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Depth and Stanley depth of the quotient rings of edge ideals of some lobster trees and unicyclic graphs

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Abstract: We compute depth and Stanley depth of the quotient rings of the edge ideals associated with different classes of graphs. These classes include some lobster trees and unicyclic graphs. We show that the values of depth and Stanley depth are equal for the classes we considered.

Key words: Depth, Stanley depth, Stanley decomposition, monomial ideal, edge ideal, lobster tree, unicyclic graph

1. Introduction

Let $S := K[x_1, \dots, x_r]$ be the polynomial ring over a field K and M be a finitely generated $\mathbb{Z}^r$-graded S-module. The K-subspace of M denoted by $\eta K[D]$ is the subspace generated by monomials of the type ηv, where η is a homogeneous element of M and v is a monomial in $K[D]$ and $D \subset \{x_1, x_2, \dots, x_r\}$. The space $\eta K[D]$ is called a Stanley space of dimension $|D|$ if it is a free $K[D]$-module, here $|D|$ refers to the number of variables in D. A Stanley decomposition is the presentation of K-vector space M as a finite direct sum of Stanley spaces

$$\mathcal{D} : M = \bigoplus_{b=1}^{h} \eta_b K[D_b].$$

The Stanley depth of decomposition $\mathcal{D}$ is $\operatorname{sdepth}\mathcal{D} = \min\{|D_b|, b = 1, \dots, h\}$. The Stanley depth of M is

$$\operatorname{sdepth}(M) = \max\{\operatorname{sdepth}\mathcal{D} : \mathcal{D} \text{ is a Stanley decomposition of } M\}.$$

If $\mathfrak{m} := (x_1, x_2, \dots, x_r)$, then $(S, \mathfrak{m})$ is Noetherian $\mathbb{Z}^r$-graded local ring. The *depth* of M is defined to be the common length of all maximal M-sequences in $\mathfrak{m}$. Equivalently, $\operatorname{depth}(M) = \min\{i : \operatorname{Ext}^i(K, M) \neq 0\}$. In [16], Stanley proposed the following conjecture for finitely generated $\mathbb{Z}^r$-graded S-modules, given by $\operatorname{depth}(M) \leq \operatorname{sdepth}(M)$, known as the Stanley's conjecture. This conjecture was proved for several special cases; see for instance [2, 3, 13]. But later Duval et al. showed in [7], that this inequality does not hold in general for modules of type S/I, where I is a monomial ideal. In [9] Herzog, Vladoiu and Zheng proved that the Stanley depth of a module can be computed in a finite number of steps using posets, when a $\mathbb{Z}^r$-graded S-module M is of the type $M = I_1/I_2$, where $I_2 \subset I_1 \subset S$ are monomial ideals. It is worth mentioning that the method of Herzog et

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al. for computing the Stanley depth of I_1/I_2 is a hard combinatorial problem, in general. Therefore, there are a few classes of modules whose Stanley depth is known; see, for instance, [1, 6, 10]. In most of the cases we have some bounds for Stanley depth and most of the existing bounds are weak, in general. So the objective of this article is to compute the precise values of depth and Stanley depth of the quotient ring of edge ideals associated with some classes of graphs. We compute the precise values of depth and Stanley depth of the quotient ring of edge ideals associated with some classes of lobster trees; see Theorems 3.3 and 3.4 of this paper. Moreover, by using Theorems 3.3 and 3.4 we compute the precise values of depth and Stanley depth of the quotient ring of edge ideals associated with some unicyclic graphs; see Theorems 4.1 and 4.2 of this paper. We observe that the Stanley's inequality hold for modules associated with these classes of graphs.

2. Definitions and notation

Let $G=(V_G,E_G)$ be a graph with vertex set $V_G=\{x_1,\dots,x_q\}$ and edge set E_G. The *edge ideal* I of the graph G is the ideal generated by monomials of the type x_ix_j, where $\{x_i,\,x_j\}\subset E_G$. Let $I\subset S$ be a monomial ideal, then $G(I)$ denotes the minimal set of monomial generators of I. For any monomial η, $\text{supp}(\eta):=\{x_j : x_j|\eta\}$ and for an ideal I generated by monomials, $\text{supp}(I)=\{x_j : x_j|\upsilon,\ \text{for some}\ \upsilon\in G(I)\}$. If there are no loops or multiple edges in a graph, it is called a *simple graph.* Throughout this article, all considered graphs are simple. For $q\geq 1$, a graph P_q with vertex set $V_{P_q}=\{x_1,\,x_2,\,\dots,\,x_q\}$ and an edge set $E_{P_q}=\{x_1x_2,\,x_2x_3,\,\dots,\,x_{q-1}x_q\}$ ($E_{P_q}=\emptyset$, if $q=1$) is called a *path* of length $q-1$ denoted by P_q. For $q\geq 3$, a *cycle* on q vertices denoted by C_q is a graph with vertex set and edge set, $V_{C_q}=\{x_1,\,x_2,\,\dots,\,x_q\}$ and $E_{C_q}=\{x_1x_2,\,x_2x_3,\dots,\,x_{q-1}x_q,\,x_1x_q\}$, respectively. A graph is said to be *connected* if there is a path between any pair of its vertices. A *unicyclic graph* is a connected graph containing exactly one cycle. If an edge connects two vertices then the vertices are said to be *neighbours* of each other. The *degree* of a vertex x in a graph is the total number of its neighbours and it is denoted as $d_G(x)$. A vertex x with $d_G(x)\geq 2$ is called an *internal vertex.* A vertex of degree one is called a *leaf* (or *pendant vertex*). A simple and connected graph is called a *tree* if there exists a unique path between any two vertices. For $q\geq 2$, a q*-star* denoted by S_q is a tree with $(q-1)$-leaves and a single vertex with degree $q-1$. A *caterpillar* is a tree with the property that the removal of pendant vertices leaves a path. A *lobster tree* is a tree with the property that the removal of pendant vertices leaves a caterpillar.

