Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 1897 – 1910
© TÜBİTAK
doi:10.55730/1300-0098.3240

On almost s-weakly regular rings

Kanchan JANGRA*, **Dinesh UDAR**
Department Of Applied Mathematics, Delhi Technological University, Delhi, India

Received: 21.08.2021 • **Accepted/Published Online:** 14.01.2022 • **Final Version:** 20.06.2022

Abstract: An element a of R is called s-weakly regular (SWR) if $a \in aRa^2R$. A ring R is called an almost SWR if for any $a \in R$, either a or $1-a$ is SWR. In this paper, we introduce almost SWR rings as the generalization of abelian von Neumann local (VNL) rings and SWR rings. We provide various properties and characterizations of almost SWR rings. We discuss various extension rings to be almost SWR. Further, we discuss SWR group rings and almost SWR group rings.

Key words: Strongly regular rings, VNL-rings, s-weakly regular rings, almost s-weakly regular rings, group rings

1. Introduction

Throughout this paper, R is assumed to be an associative ring with identity $1 \neq 0$, and all modules are unitary unless otherwise stated. We denote the ring of integers modulo n by $\mathbb{Z}_n$ and cyclic group of order n by $\mathbb{C}_n$. The symbol $J(R)$ denotes the Jacobson radical of R. We denote the ring of all upper triangular $n \times n$ matrices over a ring R by symbol $\mathbb{T}_n(R)$. The ring of formal power series in indeterminate x over a ring R is denoted by $R[[x]]$. For a nonempty subset X of R, $l(X)$ and $r(X)$ stand for left and right annihilator of X, respectively. For some usual notations, we refer to [10] and [14].

An element $a \in R$ is (strongly) regular if there exists an element $b \in R$ such that $a = aba$ $(ab = ba)$. A ring R is called (strongly) regular if every element of R is (strongly) regular. Camillo and Xiao [3] investigated weakly regular rings. A ring R is called right (left) weakly regular if for every element $a \in R$, $a \in aRaR(RaRa)$. A ring R is weakly regular if it is both right and left weakly regular. As a generalization of strongly regular rings, in [12], Gupta introduced SWR rings. An element $a \in R$ is called SWR if $a \in aRa^2R$. A ring R is said to be SWR if every element of R is SWR. The class of SWR rings lies strictly between the class of right (or left) weakly regular rings and strongly regular rings. A ring R is local if and only if for any $a \in R$, either a or $1-a$ is invertible. Contessa in [8], as a common generalization of regular rings and local rings, introduced von Neumann local (VNL) rings for commutative rings. A ring R is called VNL if for any $a \in R$, either a or $1-a$ is regular. VNL-rings for noncommutative rings were studied by Chen and Tong [6]. Moreover, Grover and Khurana [11] characterized VNL-rings in the sense of relating them to some other familiar classes of rings. For more information about VNL-rings and their related rings, one can see [5, 6, 8] and [17].

The concept of SWR rings together with the notion of local rings gives motivation for the paper. In the present paper, we discuss those elements where either a or $1-a$ is SWR. We introduce a new class of rings

*Correspondence: kanchanjangra24@gmail.com
2010 *AMS Mathematics Subject Classification:* 16E50, 16S34, 16U99

called almost SWR rings. The class of almost SWR rings is a proper generalization of the class of abelian VNL rings and SWR rings. This paper is motivated by papers [6, 12].

Definition 1.1 *A ring R is said to be an almost SWR ring if for any $a \in R$, either a or $1-a$ is SWR.*

In Section 2, we prove various properties of almost SWR rings, and some examples are provided to show that the class of almost SWR rings properly contains the classes of SWR, abelian VNL and weakly tripotent rings. A two sided ideal I in a ring R is said to be SWR ideal if each of its elements is SWR. We prove that a ring R is almost SWR if and only if, for any SWR ideal I of R, R/I is almost SWR. We characterize abelian almost SWR rings. It is proved that if e is an idempotent in an abelian almost SWR ring R, then either eRe or $(1-e)R(1-e)$ is SWR, but the converse holds if R is an exchange ring. In Section 3, we consider extensions of almost SWR rings such as triangular matrix rings, trivial extensions, and so on. In Section 4, we study semiperfect almost SWR rings. In Section 5, we prove that if RG is a commutative ring, then RG is SWR if and only if R is SWR, G is locally finite and $n \in o(G)$ is a unit in R where $o(G)$ is the set of orders of all finite subgroups of G. Let KG be a group algebra over a field K satisfying a nontrivial polynomial identity. If KG is SWR, then K is SWR and G is locally finite. It is proved that if RH is almost SWR for every finitely generated subgroup H of G, then RG is almost SWR, but the converse of this result partially holds. We prove that if $G = H \rtimes K$ is a semidirect product of finite subgroup H by a subgroup K, then almost s-weakly regularity of RG implies almost s-weakly regularity of RK. We show that for a finite group G, the group ring RG need not be almost SWR. It is also proved that if R is a commutative local ring and G an abelian p-group with $p \in J(R)$, then RG is almost SWR.

2. Basic properties and examples

We first recall some definitions. An element a of R is called tripotent if $a^3 = a$ and a ring R is tripotent if all elements in R are tripotent. In [9], Danchev introduced weakly tripotent rings. A ring R is weakly tripotent if any of its element $a \in R$ satisfies the equations $a^3 = a$ or $a^3 = -a$. A ring R is called semiregular if for each $a \in R$, there exists a regular element $b \in R$ such that $a - b \in J(R)$. Recall that a ring R is called abelian if each idempotent in R is central.

Remark 2.1 *(1) Clearly, SWR and local rings are almost SWR rings.*

(2) Every abelian VNL-ring is an almost SWR ring.

(3) Every tripotent ring and weakly tripotent ring is an almost SWR ring.

(4) For a commutative ring, $R[[x]]$ is almost SWR if and only if R is local.

(5) For $n \geq 2$ and $n = \prod_{i=1}^{m} {p_i}^{k_i}$ is a prime power decomposition, the ring $\mathbb{Z}_n$ of integers mod n is almost SWR if and only if $(pq)^2$ does not divide n, where p and q are distinct primes.

(6) If $R = \{(q_1, q_2, \ldots, q_n, a, a, \ldots) \mid n \geq 1, q_i \in \mathbb{Q}, a \in \mathbb{Z}_{(2)}\}$, where $\mathbb{Z}_{(2)}$ is the localization of $\mathbb{Z}$ at prime ideal generated by 2, then R is an abelian VNL-ring with $J(R) = 0$ but not regular. Thus, R is almost SWR but not semiregular.

