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Polynomial identities in matrix algebras with pseudoinvolution

Antonio IOPPOLO*
Department of Mathematics, University of Milano Bicocca, Milan, Italy

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Dedicated to our dear Professor Vesselin Drensky on the occasion of his seventieth anniversary.

Abstract: Let F be an algebraically closed field of characteristic zero. In this paper we deal with matrix superalgebras (i.e. algebras graded by $\mathbb{Z}_2$, the cyclic group of order 2) endowed with a pseudoinvolution. The first goal is to present the classification of the pseudoinvolutions that it is possible to define, up to equivalence, in the full matrix algebra $M_n(F)$ of $n \times n$ matrices and on its subalgebra $UT_n(F)$ of upper-triangular matrices. Along the way we shall give the generators of the T-ideal of identities for the algebras $M_2(F)$, $UT_2(F)$ and $UT_3(F)$, endowed with all possible inequivalent pseudoinvolutions.

Key words: Polynomial identities, matrix algebras, superalgebras, pseudoinvolutions.

1. Introduction

Let F be an algebraically closed fixed field of characteristic zero. Given a countable set X of variables $x_1, x_2, \dots$, the free associative algebra $F\langle X\rangle$ on X over F is the algebra of all polynomials in the variables from X and with coefficients in the field F. An F-algebra A is said to be a PI-algebra if it satisfies at least one non-trivial polynomial identity. In other words, there exists at least one non-zero polynomial $f \in F\langle X\rangle$ which vanishes under all substitutions of its variables in the elements of A. The set of all the polynomial identities satisfied by A is denoted by $\mathrm{Id}(A)$ and it is a T-ideal of the free algebra, i.e., an ideal invariant under all endomorphisms of $F\langle X\rangle$.

In this paper, in the context of the theory of polynomial identities (PI-theory for short), we shall focus our attention on the so-called superalgebras with pseudoinvolution. A superalgebra is simply an algebra graded by $\mathbb{Z}_2$, the cyclic group of order two. This means that A can be decomposed in the direct sum $A = A_0 \oplus A_1$, where the A_i's are the homogeneous components of A and they satisfy the following properties: $A_0A_0 + A_1A_1 \subseteq A_0$ and $A_0A_1 + A_1A_0 \subseteq A_1$. This kind of algebras first appeared in physics in order to have an algebraic structure representing the behaviour of the subatomic particles bosons (A_0) and fermions (A_1). The importance of superalgebras in PI-theory was highlighted by a famous theorem of Kemer ([9]). He showed that every PI-algebra is equivalent to (i.e., it has the same T-ideal of identities as) the Grassmann envelope of a finite dimensional superalgebra.

Now, a pseudoinvolution on a superalgebra $A = A_0 \oplus A_1$, is a graded linear map $* : A \to A$ such that $a^{**} = (-1)^{|a|}a$ and $(ab)^* = (-1)^{|a||b|}b^*a^*$, for any homogeneous elements $a, b \in A_0 \cup A_1$. Here $|c|$ denotes

*Correspondence: antonio.ioppolo@unimib.it
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the homogeneous degree of $c \in A_0 \cup A_1$. The existence of pseudoinvolutions of the first kind was proved in 2008 by Jaber (see [8]). Two years later, Martinez and Zelmanov used pseudoinvolutions in order to completely classify the irreducible bimodules over simple finite dimensional Jordan superalgebras (see [12]). In the PI-theory context, several papers concerning superalgebras with pseudoinvolution have been published recently ([5–7]).

Now it is time to state more clearly the goals of this paper. Let $M_n(F)$ be the algebra of $n \times n$ matrices over F. In 1950, a celebrated theorem of Amitsur and Levitzki ([1]) showed that the standard polynomial St_{2n} is an identity for the algebra $M_n(F)$ (actually, it is not hard to prove that St_{2n} is, up to a scalar, the only identity of minimal degree of $M_n(F)$). This theorem was the beginning of a new approach to PI-theory, the main objective being the description of the polynomial identities satisfied by a given algebra. The purpose of this paper goes in this direction.

In characteristic zero, it is well known that, up to isomorphism, the matrix algebra $M_n(F)$ can be endowed with only one non trivial $\mathbb{Z}_2$-grading. More precisely, if we write $n = k + h$, then $M_n(F)$ has the following decomposition as superalgebra:

$$M_n(F) = \left\{ \begin{pmatrix} X & 0 \\ 0 & T \end{pmatrix} \mid X \in M_k(F),\ T \in M_h(F) \right\} \oplus \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} \mid Y \in M_{k\times h}(F),\ Z \in M_{h\times k}(F) \right\}.$$

Such a superalgebra is denoted by $M_{k,h}(F)$. In [8], Jaber proved that, up to equivalence, there are only two classes of inequivalent pseudoinvolutions on this superalgebra: the pseudotranspose pt and the pseudosymplectic ps (we shall define them in full detail later on). The first goal of this paper is to present the generators of the ideal of identities for the superalgebra $M_{1,1}(F)$ endowed with the pseudotranspose and the pseudosymplectic pseudoinvolution.

The last part of the paper is devoted to the study of the algebra $UT_n(F)$ of $n \times n$ upper-triangular matrices. If $n = 2$, a basis of the identities of such an algebra was found by Malcev in 1971 (see [11]) whereas a solution for the general problem was given in 2010 by Latyshev ([10]). Now let us consider when the algebra $UT_n(F)$ has a structure of superalgebra. In [14], the authors completely characterized all the $\mathbb{Z}_2$-grading on this algebra by showing that, up to isomorphism, they are just the elementary ones. By taking into account these results and according to the classification of the involutions on $UT_n(F)$ given by Di Vincenzo, Koshlukov and La Scala in 2006, in 2021 the author completely classified the pseudoinvolutions on $UT_n(F)$ (see [4]). In the last sections of the paper we shall recall these results and, along the way, we shall give the generators of the T-ideal of identities of the algebras $UT_2(F)$ and $UT_3(F)$, endowed with all possible inequivalent pseudoinvolutions.

2. Preliminaries

Throughout this paper F will denote an algebraically closed field of characteristic zero and $A = A_0 \oplus A_1$ an associative superalgebra over F, i.e., an algebra graded by $\mathbb{Z}_2$, the cyclic group of order two. The elements of A_0 and A_1 are called homogeneous of degree zero (or even elements) and of degree one (or odd elements), respectively.

Let now consider a countable set of variables $X = \{x_1, x_2, \ldots\}$. Write $X = Y \cup Z$ as the disjoint union of two subsets, requiring that the variables of Y are of even homogeneous degree and the variables of Z are of odd homogeneous degree. If we denote by $\mathcal{F}_0$ the subspace of $F\langle Y \cup Z\rangle$ spanned by all monomials in the variables of $Y \cup Z$ having an even number of variables of Z and by $\mathcal{F}_1$ the subspace spanned by all monomials having an odd number of variables of Z, then it is clear that $F\langle Y \cup Z\rangle = \mathcal{F}_0 \oplus \mathcal{F}_1$ is a $\mathbb{Z}_2$-grading. Thus we can refer to $F\langle Y \cup Z\rangle$ as the free associative superalgebra.

