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A variant of Rosset's approach to the Amitsur-Levitzki theorem and some $\mathbb{Z}_2$-graded identities of $\mathrm{M}_n(E)$

Dedicated to the 70th anniversary of Vesselin Drensky

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Abstract: In the spirit of Rosset's proof of the Amitsur-Levitzki theorem, we show how the standard identiy (for matrices over a commutative base ring) and the addition of external Grassmann variables can be used to derive a certain $\mathbb{Z}_2$-graded polynomial identity of $\mathrm{M}_n(E)$.

Key words: The full matrix algebra over the infinite dimensional Grassmann algebra, the Amitsur-Levitzki theorem on $n \times n$ matrices

1. Introduction

An algebra R means a not necessarily commutative unitary algebra over a commutative ring C (or over a field K), and the notation for the full $n \times n$ matrix algebra over R is $\mathrm{M}_n(R)$.

In case of $\mathrm{char}(K) = 0$, Kemer's pioneering work (see [9], [10]) on the T-ideals of associative algebras (leading to the solution of the Specht problem) revealed the importance of the identities satisfied by $\mathrm{M}_n(E)$ and $\mathrm{M}_{n,d}(E)$, where

$$E = K\langle v_1, v_2, ..., v_i, ... \mid v_i v_j + v_j v_i = 0 \text{ for all } 1 \le i \le j\rangle = E_0 \oplus E_1 \tag{1.1}$$

is the naturally $\mathbb{Z}_2$-graded Grassmann (exterior) algebra generated by the infinite sequence of anticommutative indeterminates $(v_i)_{i \ge 1}$. The K-subspace E_0 generated by 1 and the monomials in the v_i's of even length and E_1 is the K-subspace generated by the monomials in the v_i's of odd length. We note that E_0 is a commutative subalgebra of E and E is Lie nilpotent of index 2.

Let $K\langle x_1, x_2, \ldots, x_i, \ldots\rangle$ denote the free associative K-algebra generated by the infinite sequence $x_1, x_2, \ldots, x_i, \ldots$ of noncommuting indeterminates. The prime T-ideals of this K-algebra are exactly the T-ideals of the identities satisfied by $\mathrm{M}_n(K)$ for $n \ge 1$ (see [2]). The T-prime (or verbally prime) T-ideals are the prime T-ideals plus the T-ideals of the identities of $\mathrm{M}_n(E)$ for $n \ge 1$ and of $\mathrm{M}_{n,d}(E)$ for $1 \le d \le n-1$, where $\mathrm{M}_{n,d}(E)$ is the K-subalgebra of $\mathrm{M}_n(E)$ consisting of the so-called (n,d)-supermatrices with two diagonal E_0 blocks of sizes $d \times d$ and $(n-d) \times (n-d)$ and with two E_1 blocks of sizes $d \times (n-d)$ and $(n-d) \times d$. Another remarkable result is that any T-ideal contains the T-ideal of the identities satisfied by $\mathrm{M}_n(E)$ for sufficiently large n (see p. 20 in [10]).

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The above mentioned three classes of T-prime (verbally prime) PI-algebras serve as basic building blocks in Kemer's theory, where $\mathbb{Z}_2$-graded identities also play an important role. Since the appearance of [9] and [10] considerable efforts have been concentrated on the study of the various algebraic properties of $\mathrm{M}_n(E)$ and $\mathrm{M}_{n,d}(E)$, see [1, 4–8, 11, 14–16].

The aim of the present note is to present a certain $\mathbb{Z}_2$-graded polynomial identity of the $\mathbb{Z}_2$-graded full matrix algebra $\mathrm{M}_n(E) = \mathrm{M}_n(E_0) \oplus \mathrm{M}_n(E_1)$. There is a possibility to derive the mentioned identity by using the Amitsur-Levitzki standard identity (see [3])

$$S_{2n}(x_1, x_2, \ldots, x_{2n}) = \sum_{\pi \in \mathrm{Sym}\{1,2,\ldots,2n\}} \mathrm{sgn}(\pi) x_{\pi(1)} \cdots x_{\pi(2n)} = 0 \tag{1.2}$$