Thus, the class of almost SWR rings contains the classes of SWR, abelian VNL and weakly tripotent rings. Then, we have

$$\begin{array}{c} \text{Abelian VNL} \\ \Downarrow \\ \text{SWR} \Longrightarrow \text{Almost SWR} \\ \Uparrow \\ \text{Weakly Tripotent} \end{array}$$

However, the following examples show that its reverse implication is not true.

Example 2.2 *(1) Let $R = \mathbb{Z}_4$ be the ring of intergers modulo 4. Then, R is an almost SWR ring but not SWR.*

(2) Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}_2 \right\}$. *Then,*

$$R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}.$$

If $r = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, *we can not find x, y in R such that $r = rxr^2y$ but we can easily verify that* $1 - r = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *is SWR. Thus, R is an almost SWR ring but not SWR.*

(3) Let $R = \mathbb{T}_2(\mathbb{Z}_2)$. Then, R is an almost SWR ring but not an abelian VNL because idempotents are not central in R.

(4) Consider $R = \mathbb{Z}_4$. Then, R is an almost SWR ring but not weakly tripotent.

Example 2.3 *Let ${}_RM_S$ be a bimodule. If R is SWR and S is local, then* $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ *is an almost SWR ring.*

Proof Let $\beta = \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \in T$. Since S is local, b or $1_S - b$ is invertible. Assume that b is invertible. By hypothesis, a is SWR in R. So, we have $a = axa^2y$ for some $x, y \in R$. Thus,

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \begin{pmatrix} x & -x(am+mb)b^{-2} \\ 0 & b^{-2} \end{pmatrix} \begin{pmatrix} a & m \\ 0 & b \end{pmatrix}^2 \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}.$$

It implies that β is SWR.

Assume that $1_S - b$ is invertible. Since $1_R - a$ is SWR, we have $1 - a = (1-a)z(1-a)^2w$ for some z, w in R. Similarly, $1_T - \beta = \begin{pmatrix} 1-a & -m \\ 0 & 1-b \end{pmatrix}$ is SWR in T. □

Now we elaborate some properties of almost SWR rings.

Proposition 2.4 *The following statements are true for an almost SWR ring R.*

(1) Every homomorphic image of R is almost SWR.

(2) The center of R is a VNL-ring.

(3) The corner ring eRe is almost SWR for every $e^2 = e \in R$.

Proof (1) It is straightforward.

(2) Let $C(R)$ be the center of R and $x \in C(R)$. Since R is an almost SWR ring, either x or $1-x$ is an SWR element. If x is SWR, then we have $x \in xRx^2R = x^3R$ implies immediately that $x = x(x^k y)x$ with $y \in R$ and $k \geq 1$. Moreover, for every $a \in R$, $a(x^k y) = x^{k-1}(xa)y = x^{k-1}(x^{k+1}yx)ay = x^{k-1}ya(x^{k+1}yx) = x^{k-1}y(ax) = (x^k y)a$. Hence, $x^k y \in C(R)$. Similarly, if $1-x \in C(R)$ is an SWR element in R, then $1-x$ is regular in C(R).

(3) Let $a \in eRe$. Since R is an almost SWR ring, either a or $1-a$ is SWR. If a is SWR, we have $a = axa^2y$ for some $x, y \in R$. Thus, $a = eae = eaxa^2ye = aexea^2eye$. It follows that a is SWR in eRe. Similarly, if $1-a$ is SWR in R, then $e-a$ is SWR in eRe. Hence, eRe is an almost SWR ring. □

The following result follows immediately from Proposition 2.4(2).

Corollary 2.5 *Let R be an almost SWR ring. Then, R is indecomposable as a ring if and only if its center is local.*

Remark 2.6 *In [1], r-clean rings were studied by Ashrafi and Nasibi. A ring R is called r-clean if for any element $a \in R$, we have $a = e + r$ where e is an idempotent and r is a regular element in R. If R is an r-clean ring with no zero divisor, then by [1, Corollary 2.10], R is local. Thus, R is an almost SWR ring.*

It can be easily verified that direct product of SWR rings is SWR if and only if all factors are SWR. But we observe that the direct product of almost SWR rings may not be an almost SWR ring.

Example 2.7 *The ring $\mathbb{Z}_4$ of integers modulo 4 is an almost SWR ring. But $\mathbb{Z}_4 \times \mathbb{Z}_4$ is not an almost SWR ring. By choosing $a = (\bar{2}, \bar{3})$, we can easily show that neither a nor $1-a$ is SWR, and we are done.*

For the direct product of rings to be almost SWR, we prove the following theorem.

Theorem 2.8 *Let $R = \prod_{\beta \in I} R_\beta$. Then, R is an almost SWR ring if and only if there exists $\beta_0 \in I$ such that R_{β_0} is an almost SWR ring and for each $\beta \in I \backslash \beta_0$, R_β is an SWR ring.*

Proof Let $x = (x_\beta) \in R$, $\beta \in I$. By hypothesis, x_{β_0} or $1_{R_{\beta_0}} - x_{\beta_0}$ is SWR in R_{β_0}. Assume that x_{β_0} is SWR in R_{β_0}, then x is SWR. If $1_{R_{\beta_0}} - x_{\beta_0}$ is SWR in R_{β_0}, then $1-x$ is SWR in R.

Conversely, suppose that R is an almost SWR ring. Then, R_β is also an almost SWR ring for every $\beta \in I$ by Proposition 2.4(1). Write $R = R_{\beta_0} \times S$, where $S = \prod R_\beta$, $\beta \in I \backslash \beta_0$. If neither R_{β_0} nor S is SWR, then there exist non SWR elements $a \in R_{\beta_0}$ and $b \in S$. Now choose $r = (1_{R_{\beta_0}} - a, b)$. Then, neither r nor $1 - r = (a, 1_S - b)$ is SWR in R, a contradiction. Thus, either R_{β_0} or S is SWR. If S is an SWR ring, then we are done. If S is an almost SWR ring, then the iteration of the previous technique completes the proof. □

Lemma 2.9 *Let R be an abelian almost SWR ring. Then for every idempotent $e \in R$, either eRe or $(1-e)R(1-e)$ is SWR.*

Proof Consider the Pierce decomposition

$$R \cong \begin{pmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{pmatrix}.$$

Suppose that $a \in eRe$ and $b \in (1-e)R(1-e)$ are not SWR. Then neither $r := \begin{pmatrix} a & 0 \\ 0 & 1-b \end{pmatrix}$ nor $(1-r) = \begin{pmatrix} 1-a & 0 \\ 0 & b \end{pmatrix}$ is an SWR element in R, which is a contradiction. □

The example given below reveals that the converse of the Lemma 2.9 is false.