Now suppose that the superalgebra A is endowed with a pseudoinvolution $*$. Recall that a pseudoinvolution on A is a graded linear map $* : A \longrightarrow A$ such that $a^{**} = (-1)^{|a|}a$ and $(ab)^* = (-1)^{|a||b|}b^*a^*$, for any homogeneous elements $a, b \in A_0 \cup A_1$. Here $|c|$ denotes the homogeneous degree of $c \in A_0 \cup A_1$.

Notice that, if $a \in A_1$, then $a^{**} = -a$ and so there are no symmetric or skew-symmetric elements of homogeneous degree 1. Moreover, the hypothesis that F is an algebraically closed field ensure us the existence of an element i such that $i^2 = -1$. Furthermore, since $\mathrm{char}F = 0$, we can write

$$A = A_0^+ \oplus A_0^- \oplus A_1^i \oplus A_1^{-i},$$

where $A_0^+ = \{a \in A_0 \mid a^* = a\}$ and $A_0^- = \{a \in A_0 \mid a^* = -a\}$ denote the sets of symmetric and skew elements of A_0 and $A_1^i = \{a \in A_1 \mid a^* = ia\}$ and $A_1^{-i} = \{a \in A_1 \mid a^* = -ia\}$ denote the sets of i-symmetric and i-skew elements of A_1, respectively.

We shall write $F\langle Y \cup Z, *\rangle$ for the free superalgebra with pseudoinvolution on the countable set $Y \cup Z$ over F. It is useful to regard $F\langle Y \cup Z, *\rangle$ as generated by (even) symmetric and skew variables and by (odd) i-symmetric and i-skew variables: if for $j = 1, 2, \ldots,$ we let $y_j^+ = y_j + y_j^*$, $y_j^- = y_j - y_j^*$, $z_j^i = z_j - iz_j^*$ and $z_j^{-i} = z_j + iz_j^*$, thus

$$F\langle Y \cup Z, *\rangle = F\langle y_1^+, y_1^-, z_1^i, z_1^{-i}, y_2^+, y_2^-, z_2^i, z_2^{-i}, \ldots\rangle.$$

A polynomial $f(y_1^+, \ldots, y_m^+, y_1^-, \ldots, y_n^-, z_1^i, \ldots, z_r^i, z_1^{-i}, \ldots, z_s^{-i}) \in F\langle Y \cup Z, *\rangle$ is a $*$-polynomial identity of A (or simply a $*$-identity), and we write $f \equiv 0$, if for all $u_1^+, \ldots, u_m^+ \in A_0^+$, $u_1^-, \ldots, u_n^- \in A_0^-$, $v_1^i, \ldots, v_r^i \in A_1^i$ and $v_1^{-i}, \ldots, v_s^{-i} \in A_1^{-i}$, we get

$$f(u_1^+, \ldots, u_m^+, u_1^-, \ldots, u_n^-, v_1^i, \ldots, v_r^i, v_1^{-i}, \ldots, v_s^{-i}) = 0.$$

We denote by $\mathrm{Id}^*(A) = \{f \in F\langle Y \cup Z, *\rangle \mid f \equiv 0 \text{ on } A\}$ the T_2^*-ideal of $*$-identities of A, i.e. an ideal of $F\langle Y \cup Z, *\rangle$ invariant under all graded endomorphisms of $F\langle Y \cup Z\rangle$ commuting with the pseudoinvolution $*$. Given polynomials $f_1, \ldots, f_n \in F\langle Y \cup Z, *\rangle$ we shall denote by $\langle f_1, \ldots, f_n\rangle_{T_2^*}$ the T_2^*-ideal generated by $f_1, \ldots, f_n$. Moreover, in order to simplify the notation, we shall denote by y any even variable, by z any odd variable and by x an arbitrary variable.

As in the ordinary case, in characteristic zero, every $*$-identity is equivalent to a system of multilinear $*$-identities. Hence, in order to study $\mathrm{Id}^*(A)$, one can define

$$P_n^* = \mathrm{span}_F\{w_{\sigma(1)} \cdots w_{\sigma(n)} \mid \sigma \in S_n,\ w_j = y_j^+ \text{ or } w_j = y_j^- \text{ or } w_j = z_j^i \text{ or } w_j = z_j^{-i},\ j = 1, \ldots, n\}$$

as the space of multilinear polynomials of degree n in the variables $y_1^+, y_1^-, z_1^i, z_1^{-i}, \ldots, y_n^+, y_n^-, z_n^i, z_n^{-i}$ and then analyze $P_n^* \cap \mathrm{Id}^*(A)$, for all $n \geq 1$. The non-negative integer

$$c_n^*(A) = \dim_F \frac{P_n^*}{P_n^* \cap \mathrm{Id}^*(A)},\ n \geq 1,$$

is called the n-th $*$-codimension of A.

Let $n \geq 1$ and write $n = n_1 + \cdots + n_4$ as a sum of four non-negative integers. We denote by $P_{n_1,\ldots,n_4} \subseteq P_n^*$ the vector space of multilinear polynomials in which the first n_1 variables are symmetric, the next n_2 variables are skew, the next n_3 variables are i-symmetric and the last n_4 variables are i-skew. Now if we set $c_{n_1,\ldots,n_4}(A) = \dim_F \dfrac{P_{n_1,\ldots,n_4}}{P_{n_1,\ldots,n_4} \cap \operatorname{Id}^*(A)}$ then

$$c_n^*(A) = \sum_{n_1+\cdots+n_4=n} \binom{n}{n_1,\ldots,n_4} c_{n_1,\ldots,n_4}(A), \tag{2.1}$$

where $\binom{n}{n_1,\ldots,n_4} = \frac{n!}{n_1!\cdots n_4!}$ stands for the multinomial coefficient. Hence the growth of $c_n^*(A)$ is related to the growth of multinomial coefficients and of $c_{n_1,\ldots,n_4}(A)$, for any $n = n_1 + \cdots + n_4$.

Lemma 2.1 *Let A be a superalgebra with pseudoinvolution satisfying an ordinary identity. Then, for $n \geq 1$,*

$$c_n(A) \leq c_n^*(A) \leq 4^n c_n(A).$$

Since the ordinary codimension sequence of a PI-algebra is exponentially bounded (see [13]), we get the following. Recall that those algebras satisfying at least one non-trivial ordinary identity are called PI-algebras.