of degree $2n$ (for $n \times n$ matrices over a commutative base ring). In this case, the addition of external Grassmann variables to E is essential. The ingenious idea of using (additional) Grassmann variables in an enviroment without Grassmann algebras first appeared in Rosset's short proof of the Amitsur-Levitzki theorem (see [12]). The use of a single additional Grassmann variable (out of E) in the study of $\mathrm{M}_n(E)$ appears in a certain companion matrix construction (see [13]) providing a Cayley-Hamilton identity for a matrix $A \in \mathrm{M}_n(E)$ of degree n^2 (an entirely different treatment in [14] provided a similar CH identity of the same degree). Our present work can be considered as a variation on Rosset's original theme. One of the referees caused a surprise by providing a different approach to derive the same $\mathbb{Z}_2$-graded polynomial identity of $\mathrm{M}_n(E)$ based on the use of the $*$-transform of a $\mathbb{Z}_2$-graded polynomial and the so-called Grassmann envelope. The authors decided to keep their original proof and to present the mentioned short proof of the referee at the end of the paper.

2. A $\mathbb{Z}_2$-graded identity of $\mathrm{M}_n(E)$

The Grassmann algebra

$$E = K\langle v_1, v_2, ..., v_i, ... \mid v_i v_j + v_j v_i = 0 \text{ for all } 1 \le i \le j\rangle = K\langle V\rangle \tag{2.1}$$

generated by (the countably) infinite set $V = \{v_1, v_2, \ldots, v_t, \ldots\}$ of anticommuting indeterminates can naturally be extended as

$$F = K\langle V \cup W\rangle = K\langle v_1, v_2, \ldots, v_t, \ldots, w_1, w_2, \ldots, w_t, \ldots\rangle \tag{2.2}$$

by using a bigger set $V \cup W$ of anticommuting generators, where

$$W = \{w_1, w_2, \ldots, w_t, \ldots\} \text{ and } V \cap W = \varnothing. \tag{2.3}$$

Now we have $v_i v_j + v_j v_i = 0$, $w_i w_j + w_j w_i = 0$ for all $1 \le i \le j$ and $v_i w_j + w_j v_i = 0$ for all $1 \le i, j$. The Grassmann algebra

$$G = K\langle w_1, w_2, ..., w_i, ... \mid w_i w_j + w_j w_i = 0 \text{ for all } 1 \le i \le j\rangle = K\langle W\rangle \tag{2.4}$$

generated by W is also a sub K-algebra of F. Since the cardinalities of V, W and $V \cup W$ are all equal to $\aleph_0$, the K-algebras E, G and F are isomorphic.

A $\mathbb{Z}_2$-graded K-algebra R is a pair (R_0, R_1), where R_0 and R_1 are K-subspaces of R such that $R = R_0 \oplus R_1$ is a direct sum and $R_i R_j \subseteq R_{i+j}$ for all $i, j \in \{0, 1\}$, where $i+j$ is taken modulo 2. A $\mathbb{Z}_2$-graded identity of $R = R_0 \oplus R_1$ is of the form

$$h(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_k) = 0, \tag{2.5}$$

where $h(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_k)$ is in the free polynomial K-algebra generated by the noncommuting indeterminates $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_k$. We only require that

$$h(r_1, r_2, \ldots, r_m, r'_1, r'_2, \ldots, r'_k) = 0 \tag{2.6}$$

for all substitutions such that $r_1, r_2, \ldots, r_m \in R_0$ and $r'_1, r'_2, \ldots, r'_k \in R_1$.

Thus, $\{x_1, x_2, \ldots, x_m\}$ and $\{y_1, y_2, \ldots, y_k\}$ are called the sets of even and odd variables (indeterminates) in h, respectively.

