Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 1927 – 1944
© TÜBİTAK
doi:10.55730/1300-0098.3242

# Spectral theory of B-Weyl elements and the generalized Weyl's theorem in primitive C*-algebra

**Yingying KONG, Yanxun REN*, Lining JIANG**
School of Mathematics and Statistics, Beijing Institute of Technology, Beijing, China

**Received:** 08.11.2021 • **Accepted/Published Online:** 28.02.2022 • **Final Version:** 20.06.2022

**Abstract:** Let $\mathcal{A}$ be a unital primitive C*-algebra. This paper studies the spectral theories of B-Weyl elements and B-Browder elements in $\mathcal{A}$, including the spectral mapping theorem and a characterization of B-Weyl spectrum. In addition, we characterize the generalized Weyl's theorem and the generalized Browder's theorem for an element $a \in \mathcal{A}$ and $f(a)$, where $f$ is a complex-valued function analytic on a neighborhood of $\sigma(a)$. What's more, the perturbations of the generalized Weyl's theorem under the socle of $\mathcal{A}$ and quasinilpotent element are illustrated.

**Key words:** Primitive C*-algebra, B-Browder elements, socle, the generalized Weyl's theorem, perturbation

## 1. Introduction

Throughout this paper, all algebras will be infinite dimensional unital Banach algebras over the field of complex numbers. Let $B(X)$ ($B(H)$) denote the algebra of bounded linear operators on an infinite dimensional Banach space $X$ (Hilbert space $H$).

In recent years, spectral theory, which has numerous and important applications in many parts of modern analysis and physics, has witnessed considerable development [2]. The classical Fredholm theory of bounded linear operators on a Banach space is familiar to many mathematicians [2, 3, 7, 9, 11, 13, 14, 25]. Fredholm theory in semisimple and semiprime algebras was pioneered by Barnes [4, 6]. This theory was extended to general Banach algebras by Smyth [29]. Smyth defined Fredholm elements, Weyl elements, Riesz elements and so on, and developed some of their elementary properties [29]. Subsequently, the spectral theories of Fredholm, Weyl, Browder and B-Fredholm elements have been developed by some scholars [2, 5, 16, 18, 20, 21, 23, 25, 26]. The purpose of this paper is to discuss the spectral theory of B-Weyl elements and B-Browder elements in a primitive C*-algebra $\mathcal{A}$.

P. Aiena [3] provided an introduction to some classes of operators which have their origin in the classical Fredholm theory of bounded linear operators on Banach spaces, including Fredholm operators, Weyl operators and Browder operators and so on. It is well known by the Atkinson's theorem [12, Theorem 0.2.2] that $T \in B(X)$ is a Fredholm operator if and only if $T$ is invertible modulo $F(X)$, where $F(X)$ means the set of all finite rank operators on $X$. B-Fredholm operators were introduced in [11] as a natural generalization of Fredholm operators and have been extensively studied in [7, 9, 11, 18]. Particularly, in 2001, Atkinson-type theorem for B-Fredholm

*Correspondence: renyanxun110@126.com
2010 *AMS Mathematics Subject Classification:* 16D60, 46H99, 47A05, 47A10, 47A53

operators was developed. In detail, the set of B-Fredholm operators on a Banach space $X$ is described as those bounded linear operators, which are Drazin invertible modulo the ideal of finite rank operators in $B(X)$. M. Berkani [7] defined B-Weyl operators and B-Weyl spectrum, and studied their properties. [16] argued the properties of B-Browder operators and characterized B-Fredholm spectrum, B-Weyl spectrum and B-Browder spectrum. More studies about the spectral theory of B-Fredholm operators, B-Weyl operators and B-Browder operators can be found in [7, 13–16, 25].

Now more and more scholars are devoted to generalizing the above to a general unital algebra. M. Berkani [10] defined B-Fredholm elements in a unital primitive Banach algebra $\mathcal{A}$, which are Drazin invertible modulo the socle. Meanwhile, M. Berkani [10] established a connection between B-Fredholm elements in $\mathcal{A}$ and B-Fredholm operators on $\mathcal{A}p$, where $p$ is a minimal idempotent in $\mathcal{A}$. In addition, [10] defined B-Weyl elements and described B-Fredholm elements of index 0 as the sum of a Drazin invertible element in $\mathcal{A}$ and an element in $\mathrm{Soc}(\mathcal{A})$ when $\mathcal{A}$ is a unital primitive Banach algebra satisfying the conditions in [10, Theorem 3.4]. [23] defined Browder elements, and characterized the socle of a primitive C*-algebra by B-Fredholm elements. In addition, [17] studied B-Browder elements with respect to a Banach algebra homomorphism. Motivated by all of the above, the first purpose of this paper is to consider the spectral mapping theorems and properties of B-Fredholm spectrum, B-Weyl spectrum and B-Browder spectrum.