Example 2.10 *Let $R = \{(q_1, q_2, \dots, q_n, z, z \dots) \mid q_i \in \mathbb{Q}, z \in \mathbb{Z}, n \geq 1\}$. Clearly, either eRe or $(1-e)R(1-e)$ is SWR for every $e^2 = e \in R$. But R is not an almost SWR ring because the homomorphic image $\mathbb{Z}$ of R is not almost SWR.*

An element a of R is said to be an exchange [16] if there exists an idempotent $e \in R$ such that $e \in Ra$ and $1-e \in R(1-a)$. A ring R is an exchange ring if and only if each element of R is exchange. It is easy to show that a commutative almost SWR ring is an exchange ring. The next theorem shows that the converse of Lemma 2.9 is true for an exchange ring.

Theorem 2.11 *Let R be an abelian exchange ring. Then R is almost SWR if and only if either eRe or $(1-e)R(1-e)$ is SWR for every $e^2 = e \in R$.*

Proof The 'only if' part follows by Lemma 2.9.

Conversely, suppose that R is an exchange ring. Then, for any $a \in R$, we have an idempotent $e \in R$ such that $e \in Ra$ and $1-e \in R(1-a)$. Then $Ra + R(1-e) = R$ and $Re + R(1-a) = R$. Thus, $Rae = Re$ and $R(1-a)(1-e) = R(1-e)$. So, both ae and $(1-a)(1-e)$ are SWR. Since R is an abelian, $eRe = Re$. By hypothesis, if eRe is SWR, then $(1-a)e$ is SWR. Therefore, $1-a = (1-a)e + (1-a)(1-e)$ is SWR. Similarly, if $(1-e)R(1-e)$ is SWR, then we can prove that a is SWR. □

Proposition 2.12 *Let R be a commutative ring. Then $R[x]$ is not an almost SWR ring.*

Proof Assume that $R[x]$ is an almost SWR ring. Then $R[x]$ being a commutative almost SWR ring implies that $R[x]$ is a VNL-ring, which contradicts [17, Corollary 4.8]. □

Lemma 2.13 *Let R be a ring. If $a - aza^2w$ is SWR for some $z, w \in R$, then a is SWR.*

Proof If $a - aza^2w$ is SWR, then there exist $s, t \in R$ such that

$$(a - aza^2w)s(a - aza^2w)^2t = a - aza^2w.$$

If we set $x = saz - s + z$ and $y = t - wsa^2t + waza^2wt - wsaza^2waza^2wt - za^2wt + wsa^2za^2wt + w$, then it can be verified that $axa^2y = a$. Thus, a is SWR. □

Let R be an almost SWR ring and I an ideal of R. Then, clearly, R/I is almost SWR. But in general, the converse of this result is not true (for example, let $R = \mathbb{Z}_p$ where p is a prime number, then R is almost SWR but $\mathbb{Z}$ is not almost SWR). The following theorem gives another characterization of almost SWR rings.

Theorem 2.14 *Let I be an SWR ideal of a ring R. Then, R is an almost SWR ring if and only if R/I is almost SWR.*

Proof Suppose that R is an almost SWR ring. Then, by Proposition 2.4(1), R/I is almost SWR.

Conversely, suppose that R/I is almost SWR. Then, either $a+I$ or $1-a+I$ is SWR. Thus, there exist $x,y,z,w \in R$ such that either $a-axa^2y \in I$ or $(1-a)-(1-a)z(1-a)^2w \in I$. Since I is an SWR ideal, either $a-axa^2y$ or $(1-a)-(1-a)z(1-a)^2w$ is an SWR element of R. If $a-axa^2y$ is SWR, then we have $(a-axa^2y) = (a-axa^2y)t(a-axa^2y)^2s$ for some $t,s \in R$. By Lemma 2.13, it follows that $a = aga^2h$ for some $g,h \in R$. Similarly, if $(1-a)-(1-a)z(1-a)^2w$ is SWR, then we can show that $1-a$ is SWR. □

In [12], Gupta introduced $S(R) = \{a \in R \mid (a) \text{ is a SWR ideal in } R\}$, which is the unique maximal two sided SWR ideal of R, where (a) is the principal ideal of R generated by $a \in R$ and proved that $S(R/S(R)) = 0$. Following [2], $M(R) = \{a \in R \mid (a) \text{ is a regular ideal in } R\}$ is the unique maximal two sided regular ideal of R. In [6], Chen and Tong gave a characterization of abelian VNL rings through local rings. Analogously, we characterize commutative almost SWR rings through local rings.

Proposition 2.15 *Let R be a commutative ring. Then, R is an almost SWR ring if and only if $R/S(R)$ is a local ring.*

Proof Suppose that $R/S(R)$ is a local ring. Then $R/S(R)$ is an almost SWR ring. Thus, by Theorem 2.14, R is an almost SWR ring.

Conversely, it is easy to see that a commutative almost SWR ring R is a VNL-ring. Let I be a SWR ideal in R. Then, we have

$$\begin{aligned} S(R) &= \{a \in R \mid ar \in I, r \in R\} \\ &= \{a \in R \mid ar = (ar)x(ar)^2y, x,y \in I\} \\ &= \{a \in R \mid ar = (ar)z(ar), z = x(ar)y \in I\} \\ &= M(R) \end{aligned}$$

Then, in view of [6, Lemma 2.7], $R/S(R)$ is local. □

The necessary conditions of Theorem 2.15 is not true for arbitrary rings, as shown in the following example.

Example 2.16 *Let $R = \mathbb{T}_2(\mathbb{Z}_2)$. Then R is an almost SWR ring but $R/S(R)$ is not local. Since in view of [12, Theorem 10(4)], $S(\mathbb{T}_2(\mathbb{Z}_2)) = 0$. Then, $R/S(R) = \mathbb{T}_2(\mathbb{Z}_2)$ is not local.*

Proposition 2.17 *Let L be some nonempty subset of R and $(L)_r$ be a right ideal generated by L. Then, for a commutative ring R, the following are equivalent:*

(1) R is a almost SWR ring.