For a vector $\overrightarrow{i} = (i_1, i_2, \ldots, i_k)$ with strictly increasing integer coordinates $1 \le i_1 < i_2 < \cdots < i_k \le 2n$ take

$$\Pi(\overrightarrow{i}) = \{\pi \in \mathrm{Sym}\{1, 2, \ldots, 2n\} \mid \pi(i_1), \pi(i_2), \ldots, \pi(i_k) \in \{1, 2, \ldots, k\}\}$$

and consider the complementary vector $\underset{\rightarrow}{i} = (j_1, j_2, \ldots, j_{2n-k})$ with $\{j_1, j_2, \ldots, j_{2n-k}\} = \{1, 2, \ldots, 2n\} \smallsetminus \{i_1, i_2, \ldots, i_k\}$ and $1 \le j_1 < j_2 < \cdots < j_{2n-k} \le 2n$. Now

$$\tau(\overrightarrow{i}) = \begin{pmatrix} 1 & 2 & \cdots & k & k+1 & k+2 & \cdots & 2n \\ i_1 & i_2 & & i_k & j_1 & j_2 & & j_{2n-k} \end{pmatrix} \tag{2.7}$$

defines a permutation in $\mathrm{Sym}\{1, 2, \ldots, 2n\}$. We need two more permutations

$$\pi(\overrightarrow{i}) \in \mathrm{Sym}\{1, 2, \ldots, k\} \text{ and } \pi(\underset{\rightarrow}{i}) \in \mathrm{Sym}\{k+1, k+2, \ldots, 2n\} \tag{2.8}$$

which are determined by $\pi \in \Pi(\overrightarrow{i})$ as follows:

$$\pi(\overrightarrow{i}) = \begin{pmatrix} 1 & 2 & \cdots & k \\ \pi(i_1) & \pi(i_2) & & \pi(i_k) \end{pmatrix} \tag{2.9}$$

and

$$\pi(\underset{\rightarrow}{i}) = \begin{pmatrix} k+1 & k+2 & \cdots & 2n \\ \pi(j_1) & \pi(j_2) & & \pi(j_{2n-k}) \end{pmatrix}. \tag{2.10}$$

For an integer $1 \le k \le 2n$ define a $\mathbb{Z}_2$-graded polynomial of degree $2n$ as follows:

$$\begin{aligned} f_k(X, Y) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le 2n} \mathrm{sgn}(\tau(\overrightarrow{i})) \Bigg(\sum_{\pi \in \Pi(\overrightarrow{i})} \mathrm{sgn}(\pi(\underset{\rightarrow}{i})) x_{\pi(1)} \cdots x_{\pi(i_1-1)} y_{\pi(i_1)} x_{\pi(i_1+1)} \cdots \\ \cdots x_{\pi(i_2-1)} y_{\pi(i_2)} x_{\pi(i_2+1)} \cdots x_{\pi(i_k-1)} y_{\pi(i_k)} x_{\pi(i_k+1)} \cdots x_{\pi(2n)} \Bigg), \end{aligned} \tag{2.11}$$

where

$$X = \{x_{k+1}, x_{k+2}, \ldots, x_{2n}\} \text{ and } Y = \{y_1, y_2, \ldots, y_k\}$$

are the sets of even and odd indeterminates (variables).

Theorem 2.1 *If* $1 \le k \le 2n$*, then* $f_k(X, Y) = 0$ *is a* $\mathbb{Z}_2$*-graded polynomial identity of the* $\mathbb{Z}_2$*-graded full matrix algebra* $\mathrm{M}_n(E) = \mathrm{M}_n(E_0) \oplus \mathrm{M}_n(E_1)$.

Proof (First proof of 2.1). First notice that for a permutation $\pi \in \Pi(\overrightarrow{i})$ we have $\pi(\overrightarrow{i}) \sqcup \pi(\underrightarrow{i}) = \pi \circ \tau(\overrightarrow{i})$, where

$\pi(\overrightarrow{i}) \sqcup \pi(\underrightarrow{i}) \in \mathrm{Sym}\{1, 2, \ldots, k, k+1, \ldots, 2n\}$ is the "disjoint union" of $\pi(\overrightarrow{i})$ and $\pi(\underrightarrow{i})$. Clearly, the number of even cycles of $\pi(\overrightarrow{i}) \sqcup \pi(\underrightarrow{i})$ is the sum of the numbers of the even cycles in $\pi(\overrightarrow{i})$ and in $\pi(\underrightarrow{i})$.