Definition 2.1 *Let $q,m,h\geq 1$ and P_q be a path on q vertices with vertex set $\{x_1,x_2,\dots,x_q\}$. We define a lobster tree by attaching m vertices $x_{i,f}$ at each vertex x_i, where $f=1,\dots,m$, and then attaching h pendant vertices $x_{i,f,k}$ at each vertex $x_{i,f}$, where $k=1,\dots,h$. We denote this lobster tree by $P_{q,m,h}$.*

For examples of $P_{q,m,h}$ see Figures 1–3.

Definition 2.2 *Let $q\geq 3$, $m,\,h\geq 1$ and C_q be a cycle on q vertices. We define a unicyclic graph by attaching m vertices $x_{i,f}$ at each vertex x_i of C_q where $f=1,\dots,m$, and then attaching h pendant vertices $x_{i,f,k}$ at each vertex $x_{i,f}$, where $k=1,\dots,h$. This unicyclic graph is denoted by $C_{q,m,h}$.*

For examples of $C_{q,m,h}$ see Figure 4.

Remark 2.3 $E_{C_{q,m,h}}=E_{P_{q,m,h}}\cup\{x_q,x_1\}$. *Thus* $|\mathcal{G}(I(P_{q,m,h})|=mq+q+mqh-1$ *and* $|\mathcal{G}(I(C_{q,m,h})|=mq+q+mqh$.

Let $q, m, h \geq 1$ and $P_{q,m,h}$ be a lobster tree on $q+mq+mqh$ vertices. Then the edge ideal $I_{q,m,h} = I(P_{q,m,h})$ is given by

$$I_{q,m,h} = (x_1x_{1,1}, x_1x_{1,2}, \ldots, x_1x_{1,m}, x_{1,1}x_{1,1,1}, x_{1,1}x_{1,1,2}, \ldots, x_{1,1}x_{1,1,h}, \ldots, x_{1,m}x_{1,m,1}, \ldots,$$
$$x_{1,m}x_{1,m,h}, \ldots, x_ix_{i+1}, x_{i+1}x_{i+2}, \ldots, x_{q-1}x_q, x_qx_{q,1}, x_qx_{q,2}, \ldots, x_qx_{q,m}, x_{q,1}x_{q,1,1}$$
$$, x_{q,1}x_{q,1,2}, \ldots, x_{q,1}x_{q,1,m}, \ldots, x_{q,m}x_{q,m,1}, x_{q,m}x_{q,m,2}, \ldots, x_{q,m}x_{q,m,h}).$$

Let $q \geq 3$, $m, h \geq 1$ and $C_{q,m,h}$ be a unicyclic graph on $q+mq+mqh$ vertices. Then the edge ideal $I'_{q,m,h} = I(C_{q,m,h})$ is given by

$$I'_{q,m,h} = (I_{q,m,h}, x_1x_q).$$

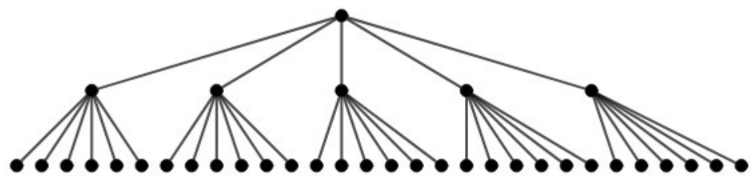

Figure 1. $P_{1,5,6}$.

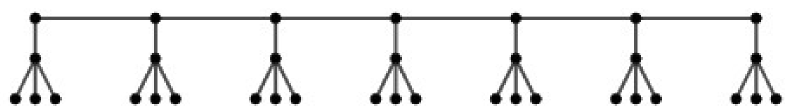

Figure 2. $P_{7,1,3}$.