(2) At least one of the element in L is SWR, whenever $(L)_r = R$.

Proof For any $a \in R$, let $L = \{a, 1-a\}$. Since $1 = a+1-a \in (L)_r$, $(L)_r = R$. Thus, either a or $1-a$ is SWR.

Conversely, if R is SWR, then the result follows. Otherwise, suppose that R is an almost SWR ring which is not SWR. Now there exist $l_1, l_2, \ldots, l_t$ in any nonempty subset L of R with $(L)_r = R$ such that $l_1R + l_2R + \cdots + l_tR = R$. Then, there exist $r_1, r_2, \ldots, r_t \in R$ satisfying $l_1r_1 + l_2r_2 + \cdots + l_tr_t = 1$, and thus $\bar{l}_1\bar{r}_1 + \bar{l}_2\bar{r}_2 + \cdots + \bar{l}_t\bar{r}_t = \bar{1}$ in $\overline{R} = R/S(R)$. And by Proposition 2.15, $\overline{R}$ is a local ring. It follows that there exists an $\bar{l}_k$ such that $\bar{l}_k \in U(\overline{R})$; thus, $\bar{l}_k$ is SWR in $\overline{R}$. So, $\bar{l}_k = \bar{l}_k\bar{x}_k(\bar{l}_k)^2\bar{y}_k$ for some $\bar{x}_k, \bar{y}_k \in \overline{R}$. Then $l_k - l_kx_k(l_k)^2y_k \in S(R)$, it implies that $l_k - l_kx_k(l_k)^2y_k = (l_k - l_kx_k(l_k)^2y_k)a_k(l_k - l_kx_k(l_k)^2y_k)^2b_k$ for some $a_k, b_k \in R$. Thus, l_k is an SWR element by Lemma 2.13. □

The following proposition shows that an almost SWR ring R is the direct summand of either $r(a)$ or $r(1-a)$ for all $a \in R$.

Proposition 2.18 *If $r(a) = r(b)$ and $r(1-a) = r(1-b)$, for each $a \in R$ and $b \in Ra^2R$. Then R is almost SWR if and only if either $r(a)$ or $r(1-a)$ is direct summand.*

Proof Let $a \in R$ and $r(a)$ be the direct summand. Then, we have an ideal $I \subset R$ such that $R = r(a) \oplus I$. So, there exist $d \in r(a)$ and $b \in I$ such that $d + b = 1$ and hence, $a = ad + ab$. Thus, $a = ab$. Since Ra^2R is a two sided ideal of R, $b \in Ra^2R$. Thus, a is SWR. If $r(1-a)$ is direct summand, then there exists an ideal $J \subset R$ such that $R = r(1-a) \oplus J$. Thus, we can prove that $1-a$ is SWR.

Conversely, let for any $a \in R$, either a or $1-a$ is SWR. If a is SWR, then there exists $b = ta^2s \in Ra^2R$ such that $a = ata^2s$ for some $t, s \in R$. Then, $a(1-ta^2s) = 0$, so $(1-ta^2s) \in r(a)$. Thus, $1 = (1-ta^2s) + ta^2s$. Hence, $R = r(a) + Ra^2R$. Now suppose that $x \in r(a) \cap Ra^2R$, then $ax = 0$ and $x = ta^2s$ for some $t, s \in R$. Thus, $ta^2s \in r(a) = r(b)$, so $bta^2s = 0$. Then, $bx = 0$ and so, $x = 0$. Therefore, $r(a) \cap Ra^2R = 0$. Hence, $R = r(a) \oplus Ra^2R$. Similarly, if $1-a$ is SWR, then we can deduce that $R = r(1-a) + R(1-a)^2R$ and $r(1-a) \cap R(1-a)^2R = 0$. Hence, $R = r(1-a) \oplus R(1-a)^2R$. □

3. Extension rings

We start this section with the necessary conditions for an upper triangular matrix ring to be almost SWR.

The proof of the following lemma is trivial.

Lemma 3.1 *Let diag$(a_1, a_2, \ldots, a_n)$ be the $n \times n$ diagonal matrix with a_i in each entry on the main diagonal. Then, diag$(a_1, a_2, \ldots, a_n)$ is SWR in $\mathbb{T}_n(R)$ if and only if $a_1, a_2, \ldots, a_n$ are all SWR in R.*

Theorem 3.2 *If $\mathbb{T}_n(R)$ is an almost SWR ring for some $n \geq 2$, then R is an SWR ring.*

Proof Let $A =$diag$(a, 1-a, 1, \ldots, 1) \in T_n(R)$. Then $I_n - A =$diag$(1-a, a, 0, \ldots, 0)$. Since $T_n(R)$ is an almost SWR ring, either A or $I_n - A$ is SWR. For any case, by Lemma 3.1, a is SWR. Thus, R is SWR. □

The example given below shows that the converse of above Theorem 3.2 may not be true.

Example 3.3 *The ring $\mathbb{T}_2(\mathbb{Z}_6)$ is not almost SWR because neither $\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ nor $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ is SWR although $\mathbb{Z}_6$ is an SWR ring.*

Proposition 3.4 *For any ring R and $n \geq 4, \mathbb{T}_n(R)$ is not an almost SWR ring.*

Proof By applying Proposition 2.4(3), we may assume that $n = 4$. Let $C = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, then neither $\mathrm{diag}(C, I_2 - C) = \left(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}\right)$ nor $diag(I_2 - C, C) = \left(\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\right)$ is SWR. Hence, $\mathbb{T}_n(R)$ is not an almost SWR ring for any $n \geq 4$. □

Let A be a ring and B a subring of ring A with $1_A \in B$. We set

$$R[A,B] = \{(c_1, c_2 \dots, c_n, d, d \dots) | c_i \in A, d \in B, n \geq 1\}$$

with addition and multiplication defined componentwise.

Theorem 3.5 *The following statements are equivalent:*

(1) $R[A,B]$ is an almost SWR ring.

(2) A is an SWR ring and B is an almost SWR ring.

Proof Construct a homomorphism $f : R[A,B] \to B$ defined by $f(c_1, c_2, \dots, c_n, d, d \dots) = d$. Then, $R[A,B]/kerf \cong B$. Thus, B is an almost SWR ring by using Proposition 2.4(1). If A is not an SWR ring, then we have a non SWR element $\alpha \in A$. Let $x = (\alpha, 1-\alpha, 1, 1, \dots) \in R[A,B]$. So, either x or $1 - x = (1-\alpha, \alpha, 0, 0, \dots) \in R[A,B]$ is SWR. If x is SWR, so is $\alpha \in A$, a contradiction. Hence, we conclude that A is an SWR ring.