It follows that

$$\mathrm{sgn}(\pi(\overrightarrow{i}))\mathrm{sgn}(\pi(\underrightarrow{i})) = \mathrm{sgn}(\pi(\overrightarrow{i}) \sqcup \pi(\underrightarrow{i})) = \mathrm{sgn}(\pi)\mathrm{sgn}(\tau(\overrightarrow{i})), \tag{2.12}$$

whence $\mathrm{sgn}(\pi)\mathrm{sgn}(\pi(\overrightarrow{i})) = \mathrm{sgn}(\tau(\overrightarrow{i}))\mathrm{sgn}(\pi(\underrightarrow{i}))$ can be derived. In order to show that $f_k(X,Y) = 0$ is a $\mathbb{Z}_2$-graded polynomial identity on $\mathrm{M}_n(E) = \mathrm{M}_n(E_0) \oplus \mathrm{M}_n(E_1)$ take the substitutions

$$x_{k+1} = A_{k+1}, x_{k+2} = A_{k+2}, \ldots, x_{2n} = A_{2n}$$

and

$$y_1 = B_1, y_2 = B_2, \ldots, y_k = B_k,$$

where $A_{k+1}, A_{k+2}, \ldots, A_{2n} \in \mathrm{M}_n(E_0)$ and $B_1, B_2, \ldots, B_k \in \mathrm{M}_n(E_1)$ and consider the "companion" matrices

$$w_1B_1, w_2B_2, \ldots, w_kB_k \in \mathrm{M}_n(F_0)$$

($w_1, w_2, \ldots, w_k$ are generators in G) over the even part F_0 of the extended Grassmann algebra $F = K\langle V \cup W\rangle$. In view of $\mathrm{M}_n(E_0) \subseteq \mathrm{M}_n(F_0)$, the application of the Amitsur-Levitzki theorem on $\mathrm{M}_n(F_0)$ yields that

$$S_{2n}(w_1B_1, \ldots, w_kB_k, A_{k+1}, A_{k+2}, \ldots, A_{2n}) = 0. \tag{2.13}$$

Any summand in

$$S_{2n}(w_1B_1, \ldots, w_kB_k, A_{k+1}, A_{k+2}, \ldots, A_{2n})$$

is a signed product of the terms $w_1B_1, \ldots, w_kB_k, A_{k+1}, A_{k+2}, \ldots, A_{2n}$ in a certain order and appears as

$$\begin{aligned}
&\mathrm{sgn}(\pi)A_{\pi(1)} \cdots A_{\pi(i_1-1)}w_{\pi(i_1)}B_{\pi(i_1)}A_{\pi(i_1+1)} \cdots A_{\pi(i_2-1)}w_{\pi(i_2)}B_{\pi(i_2)}A_{\pi(i_2+1)} \cdots \\
&\quad \cdots A_{\pi(i_k-1)}w_{\pi(i_k)}B_{\pi(i_k)}A_{\pi(i_k+1)} \cdots A_{\pi(2n)} = \\
&\mathrm{sgn}(\pi)(-1)^{1+2+\cdots+(k-1)}w_{\pi(i_1)}w_{\pi(i_2)} \cdots w_{\pi(i_k)}A_{\pi(1)} \cdots A_{\pi(i_1-1)}B_{\pi(i_1)}A_{\pi(i_1+1)} \cdots \\
&\quad \cdots A_{\pi(i_2-1)}B_{\pi(i_2)}A_{\pi(i_2+1)} \cdots A_{\pi(i_k-1)}B_{\pi(i_k)}A_{\pi(i_k+1)} \cdots A_{\pi(2n)} = \\
&\mathrm{sgn}(\pi)(-1)^{1+2+\cdots+(k-1)}\mathrm{sgn}(\pi(\overrightarrow{i}))w_1w_2 \cdots w_kA_{\pi(1)} \cdots A_{\pi(i_1-1)}B_{\pi(i_1)}A_{\pi(i_1+1)} \cdots \\
&\quad \cdots A_{\pi(i_2-1)}B_{\pi(i_2)}A_{\pi(i_2+1)} \cdots A_{\pi(i_k-1)}B_{\pi(i_k)}A_{\pi(i_k+1)} \cdots A_{\pi(2n)},
\end{aligned} \tag{2.14}$$