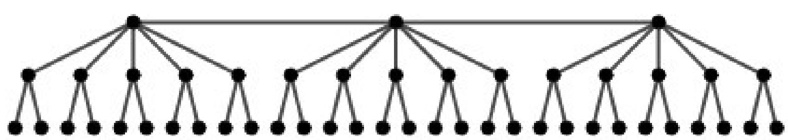

Figure 3. $P_{3,5,2}$.

Lemma 2.4 ([12], Lemma 2.4) Let $0 \to N_1 \to N_2 \to N_3 \to 0$ be a short exact sequence of $\mathbb{Z}^r$-graded S-modules. Then

$$\text{sdepth}(N_2) \geq \min\{\text{sdepth}(N_1), \text{sdepth}(N_3)\}.$$

Lemma 2.5 ([4]) (Depth Lemma) If $0 \to N_1 \to N_2 \to N_3 \to 0$ is a short exact sequence of modules over a local ring S, or a Noetherian graded ring with S_0 local, then

1. $\text{depth}(N_2) \geq \min\{\text{depth}(N_3), \text{depth}(N_1)\}$.
2. $\text{depth}(N_1) \geq \min\{\text{depth}(N_2), \text{depth}(N_3)+1\}$.

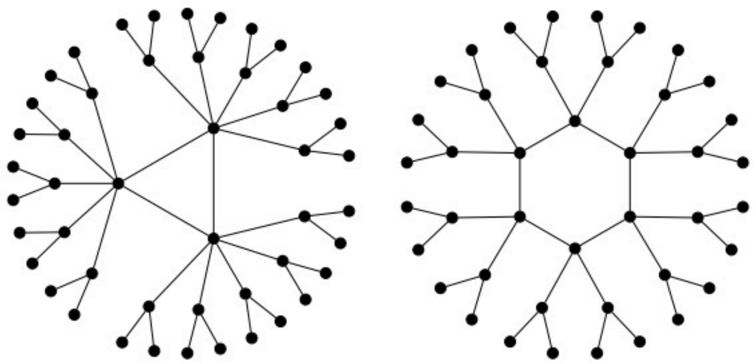

Figure 4. From left to right $C_{3,5,2}$ and $C_{6,2,2}$.

3. $\operatorname{depth}(N_3) \geq \min\{\operatorname{depth}(N_1)-1, \operatorname{depth}(N_2)\}$.

Proposition 2.6 ([1]) Let $r \geq 2$. If $I = I(S_r)$, then $\operatorname{sdepth}(S/I) = \operatorname{depth}(S/I) = 1$.

Lemma 2.7 ([11], Lemma 2.8) Let $r \geq 2$. If $I = I(P_r)$, then $\operatorname{depth}(S/I) = \lceil \frac{r}{3} \rceil$.

Proposition 2.8 Let $I \subset S$ be a monomial ideal and for any monomial $v \notin I$, we have

1. $\operatorname{depth}_S(S/(I:v)) \geq \operatorname{depth}_S(S/I)$, ([15, Corollary 1.3]).
2. $\operatorname{sdepth}_S(S/(I:v)) \geq \operatorname{sdepth}_S(S/I)$, ([5, Proposition 2.7]).

Lemma 2.9 ([9], Lemma 3.6) Let $I \subset S$ be a monomial ideal. If $S' := S \otimes_K K[x_{r+1}]$, then $\operatorname{depth}(S'/IS') = \operatorname{depth}(S/I)+1$ and $\operatorname{sdepth}(S'/IS') = \operatorname{sdepth}(S/I)+1$.

Lemma 2.10 ([6], Lemma 2.12) Let $I' \subset S' = K[x_1, \dots, x_v]$ and $I'' \subset S'' = K[x_{v+1}, \dots, x_r]$ be the monomial ideals, where $1 \leq v < r$. Then $\operatorname{depth}_S(S'/I' \otimes_K S''/I'') = \operatorname{depth}_S(S/(I'S+I''S)) = \operatorname{depth}_{S'}(S'/I') + \operatorname{depth}_{S''}(S''/I'')$.

Lemma 2.11 ([6], Lemma 2.13) Let $I' \subset S' = K[x_1, \dots, x_v]$ and $I'' \subset S'' = K[x_{v+1}, \dots, x_r]$ be the ideals generated by monomials, where $1 \leq v < r$. Then

$$\operatorname{sdepth}_S(S/(I'S + I''S)) = \operatorname{sdepth}_S(S'/I' \otimes_K S''/I'') \geq \operatorname{sdepth}_{S'}(S'/I') + \operatorname{sdepth}_{S''}(S''/I'').$$

Lemma 2.12 ([10], Lemma 3.3) Let $I \subset S$ be a square-free monomial ideal with $\operatorname{supp}(I) = \{x_1, x_2, \dots, x_r\}$. Let $w := x_{i_1}x_{i_2}\cdots x_{i_v} \in S/I$, such that, $x_m w \in I$, for all $x_m \in \{x_1, x_2, \dots, x_r\} \setminus \operatorname{supp}(w)$. Then $\operatorname{sdepth}(S/I) \leq v$.