In [31], H. Weyl proved his celebrated theorem on the structure of the spectrum of hermitian operators on a Hilbert space, which is called the Weyl's theorem. Weyl's theorem has been extended from hermitian operators to hyponormal and Toeplitz operators, and to several classes of operators including seminormal operators [13, 16]. Recently, M. Berkani and J.J. Koliha extended Weyl's theorem and Browder's theorem to generalized Weyl's theorem and generalized Browder's theorem, and they showed that $T$ satisfies the generalized Weyl's theorem whenever $T$ is a normal operator on Hilbert space. Kong et al. [24] characterized the Weyl's theorem in a semisimple Banach algebra and illustrated the perturbations of the Weyl's theorem. Illuminated by Berkani M. [7, 24], the second purpose of this paper is to discuss the generalized Weyl's theorem for an element in a primitive C*-algebra.

The structure of the paper is as follows. Section 2 lists some necessary conceptions and notations in a unital primitive C*-algebra. Section 3 gives a characterization of B-Weyl spectrum, and meanwhile, demonstrates the spectral mapping theorems of B-Weyl spectrum and B-Browder spectrum. Section 4 studies the generalized Weyl's theorem and the generalized Browder's theorem for an element $a$ and $f(a)$, respectively, where $f$ is a complex-valued function analytic on a neighborhood of $\sigma(a)$. Finally, the section explores how the generalized Weyl's theorem and the generalized Browder's theorem survive under the perturbation of $\mathrm{Soc}(\mathcal{A})$ or quasinilpotent elements.

## 2. Preliminaries

Recall that an algebra is called primitive if $\{0\}$ is a primitive ideal of $\mathcal{A}$. The socle of $\mathcal{A}$ in this case is the algebraic sum of all the minimal left ideals of $\mathcal{A}$ (which equals the sum of all the minimal right ideals), or $\{0\}$ if $\mathcal{A}$ has no minimal left ideals. Also, the socle of $\mathcal{A}$ (if it exists) denoted by $\mathrm{Soc}(\mathcal{A})$ is an ideal in $\mathcal{A}$. Recall also that it is well known that a primitive Banach algebra is a semiprime algebra.

**Definition 2.1** *[2, Page 244] Let $\mathcal{A}$ be any complex algebra. $e_0$ is called a minimal idempotent element if $e_0 \neq 0$ and $e_0^2 = e_0$ such that $e_0\mathcal{A}e_0$ is a division algebra.*

When $\mathcal{A}$ is a unital semisimple Banach algebra, the socle of $\mathcal{A}$ always exists from [28, Lemma 2.1.12]. We will assume that the socle of $\mathcal{A}$ is not reduced to $\{0\}$, so in this case, $\mathcal{A}$ possesses minimal idempotents [4]. Denote the set of all minimal idempotents of $\mathcal{A}$ by $\mathrm{Min}(\mathcal{A})$. Next we provide the concepts of Fredholm elements, Weyl elements and Browder elements.

**Definition 2.2** *[26, Page 587] Suppose that $\mathcal{A}$ is a unital semisimple Banach algebra. The element $a \in \mathcal{A}$ is called a Fredholm element if it is invertible modulo $\mathrm{Soc}(\mathcal{A})$. In other words, $a + \mathrm{Soc}(\mathcal{A})$ is an invertible element in $\mathcal{A}/\mathrm{Soc}(\mathcal{A})$. And the set of Fredholm elements is denoted by $\Phi(\mathcal{A})$.*

An element $a \in \mathcal{A}$ is called relatively regular if $aba = a$ for some $b \in \mathcal{A}$. In this case $b$ is called a pseudo-inverse of $a$. For the convenience, recall some concepts about Weyl element. For $a \in \mathcal{A}$, set $R(a) = \{x \in \mathcal{A} : ax = 0\}$, $L(a) = \{x \in \mathcal{A} : xa = 0\}$. Suppose that $J \subseteq \mathcal{A}$ is a right(left) ideal of $\mathcal{A}$. $J$ is called having finite order if it can be written as the sum of a finite number of minimal right(left) ideals of $\mathcal{A}$. The order $\Theta(J)$ of $J$ is defined to be the smallest number of minimal right(left) ideals satisfying the condition that the sum of them equals to $J$. Set $\Theta(\{0\}) = 0$, and $\Theta(J) = +\infty$ if $J$ does not have finite order [26, Page 586].

**Definition 2.3** *[26, Page 587] Suppose $\mathcal{A}$ is a semisimple Banach algebra. For $a \in \mathcal{A}$, the nullity of $a$ is defined by*

$$\mathrm{nul}(a) = \Theta(R(a)),$$

*and the defect of $a$ is defined by*

$$\mathrm{def}(a) = \Theta(L(a)).$$

Applying with [26, Proposition 3.5], $a$ is a Fredholm element if and only if it is relatively regular and $\mathrm{nul}(a) < \infty$, $\mathrm{def}(a) < \infty$. Define the index of a Fredholm element $a$ as follows: $\mathrm{index}(a) = \mathrm{nul}(a) - \mathrm{def}(a)$. We call an element $a \in \mathcal{A}$ a Weyl element if it is a Fredholm element with $\mathrm{index}(a) = 0$.