where $1 \leq i_1 < i_2 < \cdots < i_k \leq 2n$ and $\pi \in \Pi(\overrightarrow{i})$ are uniquely determined. In the above calculations we used

$$A_tw_r = w_rA_t,\ B_sw_r = -w_rB_s,\ 1 \leq r, s \leq k < t \leq 2n,$$

and

$$w_{\pi(i_1)}w_{\pi(i_2)} \cdots w_{\pi(i_k)} = \mathrm{sgn}(\pi(\overrightarrow{i}))w_1w_2 \cdots w_k.$$

Thus, we can write that

$$S_{2n}(w_1B_1,\ldots,w_kB_k,A_{k+1},A_{k+2},\ldots,A_{2n})=$$

$$\sum_{1\le i_1<i_2<\cdots<i_k\le 2n}\left(\sum_{\pi\in\Pi(\overrightarrow{i})}\operatorname{sgn}(\pi)A_{\pi(1)}\cdots A_{\pi(i_1-1)}w_{\pi(i_1)}B_{\pi(i_1)}A_{\pi(i_1+1)}\cdots\right.$$

$$\left.\cdots A_{\pi(i_2-1)}w_{\pi(i_2)}B_{\pi(i_2)}A_{\pi(i_2+1)}\cdots A_{\pi(i_k-1)}w_{\pi(i_k)}B_{\pi(i_k)}A_{\pi(i_k+1)}\cdots A_{\pi(2n)}\right)=$$

$$(-1)^{1+2+\cdots+(k-1)}w_1w_2\cdots w_k\sum_{1\le i_1<i_2<\cdots<i_k\le 2n}\operatorname{sgn}(\tau(\overrightarrow{i}))\cdot \tag{2.15}$$

$$\left(\sum_{\pi\in\Pi(\overrightarrow{i})}\operatorname{sgn}(\pi(\underrightarrow{i}))A_{\pi(1)}\cdots A_{\pi(i_1-1)}B_{\pi(i_1)}A_{\pi(i_1+1)}\cdots\right.$$

$$\left.A_{\pi(i_2-1)}B_{\pi(i_2)}A_{\pi(i_2+1)}\cdots A_{\pi(i_k-1)}B_{\pi(i_k)}A_{\pi(i_k+1)}\cdots A_{\pi(2n)}\right)=$$

$$(-1)^{1+2+\cdots+(k-1)}w_1w_2\cdots w_kf_k(A_{k+1},A_{k+2},\ldots,A_{2n},B_1,\ldots,B_k),$$

whence $f_k(A_{k+1},A_{k+2},\ldots,A_{2n},B_1,\ldots,B_k)=0$ follows. □

Remark 2.2 *The case $k=2n$ in the above Theorem 2.1 gives Rosset's key observation that*

$$f_{2n}(Y)=\sum_{\pi\in\operatorname{Sym}\{1,2,\ldots,2n\}}y_{\pi(1)}\cdots y_{\pi(2n)}=0 \tag{2.16}$$

(the multilinearization of $y^{2n}=0$) is a polynomial identity of the odd component $\mathrm{M}_n(E_1)$. The case $k=1$ has already appeared in the proof of Theorem 2.4 of [4].

A $\mathbb{Z}_2$-graded polynomial $h(x_1,x_2,\ldots,x_m,y_1,y_2,\ldots,y_k)$ which is linear in each odd variable y_i can be written as

$$h(x_1,x_2,\ldots,x_m,y_1,y_2,\ldots,y_k)=\sum_u\sum_{\sigma\in\operatorname{Sym}\{1,2,\ldots,k\}}a_{\sigma,u}u_1y_{\sigma(1)}u_2y_{\sigma(2)}\cdots u_ky_{\sigma(k)}u_{k+1}, \tag{2.17}$$

where $a_{\sigma,u}\in K$ and the u_i's are words, possibly empty, in the even variables x_j, $1\le j\le m$. The $*$-transform of h is defined as

$$h^*(x_1,x_2,\ldots,x_m,y_1,y_2,\ldots,y_k)=\sum_u\sum_{\sigma\in\operatorname{Sym}\{1,2,\ldots,k\}}\operatorname{sgn}(\sigma)a_{\sigma,u}u_1y_{\sigma(1)}u_2y_{\sigma(2)}\cdots u_ky_{\sigma(k)}u_{k+1}. \tag{2.18}$$