3. Depth and Stanley depth of cyclic modules associated with some classes of lobster trees

Let $A := \{x_1, \dots, x_q\}$, $B := \cup_{i=1}^{q}\{x_{i,1}, \dots, x_{i,m}\}$ and $C := \cup_{i=1}^{q} \cup_{f=1}^{m} \{x_{i,f,1}, x_{i,f,2}, \dots, x_{i,f,h}\}$. Then $S_{q,m,h} := K[A \cup B \cup C]$. In this section, we compute depth and Stanley depth of a cyclic module $S_{q,m,h}/I_{q,m,h}$. We show that the values of depth and Stanley depth are equal and can be given in terms of q and m, which also proves the Stanley's inequality for this module. We make the following remarks before the proof of our main theorems.

Remark 3.1 While proving our results by induction on q, sometimes we may have the description $S_{0,m,h}/I_{0,m,h}$, in that case we define $S_{0,m,h}/I_{0,m,h} := K$, hence $\text{depth}(S_{0,m,h}/I_{0,m,h}) = \text{sdepth}(S_{0,m,h}/I_{0,m,h}) = 0$.

Remark 3.2 Let I be a squarefree monomial ideal of S minimally generated by monomials of degree at most 2. We associate a graph G_I to the ideal I with $V_{G_I} = \text{supp}(I)$ and $E_{G_I} = \{\{x_i, x_j\} : x_i x_j \in G(I)\}$. Let $x_t \in S$ be a variable of the polynomial ring S such that $x_t \notin I$. Then $(I : x_t)$ and (I, x_t) are monomial ideals of S such that $G_{(I:x_t)}$ and $G_{(I,x_t)}$ are subgraphs of G_I. See Figures 5 and 6 for examples of $G_{(I_{q,m,h}, x_{q,m})}$ and $G_{(I_{q,m,h}:x_{q,m})}$, respectively. For examples of $G_{(I'_{q,m,h}:x_{q,m})}$ and $G_{(I'_{q,m,h}, x_{q,m})}$ see Figure 7. For instance, we have the following isomorphism:

$$S_{5,3,2}/(I_{5,3,2} : x_{5,3}) \cong S_{5,3,2}/I(G_{(I_{5,3,2}:x_{5,3})}) \cong S_{4,3,2}/I_{4,3,2} \otimes_K K[x_{5,1}, x_{5,1,1}, x_{5,1,2}]/(x_{5,1,1}x_{5,1}, x_{5,1}x_{5,1,2})$$
$$\otimes_K K[x_{5,2}, x_{5,2,1}, x_{5,2,2}]/(x_{5,2,1}x_{5,2}, x_{5,2}x_{5,2,2}) \otimes_K K[x_{5,3}, x_{5,3,1}, x_{5,3,2}].$$

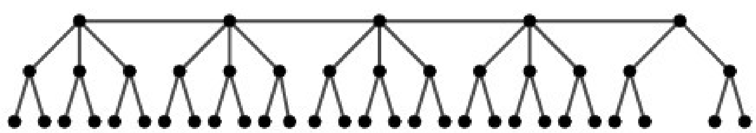

Figure 5. $G_{(I_{5,3,2}, x_{5,3})}$.

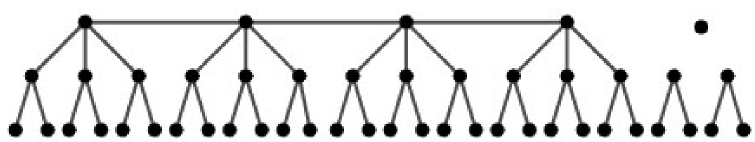

Figure 6. $G_{(I_{5,3,2}:x_{5,3})}$.

Theorem 3.3 *Let* $q, m, h \geq 1$. *Then* $\text{depth}(S_{q,m,h}/I_{q,m,h}) = mq$.

Proof We have the following short exact sequence

$$0 \longrightarrow S_{q,m,h}/(I_{q,m,h} : x_{q,m}) \xrightarrow{\cdot x_{q,m}} S_{q,m,h}/I_{q,m,h} \longrightarrow S_{q,m,h}/(I_{q,m,h}, x_{q,m}) \longrightarrow 0. \quad (3.1)$$

Let $m, h \geq 1$. We have two cases to consider, namely $q = 1$ and $q \geq 2$.