Corollary 2.2 *If A is a PI-superalgebra with pseudoinvolution, then $c_n^*(A)$, $n = 1, 2, \ldots,$ is exponentially bounded.*

3. Full matrix algebras and pseudoinvolutions

In this section we focus our attention on matrix superalgebras. Let us consider the full matrix algebra $M_n(F)$ of $n\times n$ matrices. Since we are dealing with a field F of characteristic zero, it is well known that, up to isomorphism, there is only one non trivial $\mathbb{Z}_2$-grading on $M_n(F)$. More precisely, if $n = k + h$, then $A = M_n(F)$ becomes a superalgebra $A = A_0 \oplus A_1$, where the homogeneous components are

$$A_0 = \left\{ \begin{pmatrix} X & 0 \\ 0 & T \end{pmatrix} \mid X \in M_k(F),\ T \in M_h(F) \right\} \qquad \text{and} \qquad A_1 = \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} \mid Y \in M_{k\times h}(F),\ Z \in M_{h\times k}(F) \right\}.$$

Such a superalgebra is denoted by $M_{k,h}(F)$.

In [8, Theorems 4.3 and 4.4], Jaber proved that, up to equivalence, there are only two classes of inequivalent pseudoinvolutions on $M_{k,h}(F)$. Recall that if A and B are two algebras with pseudoinvolutions $*$ and $\star$ respectively, then $(A,*)$ and $(B,\star)$ are isomorphic, as superalgebras with pseudoinvolution, if there exists an isomorphism of superalgebras $\psi : A \to B$ such that $\psi(x^*) = \psi(x)^\star$, for all $x \in A$.

For all $k \geq h \geq 0$, the superalgebra $M_{k,h}(F)$ can be endowed with the pseudotranspose pseudoinvolution pt defined by:

$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{pt} = \begin{pmatrix} X^t & Z^t \\ -Y^t & T^t \end{pmatrix},$$

where t denotes the usual transpose involution.

If $h = k$, the superalgebra $M_{k,k}(F)$ can be also endowed with the pseudosymplectic pseudoinvolutions ps_i and ps_{-i} defined, respectively, by:

$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{ps_i} = \begin{pmatrix} T^t & iY^t \\ iZ^t & X^t \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{ps_{-i}} = \begin{pmatrix} T^t & -iY^t \\ -iZ^t & X^t \end{pmatrix}.$$

In what follows we shall present the results of [2], in which the authors gave the generators of the T_2^*-ideal of identities of the superalgebra $M_{1,1}(F)$ endowed first with the pseudoinvolution pt and then with the pseudoinvolutions ps_i and ps_{-i}.

Consider first the algebra $(M_{1,1}(F), pt)$. Since $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{pt} = \begin{pmatrix} a & c \\ -b & d \end{pmatrix}$, it is not difficult to see that

$$(M_{1,1}(F), pt) = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & c \\ ic & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & c \\ -ic & 0 \end{pmatrix} \right\}.$$

Theorem 3.1 *The T_2^*-ideal of identities of $(M_{1,1}(F), pt)$ is generated by the following polynomials*

$$[y_1^+, y_2^+], \quad y^-, \quad [z_1^i, z_2^i], \quad [z_1^{-i}, z_2^{-i}], \quad z_1^i \circ z_2^{-i}, \quad z_1^{\pm i} y_2^+ z_3^{\pm i} - z_3^{\pm i} y_2^+ z_1^{\pm i}, \quad z_1^{\pm i} y_2^+ z_3^{\mp i} + z_3^{\mp i} y_2^+ z_1^{\pm i}.$$

Proof (Sketch).

Let J be the T_2^*-ideal generated by the above polynomials. It is easy to prove that $J \subseteq \mathrm{Id}^*(M_{1,1}(F), pt)$. In order to prove the opposite inclusion, let $f \in \mathrm{Id}^*(M_{1,1}(F), pt)$, $\deg f = n$, and assume, as we may, that f is multilinear. We want to show that f is the zero polynomial modulo J. By taking into account also the identities $[z_1 z_2, y_3^+] \equiv 0$ and $[y_1^+, z_2 y_3^+ z_4] \equiv 0$, it follows that f is a linear combination (modulo J) of monomials of the type

$$y_{i_1}^+ \cdots y_{i_a}^+ z_{l_1} y_{j_1}^+ \cdots y_{j_b}^+ z_{l_2} \cdots z_{l_h}, \tag{3.1}$$

where $h = 0, \ldots, n$ is the number of variables z's, $a + b = n - h$ counts the number of variables y^+'s and $i_1 < \cdots < i_a$, $j_1 < \cdots < j_b$, $l_1 < \cdots < l_h$. In what follows, we may assume that the z's are just z^+'s. Then, we can take into account that there are exactly 2^h monomials of each kind.

Now assume that $\alpha_{i_1,\ldots,i_a,j_1,\ldots,j_b,l_1,\ldots,l_h}$ is the coefficient with which the above monomial appears in f. In order to show that f is the zero polynomial, it is sufficient to consider the following evaluation:

- $y_{i_1}^+ = \cdots = y_{i_a}^+ = e_{11}$.
- $y_{j_1}^+ = \cdots = y_{j_b}^+ = e_{22}$.
- $y_s^+ = 0$, for any $s \notin \{i_1, \ldots, i_a, j_1, \ldots, j_b\}$.
- $z_{l_1}^+ = \cdots = z_{l_h}^+ = e_{12} + ie_{21}$.
- $z_t^+ = 0$, for any $t \notin \{l_1, \ldots, l_h\}$.

There are several possibilities but in any case it follows that f is the zero polynomial and we are done. □

Corollary 3.2 $c_n^*(M_{1,1}(F), pt) = 4^n - 2^n + 1$.

Proof We need just to count the monomials in (3.1). For fixed a, b and $h > 0$, it is easy to see that they are:

$$\binom{n}{h} \sum_{j=0}^{n-h} \binom{n-h}{j}.$$

Recalling that we have to consider that each z can be i-symmetric or i-skew (2^h possibilities) and by taking into account the monomial $y_1^+ \cdots y_n^+$, we get that

$$c_n^*(M_{1,1}(F), pt) = 1 + \sum_{h=0}^{n} 2^h \left(\binom{n}{h} \sum_{j=0}^{n-h} \binom{n-h}{j} \right) = 4^n - 2^n + 1.$$

□

Now let us consider the algebra $(M_{1,1}(F), ps_i)$. Since $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{ps_i} = \begin{pmatrix} d & ib \\ ic & a \end{pmatrix}$, we get

$$(M_{1,1}(F), ps_i) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

Theorem 3.3 *The T_2^*-ideal of identities of $(M_{1,1}(F), ps_i)$ is generated by the following polynomials*

$$[y^+, x], \quad [y_1^-, y_2^-], \quad y^- \circ z^i, \quad z^{-i}, \quad z_1^i z_2^i z_3^i - z_3^i z_2^i z_1^i.$$

Proof (Sketch).