Let $T \in B(X)$, we are already acquainted with the ascent of $T$ denoted by $\alpha(T)$, the descent of $T$ denoted by $\beta(T)$ and the index of $T$ denoted by $\mathrm{ind}(T)$. Next let us introduce the ascent and descent of $a \in \mathcal{A}$ illustrated by the operator ascent and descent.

Let $a \in \mathcal{A}$ and let the linear operator $L_a : \mathcal{A} \longrightarrow \mathcal{A}$ be defined by

$$L_a(x) = ax \text{ for any } x \in \mathcal{A}.$$

Put $p_l(a) = \alpha(L_a)$ and $q_l(a) = \beta(L_a)$, we call $p_l(a)$ the ascent of $a$ and $q_l(a)$ the descent of $a$.

**Definition 2.4** *[24, Definition 2.3] Suppose $\mathcal{A}$ is a unital semisimple Banach algebra and $a \in \mathcal{A}$. If $a$ is a Fredholm element, and $p_l(a) < \infty$, $q_l(a) < \infty$, then $a$ is called a Browder element.*

Recall that an element $a \in \mathcal{A}$ is said to be Drazin invertible in $\mathcal{A}$ if there exist a unique $b \in \mathcal{A}$ and some $k \in \mathbf{N}$ such that $bab = b$, $ab = ba$, $a^k ba = a^k$ [17, Page 3730]. Suppose $\mathcal{A}$ is a unital semisimple Banach algebra. Clearly, an invertible element in $\mathcal{A}$ must be a Drazin invertible element. Conversely, if $a \in \mathcal{A}$ is a Drazin invertible element with $\mathrm{nul}(a) = 0$, it is not difficult to show that $a$ must be an invertible element. The following aims to introduce B-Fredholm elements and B-Weyl elements and B-Browder elements.

The quotient algebra $\mathcal{A}/\mathrm{Soc}(\mathcal{A})$ is written as $\hat{\mathcal{A}}$. Evidently, $\hat{\mathcal{A}}$ is not a Banach algebra since $\mathrm{Soc}(\mathcal{A})$ is not closed [2, Page 245]. For $a \in \mathcal{A}$, we denote the canonical homomorphism

$$\begin{array}{rlcl} \pi: & \mathcal{A} & \to & \hat{\mathcal{A}} \\ & a & \mapsto & a + \mathrm{Soc}(\mathcal{A}), \end{array}$$

and write $\hat{a} = a + \mathrm{Soc}(\mathcal{A})$ for the coset of $a$ in $\hat{\mathcal{A}}$.

**Definition 2.5** *[10, Definition 1.2] Let $\mathcal{A}$ be a unital semiprime Banach algebra. An element $a \in \mathcal{A}$ is called a B-Fredholm element of $\mathcal{A}$ if it is Drazin invertible modulo $\mathrm{Soc}(\mathcal{A})$. In other words, $\hat{a}$ is Drazin invertible in $\hat{\mathcal{A}}$.*

**Definition 2.6** *[9, Definition 2.1] Let $I$ be an ideal in a Banach algebra $\mathcal{A}$. A function $\tau : I \to \mathbf{C}$ is called a trace on $I$ if*

*(1) $\tau(p) = 1$ if $p \in I$ is an idempotent of rank one.*

*(2) $\tau(a + b) = \tau(a) + \tau(b)$ for all $a, b \in I$.*

*(3) $\tau(\alpha a) = \alpha\tau(a)$ for all $\alpha \in \mathbf{C}$ and $a \in I$.*

*(4) $\tau(ab) = \tau(ba)$ for all $a \in I$ and $b \in \mathcal{A}$.*

**Definition 2.7** *[10, Definition 3.2] Let $\tau$ be a trace on the socle $\mathrm{Soc}(\mathcal{A})$ of a unital primitive Banach algebra $\mathcal{A}$. The index of a B-Fredholm element $a \in \mathcal{A}$ is defined by*

$$i(a) = \tau(aa_0 - a_0a) = \tau[a, a_0],$$

*where $a_0$ is a Drazin inverse of $a$ modulo the socle of $\mathcal{A}$.*

From [9, Theorem 2.3], the index of a B-Fredholm element $a \in \mathcal{A}$ is well-defined and is independent of the Drazin inverse $a_0$ of $a$ modulo the ideal $\mathrm{Soc}(\mathcal{A})$. Since invertible elements are always Drazin invertible, it follows immediately that Fredholm elements are B-Fredholm elements.

**Definition 2.8** *[10, Definition 3.3] Suppose that $\mathcal{A}$ is a unital primitive Banach algebra and $a \in \mathcal{A}$. Then $a$ is called a B-Weyl element if it is a B-Fredholm element of index 0.*

Next, it is necessary to review B-Browder elements.