Conversely, for any $(c_1, c_2, \dots, c_n, d, d, \dots) \in R[A,B]$ with each $c_i \in A$ and $d \in B$. Since A is an SWR ring, we have $c_i = c_i t_i {c_i}^2 s_i$ for some t_i, s_i in A and B is an almost SWR ring, then either d or $1 - d$ is SWR. If d is SWR, then we can find some g, h in B such that $d = dgd^2h$. Thus, $(c_1, c_2, \dots, c_n, d, d \dots) = (c_1, c_2, \dots, c_n, d, d, \dots)(t_1, t_2, \dots, t_n, g, g, \dots)(c_1, c_2, \dots, c_n, d, d, \dots)^2(s_1, s_2, \dots, s_n, h, h, \dots)$. This implies that $(c_1, c_2, \dots, c_n, d, d, \dots) \in R[A,B]$ is SWR. If $1-d$ is SWR, then we have $1-d = (1-d)y(1-d)^2z$ for some y, z in B. Thus, we get $(1, 1, \dots, 1, 1, 1, \dots) - (c_1, c_2, \dots, c_n, d, d, \dots) = (1-c_1, 1-c_2, \dots, 1-c_n, 1-d, 1-d, \dots) \in R[A,B]$ is SWR. Therefore, $R[A,B]$ is an almost SWR ring. □

Corollary 3.6 *$R[A,A]$ is an almost SWR ring if and only if A is an SWR ring.*

Let R be a ring, then the trivial extension of R over R is

$$R\Theta R = \{(s,n) \mid s \in R, n \in R\}$$

with componentwise addition and multiplication defined by $(s_1, n_1)(s_2, n_2) = (s_1 s_2, n_1 s_2 + s_1 n_2)$. Then, $R\Theta R$ is isomorphic to subring $\left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in R\right\}$ of $\mathbb{T}_2(R)$.

Theorem 3.7 *Let R be a ring. If $R\Theta R$ is an almost SWR ring, then R is almost SWR.*

Proof Let $\theta : R\Theta R \to R$ be a canonical epimorphism. Then, we have $R\Theta R/0\Theta R \cong R$. Hence, R is an almost SWR ring by Proposition 2.4(1). □

Proposition 3.8 *For a ring S and $n \geq 2$, $R = \mathbb{T}_n(S)$. Then, $R\Theta R$ is not an almost SWR ring.*

Proof Assume that $n = 2$. Let $A = (C, I_2) \in R\Theta R$, where $C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Suppose that A is SWR, then there exist (X,Y), $(V,W) \in R\Theta R$ such that $(C, I_2) = (C, I_2)(X,Y)(C,I_2)^2(V,W)$. Thus $(X + CY)C^2V + 2CXCV + CXC^2W = I_2$. Write $X = \begin{pmatrix} x_1 & x_2 \\ 0 & x_3 \end{pmatrix}, Y = \begin{pmatrix} y_1 & y_2 \\ 0 & y_3 \end{pmatrix}, V = \begin{pmatrix} v_1 & v_2 \\ 0 & v_3 \end{pmatrix}$ and $W = \begin{pmatrix} w_1 & w_2 \\ 0 & w_3 \end{pmatrix}$. Then we obtain $\left[\begin{pmatrix} x_1 & x_2 \\ 0 & x_3 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 & y_2 \\ 0 & y_3 \end{pmatrix}\right]\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}^2\begin{pmatrix} v_1 & v_2 \\ 0 & v_3 \end{pmatrix} +$ $2\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ 0 & x_3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 & v_2 \\ 0 & v_3 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ 0 & x_3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}^2\begin{pmatrix} w_1 & w_2 \\ 0 & w_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we get a contradiction by comparing the $(2,2)$ entry of matrices on both side. Similarly, we can also show that $(I_2, 0) - A$ is not SWR. Hence, $\mathbb{T}_2(S)\Theta\mathbb{T}_2(S)$ is not an almost SWR ring.

Suppose that $n \geq 3$. Let $C = \begin{pmatrix} C_1 & \alpha \\ 0 & C_2 \end{pmatrix}, D = \begin{pmatrix} D_1 & \beta \\ 0 & D_2 \end{pmatrix} \in R$, where $C_1, D_1 \in \mathbb{T}_2(S)$. If (C, D) is SWR ring in $R\Theta R$, then (C_1, D_1) is SWR in $\mathbb{T}_2(S)\Theta\mathbb{T}_2(S)$. As $\mathbb{T}_2(S)\Theta\mathbb{T}_2(S)$ is not almost SWR, neither is $R\Theta R$. □

The converse of Theorem 3.7 does not hold, which is shown in the following corollary.

Corollary 3.9 *Let $R = \mathbb{T}_2(\mathbb{Z}_2)$ be an almost SWR ring. Then, $R\Theta R$ is not almost SWR.*

Proof From Proposition 3.8, $R\Theta R = \mathbb{T}_2(\mathbb{Z}_2)\Theta\mathbb{T}_2(\mathbb{Z}_2)$ is not almost SWR. □

4. Semiperfect almost SWR rings

In this section, we consider the structure of semiperfect (see [4]) almost SWR rings. Recall that a ring R is called reduced if R has no nonzero nilpotent elements.

Lemma 4.1 *[11, Lemma 4.2]. Let e_1 and e_2 be two local idempotents of a ring R. Then, either $e_1R \cong e_2R$, or $e_1Re_2 \subseteq J(R)$ and $e_2Re_1 \subseteq J(R)$.*

Proposition 4.2 *Let R be a semiperfect ring with $1 = e_1 + e_2$, where e_1, e_2 are orthogonal primitive idempotents. If R is almost SWR, then R is isomorphic to either of the following:*

(1) $\mathbb{M}_2(C)$ for some reduced ring C;

(2) $\begin{pmatrix} A & X \\ Y & B \end{pmatrix}$ where A is a reduced ring, B is a local ring and $XY \subseteq J(A)$, $YX \subseteq J(B)$.

In particular, if $J(R) = 0$. Then, R is isomorphic to either $\mathbb{M}_2(C)$ or $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ where A_1, C are reduced and A_2 is a local ring.