Let J be the T_2^*-ideal generated by the above polynomials. It is easy to prove that $J \subseteq \mathrm{Id}^*(M_{1,1}(F), ps_i)$. Now let $f \in \mathrm{Id}^*(M_{1,1}(F), ps_i)$, $\deg f = n$, and assume, as we may, that f is multilinear. We want to show that f is the zero polynomial modulo J. Clearly, f is a linear combination (modulo J) of monomials of the type

$$y_{i_1}^+ \cdots y_{i_{n_1}}^+ y_{j_1}^- \cdots y_{j_{n_2}}^- z_{l_1}^+ \cdots z_{l_{n-n_1-n_2}}^+, \tag{3.2}$$

where $i_1 < \cdots < i_{n_1}$, $j_1 < \cdots < j_{n_2}$, $l_1 < l_3 < \cdots$, $l_2 < l_4 < \cdots$. Moreover, assume that $\alpha_{I,J,l_1,\ldots,l_{n-n_1-n2}}$ is the coefficient with which the above monomial appears in f. In order to show that f is the zero polynomial, it is sufficient to consider the following evaluation:

- $y_{i_1}^+ = \cdots = y_{i_{n_1}}^+ = e_{11} + e_{22}$ and $y_r^+ = 0$, for any $r \notin \{i_1, \ldots, i_{n_1}\}$.
- $y_{j_1}^- = \cdots = y_{j_{n_2}}^- = e_{11} - e_{22}$ and $y_s^- = 0$, for any $s \notin \{j_1, \ldots, j_{n_2}\}$.
- $z_{l_1}^+ = z_{l_3}^+ = \cdots = e_{12}$, $z_{l_2}^+ = z_{l_4}^+ = \cdots = e_{21}$ and $z_t^+ = 0$, for any $t \notin \{l_1, \ldots, l_{n-n_1-n_2}\}$.

We get $\alpha_{I,J,l_1,\ldots,l_{n-n_1-n2}} e_{11} \pm \alpha_{I,J,l_{n-n_1-n2},\ldots,l_1} e_{22}$ if $n - n_1 - n_2$ is even and $\alpha_{I,J,l_1,\ldots,l_{n-n_1-n2}} e_{12}$ otherwise. In any case we are done. □

Corollary 3.4 $c_n^*(M_{1,1}(F), ps_i) = \binom{2n+1}{n+1} \approx 4^n$.

Proof We need just to count the monomials in (3.2). For fixed n_1 and n_2, it is not difficult to see that they are:

$$\binom{n}{n_1} \binom{n-n_1}{n_2} \binom{n-n_1-n_2}{\lfloor \frac{n-n_1-n_2}{2} \rfloor}.$$

Hence we get that

$$c_n^*\left(M_{1,1}(F), ps\right) = \sum_{n_1,n_2} \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{\lfloor \frac{n-n_1-n_2}{2} \rfloor} = \binom{2n+1}{n+1} \approx 4^n.$$

□

Finally, if we consider the pseudoinvolution ps_{-i} on $M_{1,1}(F)$ given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{ps_{-i}} = \begin{pmatrix} d & -ib \\ -ic & a \end{pmatrix}$, we get

$$(M_{1,1}(F), ps_{-i}) = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right\} \oplus \left\{\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}\right\} \oplus \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\} \oplus \left\{\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\right\}.$$

With the same approach of Theorem 3.3 and of Corollary 3.4, we get the following result.

Theorem 3.5 *The T_2^*-ideal of identities of $(M_{1,1}(F), ps_{-i})$ is generated by the following polynomials*

$$[y^+, x], \quad [y_1^-, y_2^-], \quad y^- \circ z^{-i}, \quad z^i, \quad z_1^{-i} z_2^{-i} z_3^{-i} - z_3^{-i} z_2^{-i} z_1^{-i}.$$

Moreover, $c_n^*\left(M_{1,1}(F), ps_{-i}\right) = \binom{2n+1}{n+1} \approx 4^n$.

4. Pseudoinvolutions on upper-triangular matrices

In [4], the author gave a classification of the pseudoinvolutions on the superalgebra $UT_n(F)$ of $n \times n$ upper-triangular matrices. Such a classification was obtained by making use of the notion of the so-called superautomorphisms and of a strict relation between pseudoinvolutions and graded involutions. In what follows we shall recall these results.

Let us start by presenting two involutions on $UT_n(F)$. The so-called reflection involution $\circ\colon UT_n(F) \to UT_n(F)$ is defined on the matrix units by the formula:

$$e_{ij}^\circ = e_{n+1-j,n+1-i}, \quad \text{for any } 1 \leq i, j \leq n.$$

Now assume that $n = 2m$ is even and consider the matrix $J = \begin{pmatrix} I_m & 0 \\ 0 & -I_m \end{pmatrix}$. Then, for any $Y \in UT_n(F)$,

$$\begin{aligned} s\colon UT_n(F) &\longrightarrow UT_n(F) \\ Y &\longmapsto JY^\circ J \end{aligned}$$

is an involution on $UT_n(F)$. In [3], the authors proved the following.

Theorem 4.1 *[3, Proposition 2.5] If F is a field of characteristic different from 2, every involution on $UT_n(F)$ is equivalent to $\circ$ or to s. Moreover, s can occur only if n is even.*

Now let us give a structure of superalgebra to $UT_n(F)$. In this direction, we have the following result of Valenti and Zaicev, proved for any finite abelian group G and here presented just for $\mathbb{Z}_2$, the cyclic group of order 2.

Theorem 4.2 *[14] Let F be an algebraically closed field of characteristic zero and let $UT_n(F)$ be the algebra of $n \times n$ upper-triangular matrices over F graded by $\mathbb{Z}_2$. Then $UT_n(F)$, as a $\mathbb{Z}_2$-graded algebra, is isomorphic to $UT_n(F)$ with some elementary $\mathbb{Z}_2$-grading.*

Recall that an n-tuple $\mathbf{g} = (g_1, \dots, g_n) \in \mathbb{Z}_2^n$ defines an elementary $\mathbb{Z}_2$-grading on $UT_n(F)$ by setting

$$(UT_n(F))_0 = \text{span}\{e_{ij} \mid g_i + g_j = 0 \pmod 2\} \quad \text{and} \quad (UT_n(F))_1 = \text{span}\{e_{ij} \mid g_i + g_j = 1 \pmod 2\}.$$

Finally, recall that an involution $* : A \to A$ on the superalgebra $A = A_0 \oplus A_1$ is said to be a graded involution if $A_i^* \subseteq A_i$, for all $i \in \{0, 1\}$. In 2009 Valenti and Zaicev classified, up to equivalence, all graded involutions on the G-graded algebra $UT_n(F)$. In the following theorem we present such a result in the context of superalgebras.

Theorem 4.3 *[15, Theorem 5.4] Let F be an algebraically closed field of characteristic zero and let $UT_n(F)$ be the superalgebra of $n \times n$ upper-triangular matrices over F. Suppose that $UT_n(F)$ is endowed with a graded involution $\sharp$. Then $UT_n(F)$, as a $\mathbb{Z}_2$-graded algebra with graded involution, is isomorphic to $UT_n(F)$ with an elementary $\mathbb{Z}_2$-grading defined by an n-tuple $(g_1, \dots, g_n)$ such that*

$$g_1 + g_n = g_2 + g_{n-1} = \cdots = g_1 + g_n$$

and with involution $\circ$ or s. The involution s can occur only if n is even.