**Definition 2.9** *[23, Definition 3.3.1] Assume that $\mathcal{A}$ is a unital semisimple Banach algebra, $a \in \mathcal{A}$ is called a B-Browder element if there exist a Drazin invertible element $b$ and $c \in \mathrm{Soc}(\mathcal{A})$ such that $bc = cb$ and $a = b + c$.*

Suppose $\mathcal{A}$ is a unital primitive $C^*$-algebra and $a \in \mathcal{A}$, from [10, Theorem 3.4], one can see that $a$ is a B-Weyl element if and only if there exist a Drazin invertible element $b$ and $c \in \mathrm{Soc}(\mathcal{A})$ such that $a = b + c$. Hence, a B-Browder element must be a B-Weyl element, a B-Weyl element must be a B-Fredholm element. One can verify that Fredholm elements must be B-Fredholm elements. In the next section, it can be proved that Weyl elements must be B-Weyl elements and Browder elements must be B-Browder elements.

When $\mathcal{A}$ is a unital primitive C*-algebra, we will assume that the socle of $\mathcal{A}$ is not reduced to $\{0\}$, so in this case, $\mathcal{A}$ possesses minimal idempotents [4], let $p$ be a minimal idempotent in $\mathcal{A}$ in the whole paper [4].

The set of bounded linear operators on $\mathcal{A}p$ is denoted by $B(\mathcal{A}p)$. The left regular representation of $\mathcal{A}$ on the Banach space $\mathcal{A}p$ is defined by $\Gamma : \mathcal{A} \to B(\mathcal{A}p)$, such that $\Gamma(a) = L_a$ for any $a \in \mathcal{A}$. Recall that

$$\Gamma(\mathrm{Soc}(\mathcal{A})) = F(\mathcal{A}p) \subseteq B(\mathcal{A}p)$$

by [4, Theorem F.4.3], where $F(\mathcal{A}p)$ denotes the set of all finite rank operators on $\mathcal{A}p$. From [4, F.2.1], it follows that the nullity and defect of the operator $L_a \in B(\mathcal{A}p)$ are independent of the choice of $p \in \mathrm{Min}(\mathcal{A})$.

Inspired by M. Berkani [10], one can see that $a \in \mathcal{A}$ is a Fredholm (B-Fredholm) element if and only if $L_a$ is a Fredholm (B-Fredholm) operator on $\mathcal{A}p$ when $\mathcal{A}$ is a primitive $C^*$-algebra, which is the reason of considering the primitive $C^*$-algebra in this paper. Next, some related spectrums are introduced as follows:

**Definition 2.10** *[23, Definition 3.3.2] Suppose $\mathcal{A}$ is a semisimple Banach algebra with a unit $e$ and $a \in \mathcal{A}$, the spectrum of $a$, the Drazin spectrum of $a$, the Fredholm spectrum of $a$, the Weyl spectum of $a$, the Browder spectrum of $a$, the B-Fredholm spectrum of $a$, the B-Weyl spectum of $a$, the B-Browder spectrum of $a$ are defined by:*

$$\sigma(a) = \{\lambda \in \mathbf{C} : a - \lambda e \textit{ is not an invertible element}\};$$

$$\sigma_D(a) = \{\lambda \in \mathbf{C} : a - \lambda e \textit{ is not a Drazin invertible element}\};$$

$$\sigma_{ess}(a) = \{\lambda \in \mathbf{C} : a - \lambda e \textit{ is not a Fredholm element}\};$$

$$\sigma_w(a) = \{\lambda \in \mathbf{C} : a - \lambda e \textit{ is not a Weyl element}\};$$

$$\sigma_b(a) = \{\lambda \in \mathbf{C} : a - \lambda e \textit{ is not a Browder element}\};$$

$$\sigma_{BF}(a) = \{\lambda \in \mathbf{C} : a - \lambda e \textit{ is not a } B - \textit{Fredholm element}\};$$

$$\sigma_{BW}(a) = \{\lambda \in \mathbf{C} : a - \lambda e \textit{ is not a } B - \textit{Weyl element}\};$$

$$\sigma_{BB}(a) = \{\lambda \in \mathbf{C} : a - \lambda e \textit{ is not a } B - \textit{Browder element}\};$$

*respectively. Note that $a - \lambda e$ is always abbreviated to $a - \lambda$. Corresponding, set $\rho(a) = \mathbf{C} \setminus \sigma(a)$, $\rho_D(a) = \mathbf{C} \setminus \sigma_D(a)$, $\rho_{ess}(a) = \mathbf{C} \setminus \sigma_{ess}(a)$, $\rho_w(a) = \mathbf{C} \setminus \sigma_w(a)$, $\rho_b(a) = \mathbf{C} \setminus \sigma_b(a)$, $\rho_{BF}(a) = \mathbf{C} \setminus \sigma_{BF}(a)$, $\rho_{BW}(a) = \mathbf{C} \setminus \sigma_{BW}(a)$, and $\rho_{BB}(a) = \mathbf{C} \setminus \sigma_{BB}(a)$. Suppose $K \subseteq \mathbf{C}$, $\mathrm{iso}K$ denotes the set of isolated points of $K$, $\mathrm{acc}K$ denotes the set of accumulation points of $K$.*

Now some necessary conceptions and notations have been listed. In the following, the spectral theories of B-Fredholm elements, B-Weyl elements and B-Browder elements in a primitive C*-algebra will be studied.

