 end
15 for i=size(Deg,2):-1:1
16 if (Deg(i)==0)
17 Adj(i,:)=[];
18 Adj(:,i)=[];
19 Vert(i)=[];
20 Deg(i)=[];
21 end
22 end
23 fi=0;
24 for i=1:size(Deg,2)
25 $fi = fi + Deg(i)^3;$
26 end
27 fprintf("Forgotten Index: %d\n",fi);
```

In the lines 1-4 of the algorithm, an integer n for $\mathbb{Z}_n$ is requested, and the vertex set (*Vert*), the adjacency matrix (*Adj*) and the degree array (*Deg*) are initialized. Next, in line 7, all possible vertices in the graph are inserted into the set. Then, as long as the condition $i \cdot j \equiv 0 \ (mod\ n)$, the adjacency matrix is filled and the degree array is updated in lines 11-12. After that, vertices having no neighbors are removed from vertex set, degree array and adjacency matrix in lines 17-24. To compute the forgotten index of a zero divisor graph, the lines 17-24 are not obligatory. However, these lines are used to represent the zero divisor graph properly. Finally, in lines 26-30, the forgotten topological index of graph $\Gamma(\mathbb{Z}_n)$ is calculated and printed out.

In this algorithm, lines 17-24 are not obligatory. However, these lines are used to construct the graph $\Gamma(\mathbb{Z}_n)$ properly.

3. Forgotten topological index of zero-divisor graph of products of rings of integers modulo n

In this section we calculate the forgotten index of the graphs $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ and $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$ for distinct prime numbers p, q and r.

The zero-divisor graph of $\mathbb{Z}_p \times \mathbb{Z}_q$ and some graph theoretical properties of it have been studied in [2]. In the following theorem, we give the forgotten index of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$.

Theorem 3.1 *Let $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ be a zero-divisor graph and p, q be distinct prime numbers. Then the forgotten index of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ is*

$$\mathrm{FT}(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)) = (p-1)(q-1)\big((p-1)^2 + (q-1)^2\big).$$

Proof Let $x \in {\mathbb{Z}_p}^*$ and $y \in {\mathbb{Z}_q}^*$, where $x = 1, 2, ..., p-1$ and $y = 1, 2, ..., q-1$. Since $(x,0)(0,y) = (0,0)$ the edge set of $x \in {\mathbb{Z}_p}^*$ contains only the edges between the vertices $(x,0)$ and $(0,y)$.
The graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ is a complete bipartite graph $K_{p-1,q-1}$. Partitions of vertex set of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ are

$$A_1 = \{(x,0) \mid 1 \le x < p, x \in \mathbb{Z}_p\},$$

$$A_2 = \{(0,y) \mid 1 \le y < q, y \in \mathbb{Z}_q\}$$

such that $A_1 \cup A_2 = V(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ and $A_1 \cap A_2 = \emptyset$. Since $|A_1| = p-1$ and $|A_2| = q-1$, the size of this graph is $(p-1)(q-1)$. Also, $d_u = |A_2|$ for all $u \in A_1$ and $d_v = |A_1|$ for all $v \in A_2$. Hence, we obtain

$$\begin{aligned}
\mathrm{FT}(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)) &= \sum_{u \in V(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q))} {d_u}^2 \\
&= \sum_{i=1}^{2} \sum_{u \in A_i} {d_u}^3 \\
&= \mid A_1 \mid (q-1)^3 + \mid A_2 \mid (p-1)^3 \\
&= (p-1)(q-1)^3 + (p-1)^3(q-1) \\
&= (p-1)(q-1)((p-1)^2 + (q-1)^2).