Now we are almost ready to give the classification of the pseudoinvolutions on the superalgebra $UT_n(F)$, proved in [4]. To this end, recall that a bijective graded linear map $\varphi\colon A \to A$ on a superalgebra $A = A_0 \oplus A_1$ is a superautomorphism if

$$\varphi(ab) = (-1)^{|a||b|}\varphi(a)\varphi(b), \text{ for all } a, b \in A_0 \cup A_1.$$

Now we introduce a particular superautomorphism that it is possible to define on any F-superalgebra A, where F is an algebraically closed field. In fact, under this hypothesis, we may assume that there exists an element $i \in F$ such that $i^2 = -1$.

Definition 4.4 *Let $A = A_0 \oplus A_1$ be a superalgebra over an algebraically closed field F. We define*

$$\Phi\colon A_0 \oplus A_1 \longrightarrow A_0 \oplus A_1$$
$$a_0 + a_1 \longmapsto a_0 + ia_1.$$

In [4] it was proved that Φ is a superautomorphism commuting with any graded linear map on a superalgebra A and such that $\Phi^2(a) = (-1)^{|a|}a$, for any homogeneous element $a \in A_0 \oplus A_1$.

The following result will allow us to give the classification of pseudoinvolutions on $UT_n(F)$.

Theorem 4.5 *Let $A = A_0 \oplus A_1$ be a superalgebra. A map $*\colon A \to A$ is a pseudoinvolution if and only if there exists a graded involution $\sharp\colon A \to A$ such that $* = \sharp\Phi$.*

We now define two pseudoinvolutions on the upper-triangular matrix superalgebra. Let $\circ$ and s be the reflection and the symplectic graded involution defined above. We give the following definitions.

Definition 4.6 *The pseudoinvolution* $\bar{\circ}\colon UT_n(F) \to UT_n(F)$*, defined by* $\bar{\circ} = \circ\Phi$*, is called* ***pseudo-reflection****.*

Definition 4.7 *The pseudoinvolution* $\bar{s}\colon UT_n(F) \to UT_n(F)$*, defined by* $\bar{s} = s\Phi$*, is called* ***pseudo-symplectic****.*

The following theorem is the main result of this section.

Theorem 4.8 *Let* F *be an algebraically closed field of characteristic zero and let* $UT_n(F)$ *be the superalgebra of* $n \times n$ *upper-triangular matrices over* F*. Suppose that* $UT_n(F)$ *is endowed with a pseudoinvolution* $*$*. Then* $UT_n(F)$*, as a* $\mathbb{Z}_2$*-graded algebra with pseudoinvolution, is isomorphic to* $UT_n(F)$ *with an elementary* $\mathbb{Z}_2$*-grading defined by an* n*-tuple* $(g_1, \dots, g_n)$ *such that*

$$g_1 + g_n = g_2 + g_{n-1} = \cdots = g_1 + g_n$$

and with pseudoinvolution $\bar{\circ}$ *or* $\bar{s}$*. The pseudoinvolution* $\bar{s}$ *can occur only if* n *is even.*

5. Polynomial identities on $UT_2(F)$

In this section we focus our attention on 2×2 upper-triangular matrices. By Theorem 4.2, every $\mathbb{Z}_2$-grading on $UT_2(F)$ is an elementary $\mathbb{Z}_2$-grading and, among them, the different (non isomorphic) ones are induced by the following pairs of elements of $\mathbb{Z}_2$:

$$(0,0) \quad \text{and} \quad (0,1)\,.$$

First we consider the case of $UT_2(F)$ with the elementary $\mathbb{Z}_2$-grading defined by the pair $(0,0)$. From now on, $UT_2(F)_{(0,0)}$ shall denote this superalgebra with trivial $\mathbb{Z}_2$-grading. Since $\big(UT_2(F)_{(0,0)}\big)_1 = 0$, it is clear that the pseudoinvolutions on $UT_2(F)_{(0,0)}$ coincide with the involutions. In [3], the authors proved that it is possible to define on $UT_2(F)$ only the two involutions $\circ$ and s we have presented before. Here we recall the definition of such involutions in the case of 2×2 upper-triangular matrices.

Let $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \in UT_2(F)$, then we have:

$$\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}^{\circ} = \begin{pmatrix} b & c \\ 0 & a \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}^{s} = \begin{pmatrix} b & -c \\ 0 & a \end{pmatrix}.$$

It is important to remark that, since $UT_2(F)_{(0,0)}$ is a superalgebra with trivial grading, the superautomorphism Φ is the identity map and hence the pseudoinvolutions $\bar{\circ}$ and $\bar{s}$ defined before are exactly the involutions $\circ$ and s.

In the following theorems we present the results of [3] in which the authors found a basis of the T_2^*-ideal of $UT_2(F)$ in the settings of superalgebras with pseudoinvolution, for convenience of the reader.

Theorem 5.1 *[3, Theorem 3.1] Let* $\big(UT_2(F)_{(0,0)}, \circ\big)$ *be the superalgebra of* 2×2 *upper-triangular matrices with trivial* $\mathbb{Z}_2$*-grading and endowed with the pseudoinvolution* $\circ$*. Then the* T_2^**-ideal of identities of this algebra is generated, as a* T_2^**-ideal, by the following polynomials:*

$$[y_1^+, y_2^+], \quad [y_1^-, y_2^-], \quad [y_1^+, y_1^-][y_2^+, y_2^-], \quad y_1^- y^+ y_2^- - y_2^- y^+ y_1^-, \quad z^i, \quad z^{-i}.$$

Theorem 5.2 *[3, Theorem 3.2] Let $\left(UT_2(F)_{(0,0)}, s\right)$ be the superalgebra of 2×2 upper-triangular matrices with trivial $\mathbb{Z}_2$-grading and endowed with the pseudoinvolution s. Then the T_2^*-ideal of identities of this algebra is generated, as a T_2^*-ideal, by the following polynomials:*

$$[y_1^+, y_2^+], \quad [y^-, y^+], \quad [y_1^-, y_2^-][y_3^-, y_4^-], \quad y_1^- y_2^- y_3^- - y_3^- y_2^- y_1^-, \quad z^i, \quad z^{-i}.$$

Let now focus our attention on $UT_2(F)_{(0,1)}$, the superalgebra of 2×2 upper-triangular matrices with elementary $\mathbb{Z}_2$-grading defined by the pair $(0,1)$. In this case the subspaces of homogeneous elements of degree 0 and 1 are:

$$\left(UT_2(F)_{(0,1)}\right)_0 = Fe_{11} \oplus Fe_{22}, \quad \text{and} \quad \left(UT_2(F)_{(0,1)}\right)_1 = Fe_{12}.$$

Since the pair $(0,1)$ satisfies the property $0+1 = 1+0$, according to Theorem 4.8, we have that on $UT_2(F)_{(0,1)}$ we can define the pseudoinvolutions $\bar{\circ}$ and $\bar{s}$.