(1) Let $q = 1$. If $m = 1$, then $P_{1,1,h} \cong S_{h+2}$ and by Proposition 2.6 $\text{depth}(S_{1,1,h}/I_{1,1,h}) = 1 = 1 \cdot 1$, as required. If $m \geq 2$, then

$$S_{1,m,h}/(I_{1,m,h} : x_{1,m}) \cong \bigotimes_{f=1}^{m-1}{}_K K[x_{1,f}, x_{1,f,1}, \dots, x_{1,f,h}]/(x_{1,f}x_{1,f,1}, \dots, x_{1,f}x_{1,f,h}) \bigotimes_K K[x_{1,m}].$$

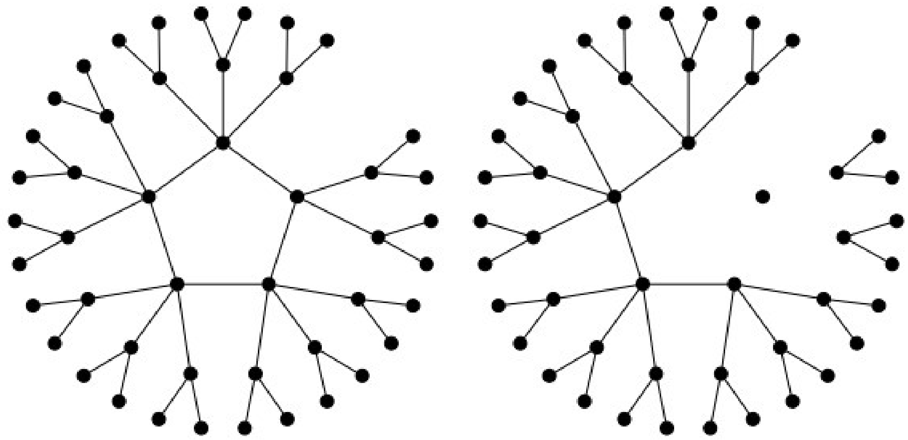

Figure 7. From left to right $G_{(I'_{5,3,2}\,,\,x_{5,3})}$ and $G_{(I'_{5,3,2}\,:\,x_{5,3})}$.

By Lemma 2.10 we have

$$\operatorname{depth}(S_{1,m,h}/(I_{1,m,h}:x_{1,m})) = \sum_{f=1}^{m-1}\operatorname{depth}(K[x_{1,f},x_{1,f,1},\dots,x_{1,f,h}]/(x_{1,f}x_{1,f,1},\dots,x_{1,f}x_{1,f,h})) + \operatorname{depth}(K[x_{1,m}]).$$

Using Proposition 2.6, we get $\operatorname{depth}(S_{1,m,h}/(I_{1,m,h}:x_{1,m})) = 1+\sum_{f=1}^{m-1}1 = 1+m-1 = m$. It is easy to see that $S_{1,m,h}/(I_{1,m,h},x_{1,m}) \cong S_{1,\,m-1,\,h}/I_{1,\,m-1,\,h}\otimes_K K[x_{1,\,m,\,1},x_{1,\,m,\,2},\,\dots,\,x_{1,\,m,\,h}]$. By Lemma 2.10

$$\operatorname{depth}(S_{1,m,h}/(I_{1,m,h},x_{1,m})) = \operatorname{depth}(S_{1,m-1,h}/I_{1,m-1,h}) + \operatorname{depth}(K[x_{1,m,1},x_{1,m,2},\,\dots,\,x_{1,m,h}]).$$

Applying induction on m, we get $\operatorname{depth}(S_{1,m,h}/(I_{1,m,h},x_{1,m})) = m-1+h$. Since

$$\operatorname{depth}(S_{1,m,h}/(I_{1,m,h}:x_{1,m})) \le \operatorname{depth}(S_{1,m,h}/(I_{1,m,h},x_{1,m})).$$

Thus by applying Depth Lemma on Eq. 3.1, $\operatorname{depth}(S_{1,m,h}/I_{1,m,h}) = \operatorname{depth}(S_{1,m,h}/(I_{1,m,h}:x_{1,m})) = m$, this completes the proof when $q=1$.

(2) Let $q\ge 2$. If $m=1$, then $S_{q,1,h}/(I_{q,1,h}:x_{q,1}) \cong S_{q-1,1,h}/I_{q-1,1,h}\otimes_K K[x_{q,1}]$. If $m\ge 2$, then

$$S_{q,m,h}/(I_{q,m,h}:x_{q,m}) \cong S_{q-1,m,h}/I_{q-1,m,h}\bigotimes_K K[x_{q,m}]\bigotimes_K \bigotimes_{f=1}^{m-1}{}_K K[x_{q,f},x_{q,f,1},\dots,x_{q,f,h}]/(x_{q,f}x_{q,f,1},\dots,x_{q,f}x_{q,f,h}).$$

By Lemma 2.10, Proposition 2.6, and induction on q, we have

$$\operatorname{depth}(S_{q,1,h}/(I_{q,1,h}:x_{q,1})) = \operatorname{depth}(S_{q-1,1,h}/I_{q-1,1,h}) + \operatorname{depth}(K[x_{q,1}]) = 1(q-1)+1 = q.$$

If $m \geq 2$, then by Lemma 2.10

$$\operatorname{depth}(S_{q,m,h}/(I_{q,m,h} : x_{q,m})) = \operatorname{depth}(S_{q-1,m,h}/I_{q-1,m,h}) + \operatorname{depth}(K[x_{q,m}]) + \sum_{f=1}^{m-1} \operatorname{depth}(K[x_{q,f}, x_{q,f,1}, \ldots, x_{q,f,h}]/(x_{q,f}x_{q,f,1}, \ldots, x_{q,f}x_{q,f,h})).$$