## 3. B-Weyl spectrum and B-Browder spectrum

From now on, we always assume that $\mathcal{A}$ is a primitive C*-algebra with a unit $e$ if there are no special instructions. The major purpose of this section is to characterize B-Weyl spectrum and study the difference between B-Weyl spectrum and B-Browder spectrum by the spectral mapping theorem. Firstly, a characterization of B-Weyl spectrum is illustrated.

**Proposition 3.1** *Let $a \in \mathcal{A}$, then $\sigma_{BW}(a) = \bigcap\limits_{c \in \mathrm{Soc}(\mathcal{A})} \sigma_D(a + c)$.*

**Proof** Firstly, we prove that $\sigma_{BW}(a) \subseteq \bigcap\limits_{c\in \mathrm{Soc}(\mathcal{A})} \sigma_D(a+c)$.

If $\lambda \notin \bigcap\limits_{c\in \mathrm{Soc}(\mathcal{A})} \sigma_D(a+c)$, there exists $c_0 \in \mathrm{Soc}(\mathcal{A})$ such that $a+c_0-\lambda$ is Drazin invertible in $\mathcal{A}$. One can get that $L_{c_0}$ is a finite rank operator in $B(\mathcal{A}p)$ since $\mathcal{A}$ is a primitive C*-algebra [4, Theorem F.4.3]. It is easy to check that $L_a + L_{c_0} - \lambda L_e$ is a Drazin invertible operator. It follows that

$$\lambda \notin \bigcap_{F\in F(\mathcal{A}p)} \sigma_D(L_a+F),$$

which implies that $\lambda \notin \sigma_{BW}(L_a)$ from the Theorem 4.3 in [7]. Hence, $L_a - \lambda$ is a B-Weyl operator, and $L_a-\lambda$ is a B-Fredholm operator, which implies $a-\lambda$ is a B-Fredholm element since $\mathcal{A}$ is a primitive $C^*$-algebra [10, Theorem 3.8]. Applying [9, Lemma 3.2], one can get that $i(a-\lambda) = \mathrm{ind}(L_a-\lambda) = 0$, and moreover $a-\lambda$ is a B-Weyl element. In other words, $\lambda \notin \sigma_{BW}(a)$.

Secondly, there is no harm in supposing $0 \notin \sigma_{BW}(a)$. In other words, $a$ is a B-Weyl element. It follows that $L_a$ is a B-Weyl operator. That is to say, $0 \notin \sigma_{BW}(L_a)$. It can be proved that there exists $F \in F(\mathcal{A}p)$ such that $L_a + F$ is a Drazin invertible operator from [7, Theorem 4.3]. One can get that there exists $t \in \mathrm{Soc}(\mathcal{A})$ such that $F = L_t$ since $\mathcal{A}$ is a primitive C*-algebra [4, Theorem F.4.3]. Hence, $L_{a+t}$ is a Drazin invertible operator in $B(\mathcal{A}p)$, which implies that $L_{a+t}$ is a Drazin invertible operator in $\Gamma(\mathcal{A})$. Since $\mathcal{A}$ is a primitive $C^*$-algebra, it follows from [4, Page 30] that the left regular representation of $\mathcal{A}$ is faithful, which implies that $a+t$ is a Drazin invertible element in $\mathcal{A}$. Therefore, $0 \notin \bigcap\limits_{c\in \mathrm{Soc}(\mathcal{A})} \sigma_D(a+c)$. Consequently, one has $\sigma_{BW}(a) = \bigcap\limits_{c\in \mathrm{Soc}(\mathcal{A})} \sigma_D(a+c)$. □

Let $a \in \mathcal{A}$, denote the set of complex-valued analytic functions on a neighborhood of $\sigma(a)$ by $Hol(a)$. Next, the spectral mapping theorem of B-Browder spectrum is stated as follows:

**Theorem 3.2** *[Spectral Mapping Theorem of B-Browder Spectrum] Suppose $a \in \mathcal{A}$, $f \in Hol(a)$, then $\sigma_{BB}(f(a)) = f(\sigma_{BB}(a))$.*