Let us start by considering $A = \left(UT_2(F)_{(0,1)}, \bar{\circ}\right)$, the superalgebra $UT_2(F)_{(0,1)}$ endowed with the pseudoinvolution $\bar{\circ}$. We have that

$$\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}^{\bar{\circ}} = \begin{pmatrix} b & ic \\ 0 & a \end{pmatrix}.$$

The four sets of symmetric and skew even elements and of i-symmetric and i-skew odd elements are:

$$A_0^+ = F\left(e_{11} + e_{22}\right), \quad A_0^- = F\left(e_{11} - e_{22}\right), \quad A_1^i = Fe_{12}, \quad A_1^{-i} = 0.$$

In the following we compute a basis for the T_2^*-ideal of identities of A. In order to simplify the notation, we denote by x an arbitrary variable.

Theorem 5.3 *The T_2^*-ideal of identities of $A = \left(UT_2(F)_{(0,1)}, \bar{\circ}\right)$ is generated by the following polynomials*

$$[y^+, x], \quad [y_1^-, y_2^-], \quad y^- z^i + z^i y^-, \quad z_1^i z_2^i, \quad z^{-i}.$$

Proof Let J be the T_2^*-ideal generated by the above polynomials. It is easy to prove that $J \subseteq \mathrm{Id}^*(A)$.

In order to prove the opposite inclusion, let $f \in \mathrm{Id}^*(A)$, $\deg f = n$, and assume, as we may, that f is multilinear and $f \in P_{n_1,\dots,n_4}$, where $n = n_1 + \cdots + n_4$. We want to show that f is the zero polynomial modulo J. Since $z^{-i} \equiv 0$ and $z_1^i z_2^i \equiv 0$, if $n_4 \neq 0$ or $n_3 \geq 2$, it is obvious that f is the zero polynomial. Then let $n_4 = 0$ and $0 \leq n_3 \leq 1$. By the other identities it is easy to see that

$$f \equiv \alpha\, y_1^+ \cdots y_{n_1}^+ y_1^- \cdots y_{n_2}^- \pmod{J}, \; n_1 + n_2 = n,$$

or

$$f \equiv \beta\, y_1^+ \cdots y_{n_1}^+ y_1^- \cdots y_{n_2}^- z^+ \pmod{J}, \; n_1 + n_2 + 1 = n.$$

In the first case, by making the evaluation $y_i^+ = e_{11} + e_{22}$, $1 \leq i \leq n_1$ and $y_j^- = e_{11} - e_{22}$, $1 \leq j \leq n_2$, we get $\alpha(e_{11} \pm e_{22}) = 0$. Thus $\alpha = 0$. Similarly, if $f \equiv \beta\, y_1^+ \cdots y_{n_1}^+ y_1^- \cdots y_{n_2}^- z^+ \pmod{J}$, by making the evaluation $y_i^+ = e_{11} + e_{22}$, $1 \leq i \leq n_1$, $y_j^- = e_{11} - e_{22}$, $1 \leq j \leq n_2$ and $z^i = e_{12}$, we get $\beta e_{12} = 0$ and so $\beta = 0$. Hence f is the zero polynomial modulo J and the proof is complete. □

An easy computation allows us to obtain the following result.

Corollary 5.4 *Let* $A = \left(UT_2(F)_{(0,1)}, \bar{\circ}\right)$ *be the superalgebra* $UT_2(F)_{(0,1)}$ *with the pseudoinvolution* $\bar{\circ}$*. Then*

$$c_n^*(A) = 2^{n-1}(n+2).$$

Let now $B = \left(UT_2(F)_{(0,1)}, \bar{s}\right)$ be the superalgebra $UT_2(F)_{(0,1)}$ with the pseudoinvolution $\bar{s}$. We have that

$$\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}^{\bar{s}} = \begin{pmatrix} b & -ic \\ 0 & a \end{pmatrix}.$$

The four sets of symmetric and skew even elements and of i-symmetric and i-skew odd elements are:

$$B_0^+ = F\left(e_{11} + e_{22}\right), \quad B_0^- = F\left(e_{11} - e_{22}\right), \quad B_1^i = 0, \quad B_1^{-i} = Fe_{12}.$$

The following result can be proved in a similar way as the preceding one, so we omit the proof.

Theorem 5.5 *The* T_2^**-ideal of identities of* $B = \left(UT_2(F)_{(0,1)}, \bar{s}\right)$ *is generated by the following polynomials*

$$[y^+, x], \quad [y_1^-, y_2^-], \quad y^- z^{-i} + z^{-i} y^-, \ z^i, \ z_1^{-i} z_2^{-i}.$$

Moreover, $c_n^*(B) = 2^{n-1}(n+2)$*.*

6. Polynomial identities on $UT_3(F)$

In this section we focus our attention on 3×3 upper-triangular matrices. By Theorem 4.2, every $\mathbb{Z}_2$-grading on UT_3 is an elementary $\mathbb{Z}_2$-grading and, among them, the different (non isomorphic) ones are induced by the following triples of elements of $\mathbb{Z}_2$:

$$(0,0,0), \quad (0,0,1), \quad (0,1,1), \quad (0,1,0).$$

First we consider the case of $UT_3(F)$ with the elementary $\mathbb{Z}_2$-grading defined by the triple $(0,0,0)$. From now on, $UT_3(F)_{(0,0,0)}$ shall denote this superalgebra with trivial $\mathbb{Z}_2$-grading. Since $\left(UT_3(F)_{(0,0,0)}\right)_1 = 0$, it follows that the pseudoinvolutions on $UT_3(F)_{(0,0,0)}$ coincide with the involutions. In [3], the authors proved that it is possible to define on $UT_3(F)$ just the involution $\circ$ (it is not possible to define the involution s since 3 is odd). Recall that

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}^{\circ} = \begin{pmatrix} f & e & c \\ 0 & d & b \\ 0 & 0 & a \end{pmatrix}, \text{ for all } \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \in UT_3(F).$$

In the following theorem we present the results of [3] in the language of superalgebras with pseudoinvolution, for convenience of the reader. In order to simplify the notation, if the variable can be indifferently y_i^+ or y_i^- we shall denote it as $y_i^{\pm}$. Moreover we put $|y_i^-| = 0$ and $|y_i^+| = 1$. Here we observe that $[y_i^+, y_j^+]$ and $[y_i^-, y_j^-]$ are skew elements while $[y_i^+, y_j^-]$ is symmetric. We shall use the notation $|y_i^{\pm} y_j^{\pm}|$ for $|[y_i^{\pm}, y_j^{\pm}]|$. In other words $|y_i^{\pm} y_j^{\pm}|$ equals 0 when the commutator $[y_i^{\pm}, y_j^{\pm}]$ is skew and it equals 1 if the commutator is symmetric.