By Proposition 2.6 and induction on q, we have

$$\operatorname{depth}(S_{q,m,h}/(I_{q,m,h} : x_{q,m})) = m(q-1) + 1 + \sum_{f=1}^{m-1} 1 = m(q-1) + 1 + (m-1) = mq.$$

Thus $\operatorname{depth}(S_{q,m,h}/(I_{q,m,h} : x_{q,m})) = mq$, for all $m \geq 1$. Let $I^*_{q,m,h} := (I_{q,m,h}, x_{q,m})$. We have the following short exact sequence

$$0 \longrightarrow S_{q,m,h}/(I^*_{q,m,h} : x_q) \xrightarrow{\cdot x_q} S_{q,m,h}/I^*_{q,m,h} \longrightarrow S_{q,m,h}/(I^*_{q,m,h}, x_q) \longrightarrow 0. \tag{3.2}$$

$$S_{q,m,h}/(I^*_{q,m,h} : x_q) \cong S_{q-2,m,h}/I_{q-2,m,h} \bigotimes_{f=1}^{m}{}_K K[x_{q,f,1}, \ldots, x_{q,f,h}] \bigotimes_K K[x_q] \bigotimes_K \bigotimes_{f=1}^{m}{}_K K[x_{q-1,f}, x_{q-1,f,1}, \ldots, x_{q-1,f,h}]/(x_{q-1,f}x_{q-1,f,1}, \ldots, x_{q-1,f}x_{q-1,f,h}),$$

and

$$S_{q,m,h}/(I^*_{q,m,h}, x_q) \cong S_{q-1,m,h}/I_{q-1,m,h} \bigotimes_K \bigotimes_{f=1}^{m-1}{}_K K[x_{q,f}, x_{q,f,1}, \ldots, x_{q,f,h}]/(x_{q,f}x_{q,f,1}, \ldots, x_{q,f}x_{q,f,h}) \bigotimes_K K[x_{q,m,1}, \ldots, x_{q,m,h}].$$

Thus by Lemma 2.10

$$\operatorname{depth}(S_{q,m,h}/(I^*_{q,m,h} : x_q)) = \operatorname{depth}(S_{q-2,m,h}/I_{q-2,m,h}) + \sum_{f=1}^{m} \operatorname{depth}(K[x_{q,f,1}, \ldots, x_{q,f,h}]) + \operatorname{depth}(K[x_q]) + \sum_{f=1}^{m} \operatorname{depth}(K[x_{q-1,f}, x_{q-1,f,1}, \ldots, x_{q-1,f,h}]/(x_{q-1,f}x_{q-1,f,1}, \ldots, x_{q-1,f}x_{q-1,f,h})),$$

and

$$\operatorname{depth}(S_{q,m,h}/(I^*_{q,m,h}, x_q)) = \operatorname{depth}(S_{q-1,m,h}/I_{q-1,m,h}) + \sum_{f=1}^{m-1} \operatorname{depth}(K[x_{q,f}, x_{q,f,1}, \ldots, x_{q,f,h}]/(x_{q,f}x_{q,f,1}, \ldots, x_{q,f}x_{q,f,h})) + \operatorname{depth}(K[x_{q,m,1}, \ldots, x_{q,m,h}]).$$

By induction on q and Proposition 2.6, we have

$$\operatorname{depth}(S_{q,m,h}/(I^*_{q,m,h} : x_q)) = m(q-2) + mh + 1 + \sum_{f=1}^{m} 1 = m(q-1) + mh + 1,$$

and

$$\operatorname{depth}(S_{q,m,h}/(I^*_{q,m,h}, x_q)) = m(q-1) + \sum_{f=1}^{m-1} 1 + h = m(q-1) + m - 1 + h = mq - 1 + h.$$

Now applying Depth Lemma on Eqs. 3.1 and 3.2, we have

$$\operatorname{depth}(S_{q,m,h}/I_{q,m,h}) \geq \min\left\{ \operatorname{depth}(S_{q,m,h}/(I_{q,m,h} : x_{q,m})), \operatorname{depth}(S_{q,m,h}/(I^*_{q,m,h} : x_q)), \operatorname{depth}(S_{q,m,h}/(I^*_{q,m,h}, x_q))\right\} = \min\{mq,\, m(q-1) + mh + 1,\, mq - 1 + h\} = mq.$$

For the upper bound using Proposition 2.8 we get $\operatorname{depth}(S_{q,m,h}/I_{q,m,h}) \leq \operatorname{depth}(S_{q,m,h}/(I_{q,m,h} : x_{q,m})) = mq$. This completes the proof. □

Theorem 3.4 *Let* $q, m, h \geq 1$*. Then* $\operatorname{sdepth}(S_{q,m,h}/I_{q,m,h}) = mq$*.*