\end{aligned}$$

□

Example 3.2 *For zero-divisor graph of $\mathbb{Z}_{11} \times \mathbb{Z}_{17}$, we attain $p = 11$ and $q = 17$. Then,* $\mathrm{FT}(\Gamma(\mathbb{Z}_{11} \times \mathbb{Z}_{17})) = 56,960$*, and the set of zero-divisors as follows:*

$$\begin{aligned}
A_1 &= \{(1,0),(2,0),(3,0),(4,0),(5,0),(6,0),(7,0),(8,0),(9,0),(10,0)\}, \\
A_2 &= \{(0,1),(0,2),(0,3),(0,4),(0,5),(0,6),(0,7),(0,8),(0,9),(0,10),(0,11), \\
&\quad (0,12),(0,13),(0,14),(0,15),(0,16)\}.
\end{aligned}$$

These sets give rise to the graph depicted in Figure 5, where $A_1 = \{(x,0) \mid 1 \le x < p, x \in \mathbb{Z}_{11}\}$ and $A_2 = \{(0,y) \mid 1 \le y < q, y \in \mathbb{Z}_{17}\}$.

Akgunes and Nacaroglu have studied some properties of zero-divisor graph of $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$ [3]. Moreover, they have calculated irregularity index and Zagreb indices of this graph. We obtain the forgotten index of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$ in the following theorem.

Theorem 3.3 *Let $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$ be a zero-divisor graph and p, q, r be distinct prime numbers. Then, the forgotten index of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$ is*

$$\begin{aligned}
\mathrm{FT}(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)) = (p-1)(q-1)(r-1)\Big[&(p-1)^2 + (q-1)^2 + (r-1)^2 \\
&+ \frac{(pq-1)^3}{(p-1)(q-1)} + \frac{(pr-1)^3}{(p-1)(r-1)} + \frac{(qr-1)^3}{(q-1)(r-1)}\Big].
\end{aligned}$$

Proof We divide the vertex set of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$ into six subsets such that $V(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)) = \cup_{i=1}^{6} A_i$

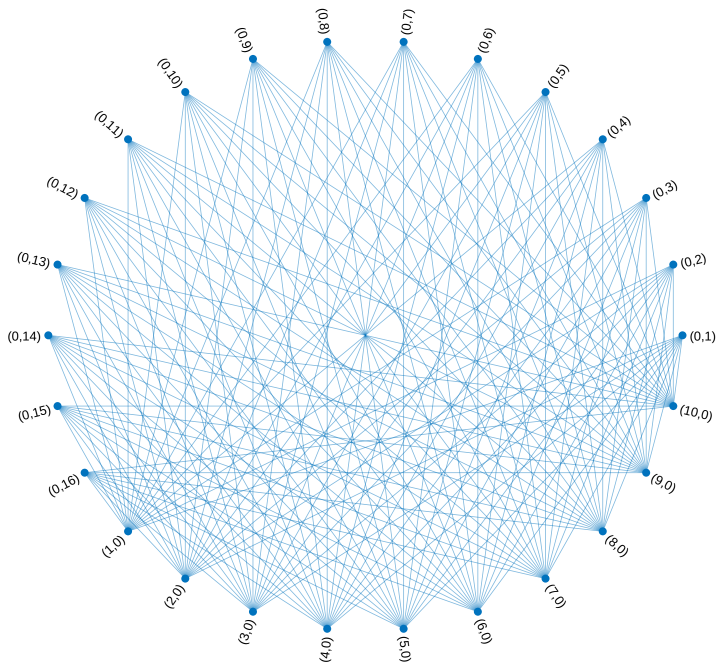

Figure 5. The graph $\Gamma(\mathbb{Z}_{11} \times \mathbb{Z}_{17})$

and $A_i \cap A_j = \emptyset$, where $i = 1, 2, ..., 5$ and $j = i + 1, ..., 6$. We show that these vertex subsets as follows:

$$A_1 = \{(x, 0, 0) \mid 1 \le x < p,\ x \in \mathbb{Z}_p\},$$
$$A_2 = \{(0, y, 0) \mid 1 \le y < q,\ y \in \mathbb{Z}_q\},$$
$$A_3 = \{(0, 0, z) \mid 1 \le z < r,\ z \in \mathbb{Z}_r\},$$
$$A_4 = \{(0, y, z) \mid 1 \le y < q,\ 1 \le z < r,\ y \in \mathbb{Z}_q,\ z \in \mathbb{Z}_r\},$$
$$A_5 = \{(x, 0, z) \mid 1 \le x < p,\ 1 \le z < r,\ x \in \mathbb{Z}_p,\ z \in \mathbb{Z}_r\},$$
$$A_6 = \{(x, y, 0) \mid 1 \le x < p,\ 1 \le y < q,\ x \in \mathbb{Z}_p,\ y \in \mathbb{Z}_q\}.$$

The number of vertices of all zero-divisor sets can be calculated as $|A_1| = (p-1)$, $|A_2| = (q-1)$, $|A_3| = (r-1)$, $|A_4| = (q-1)(r-1)$, $|A_5| = (p-1)(r-1)$, and $|A_6| = (p-1)(q-1)$. Moreover, the degree of each vertex in these zero-divisor sets can be determined as

$$d_u = \begin{cases} |A_2| + |A_3| + |A_4|, & u \in A_1 \\ |A_1| + |A_3| + |A_5|, & u \in A_2 \\ |A_1| + |A_2| + |A_6|, & u \in A_3 \\ |A_1|, & u \in A_4 \\ |A_2|, & u \in A_5 \\ |A_3|, & u \in A_6 \end{cases}.$$

According to these subsets, we calculate the forgotten index of the graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$ as follows:

$$
\begin{aligned}
\mathrm{FT}(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)) &= \sum_{u \in V(\Gamma(\mathbb{Z}_{p^2 q}))} {d_u}^3 \\
&= \sum_{i=1}^{6} \sum_{u \in A_i} {d_u}^3 \\
&= |A_1|\big(|A_2| + |A_3| + |A_4|\big)^3 + |A_2|\big(|A_1| + |A_3| + |A_5|\big)^3 \\
&+ |A_3|\big(|A_1| + |A_2| + |A_6|\big)^3 + |A_4||A_1|^3 + |A_5||A_2|^3 + |A_6||A_3|^3 \\
&= (p-1)\big((q-1)(r-1) + (q-1) + (r-1)\big)^3 \\
&+ (q-1)\big((p-1)(r-1) + (p-1) + (r-1)\big)^3 \\
&+ (r-1)\big((p-1)(q-1) + (p-1) + (q-1)\big)^3 \\
&+ (q-1)(r-1)(p-1)^3 + (p-1)(r-1)(q-1)^3 \\
&+ (p-1)(q-1)(r-1)^3 \\
&= (p-1)(q-1)(r-1)\Bigg[(p-1)^2 + (q-1)^2 + (r-1)^2 \\
&+ \frac{(pq-1)^3}{(p-1)(q-1)} + \frac{(qr-1)^3}{(q-1)(r-1)} + \frac{(pr-1)^3}{(p-1)(r-1)}\Bigg].