Theorem 6.1 *[3, Theorem 6.6] Let $\left(UT_3(F)_{(0,0,0)}, \circ\right)$ be the superalgebra of 3×3 upper-triangular matrices with trivial $\mathbb{Z}_2$-grading and endowed with the pseudoinvolution $\circ$. Then the T_2^*-ideal of identities of this algebra is generated, as a T_2^*-ideal, by the following polynomials:*

1. z,
2. $St_3(y_1^-, y_2^-, y_3^-)$,
3. $(-1)^{|y_1^\pm y_2^\pm|}[y_1^\pm, y_2^\pm][y_3^\pm, y_4^\pm] - (-1)^{|y_3^\pm y_4^\pm|}[y_3^\pm, y_4^\pm][y_1^\pm, y_2^\pm]$,
4. $y_1^-[y_3^\pm, y_4^\pm]y_2^- + (-1)^{|y_3^\pm y_4^\pm|}y_2^-[y_3^\pm, y_4^\pm]y_1^-$,
5. $[y_1^\pm, y_2^\pm]y_5^-[y_3^\pm, y_4^\pm]$,
6. $y_1^-[y_4^\pm, y_5^\pm]y_2^- y_3^\pm + (-1)^{|y_3^\pm|}y_3^\pm y_1^-[y_4^\pm, y_5^\pm]y_2^-$,
7. $(-1)^{|y_1^\pm y_2^\pm|}[y_1^\pm, y_2^\pm][y_3^\pm, y_4^\pm] - (-1)^{|y_1^\pm y_3^\pm|}[y_1^\pm, y_3^\pm][y_2^\pm, y_4^\pm] + (-1)^{|y_1^\pm y_4^\pm|}[y_1^\pm, y_4^\pm][y_2^\pm, y_3^\pm]$.

According to Theorem 4.8, it is easy to see that it is not possible to define any pseudoinvolution on $UT_3(F)_{(0,0,1)}$ and on $UT_3(F)_{(0,1,1)}$.

Finally we consider the superalgebra of 3×3 upper-triangular matrices with elementary $\mathbb{Z}_2$-grading defined by the triple $(0,1,0)$. In this case the subspaces of homogeneous elements of degree 0 and 1 are:

$$\left(UT_3(F)_{(0,1,0)}\right)_0 = Fe_{11} \oplus Fe_{22} \oplus Fe_{33} \oplus Fe_{13}, \quad \text{and} \quad \left(UT_3(F)_{(0,1,0)}\right)_1 = Fe_{12} \oplus Fe_{23}.$$

According to Theorem 4.8, on $C = UT_3(F)_{(0,1,0)}$ it is possible to define only the pseudo-symplectic pseudoinvolution $\bar{\circ} = \circ\Phi$ given by

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}^{\bar{\circ}} = \begin{pmatrix} f & ie & c \\ 0 & d & ib \\ 0 & 0 & a \end{pmatrix}.$$

The four sets of symmetric and skew even elements and of i-symmetric and i-skew odd elements are:

$$C_0^+ = F(e_{11} + e_{33}) \oplus Fe_{22} \oplus Fe_{13}, \quad C_0^- = F(e_{11} - e_{33}), \quad C_1^i = F(e_{12} + e_{23}), \quad C_1^{-i} = F(e_{12} - e_{23}).$$

In the following we compute a basis for the T_2^*-ideal of identities of C. In order to simplify the notation we denote by y_i any variable of homogeneous degree 0 and by z_i any variable of homogeneous degree 1.

Theorem 6.2 *Let $C = \left(UT_3(F)_{(0,1,0)}, \bar{\circ}\right)$ be the superalgebra of 3×3 upper-triangular matrices with elementary $\mathbb{Z}_2$-grading defined by the triple $(0,1,0)$ and endowed with the pseudoinvolution $\bar{\circ}$. Then the T_2^*-ideal of identities of this algebra is generated, as a T_2^*-ideal, by the following polynomials:*

1. $[y_1^+, y_2^+]$,
2. $[y_1^-, y_2^-]$,
3. $y_1^+ y_2^- y_3^+ - y_3^+ y_2^- y_1^+$,
4. $[y_1^+, y_2^-][y_3^+, y_4^-]$,
5. $z[y_1^+, y_2^-]$,
6. $[y_1^+, y_2^-]z$,

7. $y_1^- z y_2^-$,

8. $z_1 y^- z_2$,

9. $z_1^i y^+ z_2^i - z_2^i y^+ z_1^i$,

10. $z_1^{-i} y^+ z_2^{-i} - z_2^{-i} y^+ z_1^{-i}$,

11. $z_1^i y^+ z_2^{-i} + z_2^{-i} y^+ z_1^i$,

12. $[z_1^i, z_2^i]$,

13. $[z_1^{-i}, z_2^{-i}]$,

14. $z^i z^{-i} + z^{-i} z^i$,

15. $z_1 z_2 z_3$.

Proof Let J be the T_2^*-ideal generated by the above polynomials. It is easy to prove that $J \subseteq \mathrm{Id}^*(C)$.

In order to prove the opposite inclusion, first we find a set of generators of P_n^* modulo $P_n^* \cap J$, for all $n \geq 1$. To this end, let us consider $P_{n_1,\dots,n_4}$, where $n_1 + \cdots + n_4 = n$. Since $z_1 z_2 z_3 \equiv 0$, then it must be $n_3 + n_4 \leq 2$. Thus we have to consider three cases.

Case 1: $n_3 = n_4 = 0$.
By the Poincaré-Birkhoff-Witt theorem, any polynomial can be written as a linear combination of products of the type

$$y_{j_1}^+ \cdots y_{j_p}^+ y_{k_1}^- \cdots y_{k_q}^- w_1 \cdots w_m,$$

where $w_1, \dots, w_m$ are left normed commutators in the y_i^+s and y_i^-s, $j_1 < j_2 < \cdots < j_p$ and $k_1 < k_2 < \cdots < k_q$. By the identities $1. - 4.$ plus the identities

$$[y_1^+, y^-, y_2^+] \equiv 0, \qquad [y^+, y_1^-, y_2^-] - [y^+, y_2^-, y_1^-] \equiv 0, \qquad [y^+, y_2^-, y_1^-] + 2y_1^-[y^+, y_2^-] \equiv 0,$$

it is not difficult to prove that $P_{n_1,n_2,0,0}$ is generated modulo $P_{n_1,n_2,0,0} \cap J$ by the polynomials

$$y_1^+ \cdots y_{n_1}^+ y_1^- \cdots y_{n_2}^- \qquad \text{and} \qquad y_1^+ \cdots \widehat{y_l^+} \cdots y_{n_1}^+ [y_l^+, y_1^-, \dots, y_{n_2}^-], \tag{6.1}$$

where $1 \leq l \leq n_1$ and $\widehat{y_l^+}$ means that the variable y_l^+ is omitted. We next show that these polynomials are linearly independent modulo $\mathrm{Id}^*(C)$. To this end, let $f \in \mathrm{Id}^*(C)$ be a linear combination of the above polynomials and write

$$f \equiv \alpha y_1^+ \cdots y_{n_1}^+ y_1^- \cdots y_{n_2}^- + \sum_{l=1}^{n_1} \beta_l y_1^+ \cdots \widehat{y_l^+} \cdots y_{n_1}^+ [y_l^+, y_1^-, \dots, y_{n_2}^-] \ (\text{mod J}). \tag{6.2}$$

By making the evaluation $y_i^+ = e_{11} + e_{33}$, $i = 1, \dots, n_1$ and $y_j^- = e_{11} - e_{33}$, $1 \leq j \leq n_2$, one gets $\alpha(e_{11} \pm e_{33}) = 0$ and so $\alpha = 0$. Now, for any $1 \leq l \leq n_1$, consider the evaluation

$$y_i^+ = e_{11} + e_{33},\ i \in \{1, \dots, n_1\} \setminus \{l\}, \qquad y_l^+ = e_{13}, \qquad y_m^- = e_{11} - e_{33},\ m \in \{1, \dots, n_2\}.$$

We get $\beta_l(-2)^{n_2} e_{13} = 0$ and so $\beta_l = 0$. Therefore the polynomials in (6.1) are linearly independent modulo $P_{n_1,n_2,0,0} \cap \mathrm{Id}^*(C)$ and we are done in this case.