Lemma 19.4.10 (in [1]) asserts that $h=0$ is a $\mathbb{Z}_2$-graded identity of the $\mathbb{Z}_2$-graded K-algebra $R=R_0\oplus R_1$ if and only if $h^*=0$ is a $\mathbb{Z}_2$-graded identity of the Grassmann envelope $G(R)=(R_0\otimes E_0)\oplus(R_1\otimes E_1)=(R\otimes E)_0$ (the even part of $R\otimes E$).

Proof (Second proof of 2.1). Take $R=\mathrm{M}_n(K\oplus cK)$ with $R_0=\mathrm{M}_n(K)$ and $R_1=c\mathrm{M}_n(K)$, where $K\oplus cK\cong K[c]/(c^2-1)$ is the commutative group algebra of the two element group $\{1,c\}$ with $c^2=1$. Clearly

$M_n(E)$ can be naturally identified with the Grassmann envelope $G(R)$. Since the Amitsur-Levitzki theorem trivially ensures that

$$f(x_{k+1}, x_{k+2}, \ldots, x_{2n}, y_1, y_2, \ldots, y_k) = S_{2n}(y_1, y_2, \ldots, y_k, x_{k+1}, x_{k+2}, \ldots, x_{2n}) = 0 \tag{2.19}$$

is a $\mathbb{Z}_2$-graded identity of $R = R_0 \oplus R_1$, the application of the above Lemma 19.4.10 gives that $M_n(E)$ satisfies the $\mathbb{Z}_2$-graded identity $f^*(x_{k+1}, x_{k+2}, \ldots, x_{2n}, y_1, y_2, \ldots, y_k) = 0$. In view of

$$\begin{aligned} f &= S_{2n}(y_1, y_2, \ldots, y_k, x_{k+1}, x_{k+2}, \ldots, x_{2n}) = \\ &\sum_{1 \le i_1 < i_2 < \cdots < i_k \le 2n} \Bigg(\sum_{\pi \in \Pi(\vec{i})} \mathrm{sgn}(\pi) x_{\pi(1)} \cdots x_{\pi(i_1 - 1)} y_{\pi(\vec{i})(1)} x_{\pi(i_1+1)} \cdots \\ &\cdots x_{\pi(i_2-1)} y_{\pi(\vec{i})(2)} x_{\pi(i_2+1)} \cdots x_{\pi(i_k-1)} y_{\pi(\vec{i})(k)} x_{\pi(i_k+1)} \cdots x_{\pi(2n)} \Bigg), \end{aligned} \tag{2.20}$$

we obtain that

$$\begin{aligned} f^* = &\sum_{1 \le i_1 < i_2 < \cdots < i_k \le 2n} \Bigg(\sum_{\pi \in \Pi(\vec{i})} \mathrm{sgn}(\pi(\vec{i})) \mathrm{sgn}(\pi) x_{\pi(1)} \cdots x_{\pi(i_1 - 1)} y_{\pi(\vec{i})(1)} x_{\pi(i_1+1)} \cdots \\ &\cdots x_{\pi(i_2-1)} y_{\pi(\vec{i})(2)} x_{\pi(i_2+1)} \cdots x_{\pi(i_k-1)} y_{\pi(\vec{i})(k)} x_{\pi(i_k+1)} \cdots x_{\pi(2n)} \Bigg). \end{aligned} \tag{2.21}$$

Now $f^* = f_k(X, Y)$ is a consequence of $\mathrm{sgn}(\pi(\vec{i}))\mathrm{sgn}(\pi) = \mathrm{sgn}(\tau(\vec{i}))\mathrm{sgn}(\pi(\underset{\to}{i}))$. □

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