**Proof** Suppose $\mu \notin f(\sigma_{BB}(a))$, there is no harm in assuming that

$$f(\lambda) - \mu = (\lambda-\lambda_1)^{n_1}(\lambda-\lambda_2)^{n_2}\cdots(\lambda-\lambda_k)^{n_k}h(\lambda),$$

where $\lambda_1, \lambda_2, \cdots, \lambda_k \in \sigma(a)$, $n_1, n_2, \cdots, n_k \in \mathbf{N}$, $h \in Hol(a)$ and $h(a)$ is an invertible element in $\mathcal{A}$. Hence,

$$f(a) - \mu = (a-\lambda_1)^{n_1}(a-\lambda_2)^{n_2}\cdots(a-\lambda_k)^{n_k}h(a),$$

where $\lambda_1, \lambda_2, \cdots, \lambda_k \in \sigma(a)$, $n_1, n_2, \cdots, n_k \in \mathbf{N}$. Evidently, $f(\lambda_i) = \mu$ for $i = 1, 2, \cdots, k$. One can get that $a - \lambda_i$ is a B-Browder element for $i = 1, 2, \cdots, k$, which implies that $L_{a-\lambda_i}$ is a B-Browder operator since $\mathcal{A}$ is a primitive $C^*$-algebra [23, Lemma 3.3.4]. According to the fact that $L_a - \lambda_i$ is a B-Browder operator, then $\lambda_i \notin \sigma_{BB}(L_a)$. Therefore, $f(\lambda_i) \notin f(\sigma_{BB}(L_a))$. Applying Theorem 2.9 in [16], one can obtain that $f(\lambda_i) \notin \sigma_{BB}(f(L_a))$. In other words, $f(L_a) - \mu$ is a B-Browder operator. Associated with the relation $p(L_a) = L_{p(a)}$ holds for any polynomial $p$ with complex coefficients, it follows that $f(L_a) = L_{f(a)}$ for any $f \in Hol(a)$ by [30, Page 269], which implies that $L_{f(a)-\mu}$ is a B-Browder operator. This leads to the conclusion that $f(a) - \mu$ is a B-Browder element [23, Lemma 3.3.4]. Therefore, $\mu \notin \sigma_{BB}(f(a))$.

Conversely, suppose $\mu \notin \sigma_{BB}(f(a))$, then $f(a)-\mu$ is a B-Browder element. It follows that $L_{f(a)-\mu}$ is a B-Browder operator [23, Lemma 3.3.4]. In other words, one can see that $\mu \notin \sigma_{BB}(L_{f(a)}) = \sigma_{BB}(f(L_a)) = f(\sigma_{BB}(L_a))$ referred to [16, Theorem 2.7] and [18, Theorem 2.2]. Suppose

$$f(L_a)-\mu = (L_a-\lambda_1)^{n_1}(L_a-\lambda_2)^{n_2}\cdots(L_a-\lambda_k)^{n_k}h(L_a),$$

where $\lambda_1,\lambda_2,\cdots,\lambda_k \in \sigma(a)$, $n_1,n_2,\cdots,n_k \in \mathbf{N}$, $h \in Hol(a)$ and $h(a)$ is an invertible element in $\mathcal{A}$. One can show that $\lambda_i \notin \sigma_{BB}(L_a)$ for any $i=1,2,\cdots,k$. Then $L_a-\lambda_i$ is a B-Browder operator, which implies that $a-\lambda_i$ is a B-Browder element for $i=1,2,\cdots,k$ because $\mathcal{A}$ is a primitive $C^*$-algebra [23, Lemma 3.3.4]. Hence, $\lambda_i \notin \sigma_{BB}(a)$. This leads to the conclusion that

$$f(\lambda_i)=\mu \notin f(\sigma_{BB}(a)).$$

Therefore, the relation $\sigma_{BB}(f(a)) = f(\sigma_{BB}(a))$ holds. □

We have shown the spectral mapping theorem of B-Browder spectrum. The following will give a necessary and sufficient condition that the spectral mapping theorem of B-Weyl spectrum holds.

**Proposition 3.3** *Suppose $\mathcal{A}$ is a unital primitive Banach algebra. If $a \in \mathcal{A}$, $f \in Hol(a)$, then $\sigma_{BW}(f(a)) \subseteq f(\sigma_{BW}(a))$.*

**Proof** Suppose $\mu \notin f(\sigma_{BW}(a))$, let us assume that

$$f(\lambda)-\mu = (\lambda-\lambda_1)^{n_1}(\lambda-\lambda_2)^{n_2}\cdots(\lambda-\lambda_k)^{n_k}h(\lambda),$$

where $\lambda_1,\lambda_2,\cdots,\lambda_k \in \sigma(a)$, $n_1,n_2,\cdots,n_k \in \mathbf{N}$, $h \in Hol(a)$ and $h(a)$ is an invertible element in $\mathcal{A}$. Hence,

$$f(a)-\mu = (a-\lambda_1)^{n_1}(a-\lambda_2)^{n_2}\cdots(a-\lambda_k)^{n_k}h(a),$$