\end{aligned}
$$

□

Example 3.4 *For zero-divisor graph of $\mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_{13}$, we have $p = 3$, $q = 5$ and $r = 13$. Then, $\mathrm{FT}(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_{13})) = 792,448$, and the following are the set of zero-divisors of this ring. The zero-divisor graph of this ring can be seen in Figure 6:*

$$
\begin{aligned}
A_1 &= \{(1,0,0), (2,0,0)\}, \\
A_2 &= \{(0,1,0), (0,2,0), (0,3,0), (0,4,0)\}, \\
A_3 &= \{(0,0,1), (0,0,2), (0,0,3), (0,0,4), (0,0,5), (0,0,6), (0,0,7), (0,0,8), (0,0,9), \\
&\quad (0,0,10), (0,0,11), (0,0,12)\}, \\
A_4 &= \{(0,1,1), (0,1,2), (0,1,3), (0,1,4), (0,1,5), (0,1,6), (0,1,7), (0,1,8), (0,1,9), \\
&\quad (0,1,10), (0,1,11), (0,1,12), (0,2,1), (0,2,2), (0,2,3), (0,2,4), (0,2,5), (0,2,6), \\
&\quad (0,2,7), (0,2,8), (0,2,9), (0,2,10), (0,2,11), (0,2,12), (0,3,1), (0,3,2), (0,3,3), \\
&\quad (0,3,4), (0,3,5), (0,3,6), (0,3,7), (0,3,8), (0,3,9), (0,3,10), (0,3,11), (0,3,12), \\
&\quad (0,4,1), (0,4,2), (0,4,3), (0,4,4), (0,4,5), (0,4,6), (0,4,7), (0,4,8), (0,4,9), \\
&\quad (0,4,10), (0,4,11), (0,4,12)\}, \\
A_5 &= \{(1,0,1), (1,0,2), (1,0,3), (1,0,4), (1,0,5), (1,0,6), (1,0,7), (1,0,8), (1,0,9), \\
&\quad (1,0,10), (1,0,11), (1,0,12), (2,0,1), (2,0,2), (2,0,3), (2,0,4), (2,0,5), (2,0,6), \\
&\quad (2,0,7), (2,0,8), (2,0,9), (2,0,10), (2,0,11), (2,0,12)\}, \\
A_6 &= \{(1,1,0), (1,2,0), (1,3,0), (1,4,0), (2,1,0), (2,2,0), (2,3,0), (2,4,0)\}.
\end{aligned}
$$

By the above arguments, we give the following corollary.

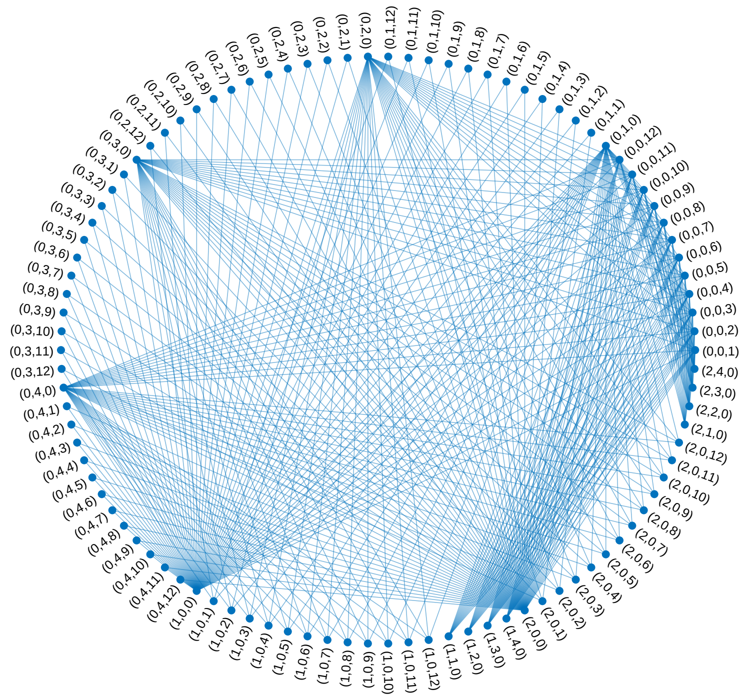

Figure 6. The graph $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_{13})$

Corollary 3.5 *Let* $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$, $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$, $\Gamma(\mathbb{Z}_{pq})$, *and* $\Gamma(\mathbb{Z}_{pqr})$ *be zero-divisor graphs where* p, q, *and* r *are distinct prime numbers. Then the followings hold:*

(i) $\mathrm{FT}(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)) = \mathrm{FT}(\Gamma(\mathbb{Z}_{pq}))$,

(ii) $\mathrm{FT}(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)) = \mathrm{FT}(\Gamma(\mathbb{Z}_{pqr}))$.

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