Proof Consider the Pierce decomposition

$$R \cong \begin{pmatrix} e_1Re_1 & e_1Re_2 \\ e_2Re_1 & e_2Re_2 \end{pmatrix}$$

If $e_1R \cong e_2R$, then $R \cong \mathbb{M}_2(e_1Re_1)$, where e_1Re_1 is a local ring. By using Lemma 2.9, e_1Re_1 is an SWR ring. Then in view of [12, Theorem 5], e_1Re_1 is reduced. If $e_1R \ncong e_2R$, then e_1Re_2 and e_2Re_1 are contained in $J(R)$ by Lemma 4.1. Again by Lemma 2.9, either e_1Re_1 or e_2Re_2 is SWR. It follows that either e_1Re_1 or e_2Re_2 is a reduced ring. We assume that e_1Re_1 is a reduced ring. Note that $e_1Re_2Re_1 \subseteq J(R) \cap e_1Re_1 = J(e_1Re_1)$ and $e_2Re_1Re_2 \subseteq J(R) \cap e_2Re_2 = J(e_2Re_2)$. So take $A = e_1Re_1$, $B = e_2Re_2$, $X = e_1Re_2$ and $Y = e_2Re_1$, we obtain that $R \cong \begin{pmatrix} A & X \\ Y & B \end{pmatrix}$. □

Proposition 4.3 *Let R be a semiperfect ring with $1 = e_1 + e_2 + e_3$, where $\{e_1, e_2, e_3\}$ is a orthogonal set of primitive idempotents. If R is almost SWR, then R is isomorphic to one of the followings:*

(1) $\mathbb{M}_3(C)$ for some reduced ring C;

(2) $\begin{pmatrix} R_1 & X \\ Y & R_2 \end{pmatrix}$ where R_1 is a reduced ring, R_2 is a local ring and $XY \subseteq J(R_1)$, $YX \subseteq J(R_2)$;

(3) $\begin{pmatrix} R_1 & X \\ Y & R_2 \end{pmatrix}$ where R_1 is semiprime, R_2 is a local ring and $XY \subseteq J(R_1)$, $YX \subseteq J(R_2)$;

(4) $\begin{pmatrix} A & X \\ Y & C \end{pmatrix}$ with $A \cong \begin{pmatrix} R_1 & X_1 \\ Y_1 & R_2 \end{pmatrix}$ and R_1, R_2, C are reduced rings, $X_1Y_1 \subseteq J(R_1)$, $Y_1X_1 \subseteq J(R_2)$.

Proof **Case 1**. If $e_iR \cong e_jR$ for all i, j, then $R \cong \mathbb{M}_3(e_1Re_1)$ where e_1Re_1 is a local ring. By Lemma 2.9, e_1Re_1 is a reduced ring.

Now we consider Pierce decomposition

$$R \cong \begin{pmatrix} (1-e_1)R(1-e_1) & (1-e_1)Re_1 \\ e_1R(1-e_1) & e_1Re_1 \end{pmatrix}$$

Case 2. Assume that e_1Re_1 is local but not a reduced ring, then $(1-e_1)R(1-e_1)$ is a reduced ring by [12, Theorem 5]. Thus, e_2Re_2 and e_3Re_3 are reduced rings. So by Lemma 4.1, $e_1Re_2, e_2Re_1, e_1Re_3$ and e_3Re_1 are all contained in $J(R)$. Thus $(1-e_1)Re_1R(1-e_1) \subseteq J(R) \cap (1-e_1)R(1-e_1) = J((1-e_1)R(1-e_1))$ and $e_1R(1-e_1)Re_1 \subseteq J(R) \cap e_1Re_1 = J(e_1Re_1)$. Hence, R is as in (2) above.

Case 3. Suppose that all e_iRe_i are reduced rings. If $e_2R \cong e_3R$ but $e_1R \ncong e_2R$, then $(1-e_1)R(1-e_1) \cong \mathbb{M}_2(C)$ for some reduced ring C, and so C is a semiprime ring. Then $\mathbb{M}_2(C)$ is semiprime by [13, Proposition 10.20]. Hence, $(1-e_1)R(1-e_1)$ is semiprime. By Lemma 4.1, $(1-e_1)Re_1R(1-e_1) \subseteq J((1-e_1)R(1-e_1))$ and $e_1R(1-e_1)Re_1 \subseteq J(e_1Re_1)$. Thus, R is as in (3).

Case 4. Suppose that e_iRe_i is a reduced ring for all i=1,2,3 and $e_1R \ncong e_2R \ncong e_3R$. Then

$$(1-e_1)R(1-e_1) \cong \begin{pmatrix} e_2Re_2 & e_2Re_3 \\ e_3Re_2 & e_3Re_3 \end{pmatrix},$$

where $e_2Re_3Re_2 \subseteq J(e_2Re_2)$ and $e_3Re_2Re_3 \subseteq J(e_3Re_3)$. Now note that $(1-e_1)Re_1R(1-e_1) \subseteq J(R) \cap (1-e_1)R(1-e_1) = J((1-e_1)R(1-e_1))$. So by taking $e_2Re_2 = R_1, e_3Re_3 = R_2, e_3Re_2 = Y_1, e_2Re_3 = X_1, e_1Re_1 = C, (1-e_1)Re_1 = Y, e_1R(1-e_1) = X$. Thus (3) follows. □

5. Almost SWR group rings

Let G be a group and R be a ring, then the group ring of G and R is denoted by RG. If R is commutative, then RG is called a group algebra. The augmentation ideal ωG of RG is generated by $\{1 - g \mid g \in G\}$. Then, ωG is the kernel of the augmentation map, $\omega : RG \to R$ defined by $\omega(\sum_{g \in G} a_g g) = \sum_{g \in G} a_g$ and $RG/\omega G \cong R$. If every element $g \in G$ has a finite number of conjugates in G, the group is called an FC group. If the set of all the element $g \in G$ has a finite number of conjugates in G, then G is called FC-center $\Delta(G)$ of G. Note that $\Delta(G) = \{g \in G \mid |G : C_G(g)| < \infty\}$ is a normal subgroup of G. For further results of group rings, we refer to Passman [18] and Connell [7].

We start this section with the necessary conditions for RG to be SWR.