Proof The proof of the inequality, $\operatorname{sdepth}(S_{q,m,h}/I_{q,m,h}) \geq mq$, is similar to the proof of the same inequality for depth in Theorem 3.3. We have to use Lemma 2.11 instead of Lemma 2.10 and to apply Lemma 2.4 instead of Depth Lemma on Eqs. 3.1 and 3.2. It remains to show that $\operatorname{sdepth}(S_{q,m,h}/I_{q,m,h}) \leq mq$. Since $w = x_{1,1}x_{1,2}\cdots x_{1,m}x_{2,1}x_{2,2}\cdots x_{2,m}\cdots x_{q,1}x_{q,2}\cdots x_{q,m} \in S_{q,m,h}/I_{q,m,h}$, and $x_l w \in I_{q,m,h}$, for all $x_l \in V_{P_{q,m,h}} \setminus \operatorname{supp}(w)$, therefore by using Lemma 2.12, we have $\operatorname{sdepth}(S_{q,m,h}/I_{q,m,h}) \leq mq$. This completes the proof. □

Corollary 3.5 *Stanley's inequality holds for cyclic module* $S_{q,m,h}/I_{q,m,h}$*.*

4. Depth and Stanley depth of cyclic modules associated with some classes of unicyclic graphs

In this section, we use the values of depth and Stanley depth of cyclic module $S_{q,m,h}/I_{q,m,h}$ for finding the values of depth and Stanley depth of cyclic module $S_{q,m,h}/I'_{q,m,h}$. We also show that the values of depth and Stanley depth are equal which proves the Stanley's inequality for the cyclic module $S_{q,m,h}/I'_{q,m,h}$.

Theorem 4.1 *Let* $q \geq 3$ *and* $m, h \geq 1$*. Then* $\operatorname{depth}(S_{q,m,h}/I'_{q,m,h}) = mq$*.*

Proof We have the following short exact sequence

$$0 \longrightarrow S_{q,m,h}/(I'_{q,m,h} : x_{q,m}) \xrightarrow{\cdot x_{q,m}} S_{q,m,h}/I'_{q,m,h} \longrightarrow S_{q,m,h}/(I'_{q,m,h}, x_{q,m}) \longrightarrow 0. \tag{4.1}$$

Clearly, $S_{q,m,h}/(I'_{q,m,h} : x_{q,m}) = S_{q,m,h}/(I_{q,m,h} : x_{q,m})$, thus by Theorem 3.3 $\operatorname{depth}(S_{q,m,h}/(I'_{q,m,h} : x_{q,m})) = \operatorname{depth}(S_{q,m,h}/(I_{q,m,h} : x_{q,m})) = mq$. Let $I^{**}_{q,m,h} := (I'_{q,m,h}, x_{q,m})$. We consider another short exact sequence

$$0 \longrightarrow S_{q,m,h}/(I^{**}_{q,m,h} : x_q) \xrightarrow{\cdot x_q} S_{q,m,h}/I^{**}_{q,m,h} \longrightarrow S_{q,m,h}/(I^{**}_{q,m,h}, x_q) \longrightarrow 0. \tag{4.2}$$

We have the following isomorphism

$$S_{q,m,h}/(I^{**}_{q,m,h} : x_q) \cong S_{q-3,m,h}/I_{q-3,m,h} \bigotimes_K \bigotimes_{f=1}^{m} {}_K K[x_{1,f}, x_{1,f,1}, \dots, x_{1,f,h}]/(x_{1,f}x_{1,f,1}, \dots, x_{1,f}x_{1,f,h}) \bigotimes_K \bigotimes_{f=1}^{m-1} {}_K K[x_{q-1,f}, x_{q-1,f,1}, \dots, x_{q-1,f,h}]/(x_{q-1,f}x_{q-1,f,1}, \dots, x_{q-1,f}x_{q-1,f,h}) \bigotimes_{f=1}^{m} {}_K K[x_{q,f,1}, \dots, x_{q,f,h}] \bigotimes_K K[x_q],$$

by Lemma 2.10

$$\begin{aligned}\operatorname{depth}(S_{q,m,h}/(I^{**}_{q,m,h} : x_q)) = {} & \operatorname{depth}(S_{q-3,m,h}/I_{q-3,m,h}) + \\ & + \sum_{f=1}^{m} \operatorname{depth}(K[x_{1,f}, x_{1,f,1}, \dots, x_{1,f,h}]/(x_{1,f}x_{1,f,1}, \dots, x_{1,f}x_{1,f,h})) \\ & + \sum_{f=1}^{m} \operatorname{depth}(K[x_{q-1,f}, x_{q-1,f,1}, \dots, x_{q-1,f,h}]/(x_{q-1,f}x_{q-1,f,1}, \dots, x_{q-1,f}x_{q-1,f,h})) \\ & + \sum_{f=1}^{m} \operatorname{depth}(K[x_{q,f,1}, \dots, x_{q,f,h}]) + \operatorname{depth}(K[x_q]).\end{aligned}$$