where $\lambda_1,\lambda_2,\cdots,\lambda_k \in \sigma(a)$, $n_1,n_2,\cdots,n_k \in \mathbf{N}$. One can get that $\lambda_i \in \rho_{BW}(a)$ by means of $\mu \notin f(\sigma_{BW}(a))$, where $i=1,2,\cdots,k$. In other words, $a-\lambda_i$ is a B-Weyl element for any $i=1,2,\cdots,k$. Since $\mathcal{A}$ is a primitive Banach algebra, it follows from [9, Proposition 3.3] that $f(a)-\mu$ is a B-Fredholm element and

$$i(f(a)-\mu) = \sum_{i=1}^{k} n_i \cdot i(a-\lambda_i) + i(h(a)) = 0.$$

Thus, $f(a)-\mu$ is a B-Weyl element and $\mu \notin \sigma_{BW}(f(a))$. □

However, the spectral mapping theorem of B-Weyl spectrum does not hold; in other words, "$\sigma_{BW}(f(a)) = f(\sigma_{BW}(a))$" does not hold for $a \in \mathcal{A}$, $f \in Hol(a)$. One can refer to the following example.

**Example 3.4** *Let $U \in B(l^2)$ be the unilateral shift and consider the operator $T = U \bigoplus (U^*+2)$. Let $p(z) = z(z-2)$, then $p \in Hol(T)$. From [16, Example 2.8], one can get that $0 \notin \sigma_{BW}(p(T))$, but $0 \in p(\sigma_{BW}(T))$. Hence, $\sigma_{BW}(p(T)) \neq p(\sigma_{BW}(T))$.*

In the following, we consider when the spectral mapping theorem of B-Weyl spectrum holds.

**Theorem 3.5** *[Spectral Mapping Theorem of B-Weyl Spectrum] Suppose $a \in \mathcal{A}$, $f \in Hol(a)$, then the following statements are equivalent:*

*(1) $\sigma_{BW}(f(a)) = f(\sigma_{BW}(a))$ for all $f \in Hol(a)$.*

*(2) either $i(a-\lambda) \geq 0$ for all $\lambda \in \rho_{BF}(a)$ or $i(a-\lambda) \leq 0$ for all $\lambda \in \rho_{BF}(a)$.*

**Proof** (1) $\Rightarrow$ (2). Suppose that there exist $\lambda_0, \mu_0 \in \rho_{BF}(a)$ such that

$$0 < m = i(a-\lambda_0),\ i(a-\mu_0) = n < 0.$$

Define $f(x) = (x-\lambda_0)^n(x-\mu_0)^m$, one can get

$$f(a) = (a-\lambda_0)^n(a-\mu_0)^m.$$

It follows from [9, Proposition 3.3] that $f(a)$ is a B-Fredholm element and $i(f(a)) = 0$. Therefore, $f(a)$ is a B-Weyl element. In other words, $0 \notin \sigma_{BW}(f(a))$. Since $\sigma_{BW}(f(a)) = f(\sigma_{BW}(a))$ for all $f \in Hol(a)$, it is clear that $0 \notin f(\sigma_{BW}(a))$. However, $f(\lambda_0) = f(\mu_0) = 0$ and $\lambda_0, \mu_0 \in \sigma_{BW}(a)$, in other words, $0 \in f(\sigma_{BW}(a))$, which is a contradiction with $0 \notin f(\sigma_{BW}(a))$. Hence, either $i(a-\lambda) \geq 0$ for all $\lambda \in \rho_{BF}(a)$ or $i(a-\lambda) \leq 0$ for all $\lambda \in \rho_{BF}(a)$.

(2) $\Rightarrow$ (1). From Proposition 3.3, it suffices to prove that $f(\sigma_{BW}(a)) \subseteq \sigma_{BW}(f(a))$. Suppose $\mu \notin \sigma_{BW}(f(a))$; in other words, $f(a)-\mu$ is a B-Weyl element. Suppose

$$f(\lambda)-\mu = (\lambda-\lambda_1)^{n_1}(\lambda-\lambda_2)^{n_2}\cdots(\lambda-\lambda_k)^{n_k}h(\lambda),$$

where $\lambda_1, \lambda_2, \cdots, \lambda_k \in \sigma(a)$, $n_1, n_2, \cdots, n_k \in \mathbf{N}$, $h \in Hol(a)$ and $h(a)$ is an invertible element in $\mathcal{A}$. Hence,

$$f(a)-\mu = (a-\lambda_1)^{n_1}(a-\lambda_2)^{n_2}\cdots(a-\lambda_k)^{n_k}h(a),$$

where $\lambda_1, \lambda_2, \cdots, \lambda_k \in \sigma(a)$, $n_1, n_2, \cdots, n_k \in \mathbf{N}$. Evidently, $f(a)-\mu$ is a B-Fredholm element, then $L_{f(a)-\mu}$ is a B-Fredholm operator [10, Theorem 3.6]. It follows that

$$L_{f(a)-\mu} = L_{f(a)} - L_{\mu} = (L_a - L_{\lambda_1})^{n_1}(L_a - L_{\lambda_2})^{n_2}\cdots(L_a - L_{\lambda_k})^{n_k}L_{h(a)}.$$