Theorem 5.1 *If RG is an SWR ring. Then, R is an SWR ring and G is a torsion group.*

Proof By the augmentation map, R is an image of RG. Since homomorphic image of an SWR ring is SWR, R is SWR. Let $g(\neq 1) \in G$. Since RG is SWR, $1-g = (1-g)x$ where $x \in ((1-g)^2)$. Then, $(1-g)(1-x) = 0$. This implies that $1 = x \in \omega G$, which is a contradiction. Thus, $1 - g$ is a zero divisor, and hence g is of finite order by [7, Proposition 6]. Thus, G is a torsion group. □

Recall that an abelian torsion group is locally finite.

Corollary 5.2 *Let G be an abelian group. If RG is SWR, then R is SWR, and G is locally finite.*

Theorem 5.3 *If RG is an SWR ring. Then for each $n \in o(G)$, n is a unit in R, where $o(G)$ denotes the set of orders of all finite subgroups of G.*

Proof Let n be the order of $g \in G$. We will show that n is a unit in R. Since RG is SWR, there exist $x, y \in RG$ such that $(1-g)(1-x(1-g)^2y) = 0$. By using [7, Proposition 6], $(1-x(1-g)^2y) = (1+g+g^2+\cdots+g^{n-1})r$ for some $r \in RG$ and by applying augmentation map $\omega : RG \to R$ on above equation, we get $1 = n\omega(r)$, where $\omega(r) \in R$. □

The following example shows that the converse of Theorem 5.1 is not true.

Example 5.4 *Let $R = \mathbb{Z}_2\mathbb{C}_2$. Then, $\mathbb{Z}_2$ is SWR and $\mathbb{C}_2$ is torsion but R is not SWR.*

If RG is commutative, then we have necessary and sufficient conditions for RG to be SWR.

Theorem 5.5 *Let RG be a commutative ring. Then RG is SWR if and only if*

(1) R is SWR.

(2) G is locally finite.

(3) for each $n \in o(G)$, n is a unit in R.

Proof The necessity follows from Theorem 5.1 and Theorem 5.3, and the sufficiency follows from the fact that commutative SWR rings are VNL and by [7, Theorem 3]. □

The next result gives necessary conditions for group algebra KG over a field K satisfying a nontrivial polynomial identity to be SWR.

Theorem 5.6 *Let KG be a group algebra over a field K satisfying a nontrivial polynomial identity. If KG is SWR, then K is SWR, and G is locally finite.*

Proof Suppose that KG is SWR. Then, since homomorphic image of an SWR ring is SWR, K is SWR. In view of Theorem 5.1, G is a torsion group, and by [19, Theorem 5.5], we have $|G : \Delta(G)| < \infty$. Let H be a finitely generated subgroup of G. Then $|H : H \cap \Delta(G)| < \infty$, and in view of [19, Lemma 6.1], $H \cap \Delta(G)$ is a finitely generated subgroup of $\Delta(G)$. Since by [19, Lemma 2.2], the center $C(H \cap \Delta(G))$ of $H \cap \Delta(G)$ is a subgroup of finite index, $|H : C(H \cap \Delta(G))| < \infty$. Thus, again, by [19, Lemma 6.1], $C(H \cap \Delta(G))$ is a finitely generated torsion group. So, $C(H \cap \Delta(G))$ is finite. Hence, H is finite. □

Remark 5.7 *The condition in Theorem 5.3 is not necessary for RG to be almost SWR since $R = \mathbb{Z}_4\mathbb{C}_2$ is almost SWR, but 2 is not a unit in $\mathbb{Z}_4$.*

Let p be a prime number. A group G is called p-group if the order of each element $g \in G$ is a power of p.

Theorem 5.8 *Let R be a commutative local ring and G an abelian p-group with $p \in J(R)$. Then, RG is an almost SWR ring.*

Proof Suppose that R is a commutative ring and G an abelian p-group with $p \in J(R)$. Following [20, Lemma 2.1] we get that $\omega G \subseteq J(RG)$. Then, R being local implies that RG is local by [15]. Hence, RG is an almost SWR ring. □

Example 5.9 *Let $R = \mathbb{Z}_{(p)} = \left\{ \frac{b}{a} \mid b, a \in \mathbb{Z}, gcd(a,p) = 1 \right\}$ and $G = C_p$. The group ring RG is almost SWR.*

Lemma 5.10 *Let G be a group. If RH is almost SWR for every finitely generated subgroup H of G, then RG is almost SWR.*

Proof Let $\alpha \in RG$ and H be a subgroup generated by the support of α. Then H is a finitely generated subgroup of G. Thus, either α or $1 - \alpha$ is SWR in RH. Assume that α is SWR, then we have $\alpha \in \alpha RH \alpha^2 RH \subseteq \alpha RG \alpha^2 RG$. It follows that α is SWR in RG. Similarly, if $1 - \alpha$ is SWR in RH, then $1 - \alpha \in (1 - \alpha)RH(1 - \alpha)^2 RH \subseteq (1 - \alpha)RG(1 - \alpha)^2 RG$. Thus, $1 - \alpha$ is an SWR element in RG. Hence, RG is an almost SWR ring. □

If H and K are subgroups of G such that: $H \triangleleft G, H \cap K = \{1\}$ and $HK = G$, then G is called a semidirect product of H by K, denoted by $G = H \rtimes K$. The following result shows that the converse of Lemma 5.10 partially holds.

Theorem 5.11 *Let $G = H \rtimes K$, $|H| < \infty$. If RG is an almost SWR ring, then RK is an almost SWR ring.*

Proof For any $\alpha \in RK$, either α or $1 - \alpha$ is SWR in RG. Assume that α is SWR, then we have $\alpha = \alpha a \alpha^2 b$ for some $a, b \in RG$. Let $a = \sum a_i k_i$ and $b = \sum b_i k_i$, where $a_i, b_i \in RH$, $k_i \in K$ and let $\alpha = \sum \alpha_j k_j$, where $\alpha_j \in R$. Denote $x = \sum \omega(a_i) k_i, y = \sum \omega(b_i) k_i$, so $x, y \in RK$. We will show that $\alpha = \alpha x \alpha^2 y$ for some $x, y \in RK$.

Let $\xi : G \to G/H$ stand for the natural group homomorphism and then extend ξ to a ring homomorphism $\overline{\xi} : RG \to R(G/H)$, defined by $\overline{\xi}(\sum \alpha_i g_i) = \sum \alpha_i \overline{\xi}(g_i)$. Obviously, $Ker(\overline{\xi}) \cap RK = 0$ and $\overline{\xi}(z) = \omega(z)$ for all $z \in RH$.