Using Theorem 3.3 and Proposition 2.6 we have

$$\operatorname{depth}(S_{q,m,h}/(I^{**}_{q,m,h} : x_q)) = (q-3)m + \sum_{f=1}^{m} 1 + \sum_{f=1}^{m} 1 + mh + 1 = m(q-1) + mh + 1.$$

Depending the values of m we have the following isomorphisms:

$$S_{q,1,h}/(I^{**}_{q,1,h}, x_q) \cong S_{q-1,1,h}/I_{q-1,1,h} \otimes_K K[x_{q,1,1}, \dots, x_{q,1,h}],$$

and for $m \geq 2$,

$$S_{q,m,h}/(I^{**}_{q,m,h}, x_q) \cong S_{q-1,m,h}/I_{q-1,m,h} \bigotimes_K \bigotimes_{f=1}^{m-1} {}_K K[x_{q,f}, x_{q,f,1}, \dots, x_{q,f,h}]/(x_{q,f}x_{q,f,1}, \dots, x_{q,f}x_{q,f,h}) \bigotimes_K K[x_{q,m,1}, \dots, x_{q,m,h}].$$

Again by Lemma 2.10,

$$\operatorname{depth}(S_{q,1,h}/(I^{**}_{q,1,h}, x_q)) = \operatorname{depth}(S_{q-1,1,h}/I_{q-1,1,h}) + \operatorname{depth}(K[x_{q,m,1}, \ldots, x_{q,m,h}]),$$

and for $m \geq 2$,

$$\operatorname{depth}(S_{q,m,h}/(I^{**}_{q,m,h}, x_q)) = \operatorname{depth}(S_{q-1,m,h}/I_{q-1,m,h}) + \sum_{f=1}^{m-1} \operatorname{depth}(K[x_{q,f}, x_{q,f,1}, \ldots, x_{q,f,h}]/(x_{q,f}x_{q,f,1}, \ldots, x_{q,f}x_{q,f,h})) + \operatorname{depth}(K[x_{q,m,1}, \ldots, x_{q,m,h}]).$$

From Theorem 3.3 and Proposition 2.6, we have

$$\operatorname{depth}(S_{q,1,h}/(I^{**}_{q,1,h}, x_q)) = (q-1) + h = q - 1 + h,$$

and for $m \geq 2$,

$$\operatorname{depth}(S_{q,m,h}/(I^{**}_{q,m,h}, x_q)) = (q-1)m + \sum_{f=1}^{m-1} 1 + h = mq - 1 + h.$$

Thus we have $\operatorname{depth}(S_{q,m,h}/(I^{**}_{q,m,h}, x_q)) = mq - 1 + h$, for all $m \geq 1$. Now applying Depth Lemma on Eqs. 4.1 and 4.2, we have

$$\operatorname{depth}(S_{q,m,h}/(I'_{q,m,h}, x_{q,m})) \geq \min\big\{ \operatorname{depth}(S_{q,m,h}/(I'_{q,m,h} : x_{q,m})), \operatorname{depth}(S_{q,m,h}/(I^{**}_{q,m,h} : x_q)), \operatorname{depth}(S_{q,m,h}/(I^{**}_{q,m,h}, x_q))\big\} = \min\{mq, m(q-1) + mh + 1, mq - 1 + h\} = mq.$$

Now by Proposition 2.8 we have $\operatorname{depth}(S_{q,m,h}/I'_{q,m,h}) \leq \operatorname{depth}(S_{q,m,h}/(I'_{q,m,h} : x_{q,m})) = mq$. This completes the proof. □

Theorem 4.2 *Let $q \geq 3$ and $m, h \geq 1$. Then* $\operatorname{sdepth}(S_{q,m,h}/I'_{q,m,h}) = mq$.

Proof We use Lemma 2.11 and Theorem 3.4 instead of Lemma 2.10 and Theorem 3.3, respectively and repeat the proof of Theorem 4.1. Then applying Lemma 2.4 instead of Depth Lemma on Eqs. 4.1 and 4.2 we get $\operatorname{sdepth}(S_{q,m,h}/I'_{q,m,h}) \geq mq$. Since $w = x_{1,1}x_{1,2} \cdots x_{1,m}x_{2,1}x_{2,2} \cdots x_{2,m} \cdots x_{q,1}x_{q,2} \cdots x_{q,m} \in S_{q,m,h}/I'_{q,m,h}$ and $x_l w \in I'_{q,m,h}$, for all $x_l \in V_{Cq,m,h} \backslash \operatorname{supp}(w)$, thus by Lemma 2.12 we have $\operatorname{sdepth}(S_{q,m,h}/I'_{q,m,h}) \leq mq$. This completes the proof.

□

Corollary 4.3 *Stanley's inequality holds for cyclic module $S_{q,m,h}/I'_{q,m,h}$.*

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