Case 2: $n_3 + n_4 = 1$. We suppose that $n_3 = 1$, $n_4 = 0$ (the case $n_3 = 0$, $n_4 = 1$ is analogous).
Since we are considering monomials in which a z^i appears, according to the identities 5. and 6. we can order also the variables y^+'s and y^-'s, both on the left and on the right. But by identity 7., it is not possible to have a variable y^- both on the left and on the right of the z^i. By making use also of the identities 1. and 2., we get that each monomial of $P_{n_1,n_2,1,0}$ can be written modulo J as

$$y_{i_1}^+ \cdots y_{i_r}^+ y_1^- \cdots y_{n_2}^- z^i y_{j_1}^+ \cdots y_{j_s}^+ \qquad \text{or} \qquad y_{i_1}^+ \cdots y_{i_r}^+ z^i y_{j_1}^+ \cdots y_{j_s}^+ y_1^- \cdots y_{n_2}^-, \tag{6.3}$$

where $r+s=n_1$, $i_1<i_2<\cdots<i_r$ and $j_1<j_2<\cdots<j_s$. So the monomials above span $P_{n_1,n_2,1,0}$ modulo J. Let us show that they are linearly independent modulo $\mathrm{Id}^*(C)$. To this end, let $f\in\mathrm{Id}^*(C)$ be a linear combination of these monomials and write

$$f\equiv\sum_{l,I,J}\alpha_{l,I,J}\,y^+_{i_1}\cdots y^+_{i_r}y^-_1\cdots y^-_{n_2}z^i_l y^+_{j_1}\cdots y^+_{j_s}+\sum_{m,H,K}\beta_{m,H,K}\,y^+_{h_1}\cdots y^+_{h_t}z^+_m y^+_{k_1}\cdots y^+_{k_u}y^-_1\cdots y^-_{n_2}\ (\mathrm{mod\ J}),$$

with $r+s=t+u=n_1$, $I=\{i_1,\dots,i_r\}$, $J=\{j_1,\dots,j_s\}$, $H=\{h_1,\dots,h_t\}$, $K=\{k_1,\dots,k_u\}$ and $i_1<\cdots<i_r$, $j_1<\cdots<j_s$, $h_1<\cdots<h_t$, $k_1<\cdots<k_u$. Suppose that there exists $\alpha_{l,I,J}\neq 0$ or $\beta_{l,I,J}\neq 0$ for some l, I and J. The evaluation $y^+_{i_a}=e_{11}+e_{33}$, $a=1,\dots,r$, $y^-_b=e_{11}-e_{33}$, $b=1,\dots,n_2$, $z^i_l=e_{12}+e_{23}$, $y^+_{i_c}=e_{22}$, $c=1,\dots,s$, gives $\alpha_{l,I,J}e_{12}\pm\beta_{l,J,I}e_{23}=0$. Thus $\alpha_{l,I,J}=\beta_{l,J,I}=0$, a contradiction. Therefore the monomials in (6.3) are linearly independent modulo $P_{n_1,n_2,1,0}\cap\mathrm{Id}^*(C)$.

Case 3: $n_3+n_4=2$. We consider $n_3=2$, $n_4=0$ (the cases $n_3=0$, $n_4=2$ and $n_3=n_4=1$ are analogous). In this case, by using identities 1., 2., 5., 6. plus the identities from 8. to 12. (notice that, instead of identity 12. in the case $n_3=0$, $n_4=2$ we will use identity 13. and in the case $n_3=n_4=1$ we will use identity 14.), each monomial in $P_{n_1,n_2,2,0}$ can be written modulo J as

$$y^+_{i_1}\cdots y^+_{i_t}y^-_1\cdots y^-_k z^i_1 y^+_{j_1}\cdots y^+_{j_s}z^i_2 y^+_{i_{t+1}}\cdots y^+_{i_{n_1-s}}y^-_{k+1}\cdots y^-_{n_2},\tag{6.4}$$

where $i_1<\cdots<i_t<i_{t+1}<\cdots<i_{n_1-s}$ and $j_1<\cdots<j_s$. Thus these monomials span $P_{n_1,n_2,2,0}$ modulo J. Let us show now that they are linearly independent modulo $\mathrm{Id}^*(C)$. To this end, let $f\in\mathrm{Id}^*(C)$ be a linear combination of the above monomials and write

$$f\equiv\sum_{\substack{a,b,I,J\\ a<b}}\alpha_{a,b,I,J}\,y^+_{i_1}\cdots y^+_{i_t}y^-_1\cdots y^-_k z^i_a y^+_{j_1}\cdots y^+_{j_s}z^i_b y^+_{i_{t+1}}\cdots y^+_{i_{n_1-s}}y^-_{k+1}\cdots y^-_{n_2},\tag{6.5}$$

where $I=\{i_1,\dots,i_{n_1-s}\}$, $J=\{j_1,\dots,j_s\}$ and $i_1<\cdots<i_t<i_{t+1}<\cdots<i_{n_1-s}$, $j_1<\cdots<j_s$.
Suppose that $\alpha_{a,b,I,J}\neq 0$ for some $a<b$, I and J. The evaluation $y^+_{i_h}=e_{11}+e_{33}$, $h=1,\dots,n_1-s$, $y^-_k=e_{11}-e_{33}$, $k=1,\dots,n_2$, $z^i_a=z^i_b=e_{12}+e_{23}$, $y^+_{i_l}=e_{22}$, $l=1,\dots,s$, gives $\pm\alpha_{a,b,I,J}e_{13}=0$. Therefore $\alpha_{a,b,I,J}=0$, a contradiction. In conclusion the monomials in (6.4) are linearly independent modulo $P_{n_1,n_2,2,0}\cap\mathrm{Id}^*(C)$.

By putting together the previous results, we get that the polynomials in (6.1), (6.3) and (6.4) are linearly independent modulo $P^*_n\cap\mathrm{Id}^*(C)$, $n=n_1+\cdots+n_4$, and since $P^*_n\cap\mathrm{Id}^*(C)\supseteq P^*_n\cap J$, they form a basis of P^*_n $(\mathrm{mod}\ P^*_n\cap\mathrm{Id}^*(C))$ and so $J=\mathrm{Id}^*(C)$. □

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