Since $0 = \alpha - \alpha a \alpha^2 b$, we have

$$\begin{aligned}
0 &= \overline{\xi}(\alpha) - \overline{\xi}(\alpha)\overline{\xi}(a)\overline{\xi}(\alpha^2)\overline{\xi}(b) \\
&= \overline{\xi}(\alpha) - \overline{\xi}(\alpha)\overline{\xi}(\sum a_i k_i)\overline{\xi}(\alpha^2)\overline{\xi}(\sum b_i k_i) \\
&= \overline{\xi}(\alpha) - \overline{\xi}(\alpha)\sum \omega(a_i)\overline{\xi}(k_i)\overline{\xi}(\alpha^2)\sum \omega(b_i)\overline{\xi}(k_i) \\
&= \overline{\xi}(\alpha) - \overline{\xi}(\alpha)\overline{\xi}(\sum \omega(a_i)k_i)\overline{\xi}(\alpha^2)\overline{\xi}(\sum \omega(b_i)k_i) \\
&= \overline{\xi}(\alpha) - \overline{\xi}(\alpha)\overline{\xi}(x)\overline{\xi}(\alpha^2)\overline{\xi}(y) \\
&= \overline{\xi}(\alpha - \alpha x \alpha^2 y).
\end{aligned}$$

Then, $\alpha - \alpha x \alpha^2 y \in Ker(\overline{\xi}) \cap RK = 0$, so we have $\alpha = \alpha x \alpha^2 y$. Similarly, if $1 - \alpha$ is SWR, then we can find $t = \sum \omega(t_i)k_i$ and $s = \sum \omega(s_i)k_i$ in RK such that $1 - \alpha = (1-\alpha)t(1-\alpha)^2 s$. □

Remark 5.12 *An artinian ring RG may not be an almost SWR ring.*

Example 5.13 *The group ring $(\mathbb{Z}_4 \times \mathbb{Z}_4)\mathbb{C}_2$ is artinian but not almost SWR.*

For any nontrivial finite group G, group ring RG may or may not be an almost SWR ring.

Example 5.14 *$\mathbb{Z}G$ is not almost SWR for any nontrivial finite group G.*

Example 5.15 *Let $\mathbb{Z}_2 = \{0, 1\}$ and $G = \langle g | g^2 = 1 \rangle$. An element $1 + g \in \mathbb{Z}_2 G$ is not SWR but $1 - (1 + g)$ is SWR. So $\mathbb{Z}_2 G$ is an almost SWR ring.*

Proposition 5.16 *Let K be a field of $char(K) = p > 0$ and G a finite p-group. Then group algebra KG is almost SWR.*

Proof Suppose that K be a field of $char(K) = p > 0$ and G a finite p-group. Then by [13, Corollary 8.8], jacobson radical of group algebra $J(KG)$ is equal to augmentation ideal ωG with $J(KG)^{|G|} = 0$. It follows that $KG/J(KG) \cong K$. Since K is a division ring, KG is local. Thus, KG is an almost SWR ring. □

Acknowledgment

The authors are grateful to the referee for his/her helpful suggestions and comments.

References

[1] Ashrafi N, Nasibi E. Rings in which elements are the sum of an idempotent and a regular element. Bulletin of the Iranian Mathematical Society 2013; 39 (3): 579-588.

[2] Brown B, McCoy NH. The maximal regular ideal of a ring. Proceedings of the American Mathematical Society 1950; 1: 165-171. doi.org/10.2307/2031919

[3] Camillo V, Xiao YF. Weakly regular rings. Communications in Algebra 1950; 22 (10): 4095-4112. doi.org/10.1080/00927879408825068

[4] Camillo VP, Yu HP. Exchange rings, units and idempotents. Communications in Algebra 1994; 22 (12): 4737-4749. doi.org/10.1080/00927879408825098

[5] Chen H. On almost unit-regular rings. Communications in Algebra 2012; 40 (9): 3494-3506. doi.org/10.1080/00927872.2011.590953

[6] Chen W, Tong W. On noncommutative VNL-rings and GVNL-rings. Glasgow Mathematical Journal 2006; 48 (1): 11-17. doi.org/10.1017/S0017089505002806

[7] Connell IG. On the group ring. Canadian Journal of Mathematics 1963; 15: 650-685.doi.org/10.4153/CJM-1963-067-0

[8] Contessa M. On certain classes of pm-rings. Communications in Algebra 1984; 12 (11-12): 1447-1469. doi.org/10.1080/00927878408823063

[9] Danchev PV. Weakly tripotent rings. Kragujevac Journal of Mathematics 2019; 43 (3): 465-469.

[10] Goodearl KR. Von Neumann regular rings. Monographs and Studies in Mathematics, 4, Boston: Pitman (Advanced Publishing Program), MA, 1979.

[11] Grover HK, Khurana D. Some characterizations of VNL-rings. Communications in Algebra 2009; 37 (9): 3288-3305. doi.org/10.1080/00927870802502761

[12] Gupta V. A generalization of strongly regular rings. Acta Mathematica Hungarica 1984; 43 (1-2): 57-61. doi.org/10.1007/BF01951326

[13] Lam TY. A first course in noncommutative rings. Graduate Texts in Mathematics, 131, New York: Springer-Verlag, 1991.

[14] Lam TY. Lectures on modules and rings. Graduate Texts in Mathematics, 189, New York: Springer-Verlag, 1999.

[15] Nicholson WK. Local group rings. Canadian Mathematical Bulletin 1972; 15: 137-138. doi.org/10.4153/CMB-1972-025-1

[16] Nicholson WK. Lifting idempotents and exchange rings. Transactions of the American Mathematical Society 1977; 229: 269-278. doi.org/10.1090/S0002-9947-1977-0439876-2

[17] Osba EA, Henriksen M, Alkam O. Combining local and von Neumann regular rings. Communications in Algebra 2004; 32 (7): 2639-2653. doi.org/10.1081/AGB-120037405

[18] Passman DS. The algebraic structure of group rings. Pure and Applied Mathematics, New York: Wiley-Interscience, 1977.

[19] Passman DS. Infinite group rings. Pure and Applied Mathematics, 6, New York: Marcel Dekker, Inc., 1971.

[20] Wang X, You H. Cleanness of the group ring of an abelian p-group over a commutative ring. Algebra Colloquium 2012; 19 (3): 539-544. doi.org/10.1142/S1005386712000405