According to Theorem 3.4 in [11], one can get that $L_a - L_{\lambda_i}$ is a B-Fredholm operator for $i = 1, 2, \cdots, k$, and thus $a-\lambda_i$ $(i = 1, 2, \cdots, k)$ is a B-Fredholm element since $\mathcal{A}$ is a primitive C*-algebra. Without loss of generality, one can suppose $i(a-\lambda) \geq 0$ for all $\lambda \in \rho_{BF}(a)$. Since $f(a)-\mu$ is a B-Fredholm element, it follows from [9, Proposition 3.3] that

$$i(f(a)-\mu) = \sum_{i=1}^{k} n_i \cdot i(a-\lambda_i) + i(h(a)) = 0,$$

which implies that $i(a-\lambda_i) = 0$ since $i(a-\lambda) \geq 0$ for all $\lambda \in \rho_{BF}(a)$. Hence, $a-\lambda_i$ $(i = 1, 2, \cdots, k)$ is a B-Weyl element. In other words, $\lambda_i \notin \sigma_{BW}(a)$. Consequently, $\mu \notin f(\sigma_{BW}(a))$. This completes the proof. □

Next, an example satisfying the condition (2) in Theorem 3.5 is provided. Let $T \in B(l^2)$ be defined by $T(x_1, x_2, \cdots) = (\frac{x_2}{2}, \frac{x_3}{3}, \cdots)$. From [32, Remark 2.1] and [25, Theorem 2.1], it follows that $T$ satisfies the conditions (1) and (2) in Theorem 3.5.

**Remark 3.6** *We can classify B-Fredholm elements according to Theorem 3.5. P. Aiena defined the upper semi-B-Weyl operators and the lower semi-B-Weyl operators and studied their properties [3, Definition 3.45]. Inspired by P. Aiena and based on Theorem 3.5, one can divide B-Fredholm elements into two classes, one is upper semi-B-Weyl elements and the other is lower semi-B-Weyl elements. In detail, suppose $\mathcal{A}$ is a unital primitive $C^*$-algebra and $a \in \mathcal{A}$, $a$ is called an upper (lower) semi-B-Fredholm element if $L_a$ is an upper (lower) semi-B-Fredholm operator. We call the element $a$ an upper semi-B-Weyl element if it is an upper semi-B-Fredholm element with $i(a) \leq 0$. Similarly, we call the element $a$ a lower semi-B-Weyl element if it is a lower semi-B-Fredholm element with $i(a) \geq 0$. Denote the upper (lower) semi-B-Weyl elements by $USB(\mathcal{A})$ ($LSB(\mathcal{A})$), respectively.*

This section considers the properties of B-Weyl spectrum and B-Browder spectrum. Based on the above spectral theories, the generalized Weyl's theorem and the generalized Browder's theorem for an element in a primitive C*-algebra will be discussed in the next section, which generalizes the operator situation in $B(H)$.

## 4. Generalized Weyl's theorem and generalized Browder's theorem in primitive C*-algebra

This section aims to consider the generalized Weyl's theorem and the generalized Browder's theorem for $a \in \mathcal{A}$ and $f(a)$, where $f \in Hol(a)$. Meanwhile, the perturbations of the generalized Weyl's theorem and the generalized Browder's theorem are investigated.

In this section, we always assume that $\mathcal{A}$ is a primitive C*-algebra with a unit $e$ if there are no special instructions. Let $T \in B(H)$, denote the dimension of the null space of $T$ and the co-dimension of the range of $T$ by $n(T)$ and $d(T)$, respectively. If $\mathcal{A}$ is a primitive C*-algebra and $a \in \mathcal{A}$, we define the rank of $a$ by $\text{rank}(a) = \text{rank}(L_a)$. In order to study the generalized Weyl's theorem and the generalized Browder's theorem for an element in a primitive C*-algebra, some lemmas will be required as follows.

**Lemma 4.1** *[4, F.2.4] Suppose $\mathcal{A}$ is a unital primitive $C^*$-algebra, then $\text{Soc}(\mathcal{A}) = \{x \in \mathcal{A} : \text{rank}(x) < \infty\}$.*

**Definition 4.2** *[19, Definition 3.9] Suppose $\mathcal{A}$ is a semisimple Banach algebra with a unit $e$. An idempotent $s$ is called a left Barnes idempotent for $a \in \mathcal{A}$ if $a\mathcal{A} = (e-s)\mathcal{A}$ and a right Barnes idempotent for $a \in \mathcal{A}$ if $\mathcal{A}a = \mathcal{A}(e-s)$.*

Recall that $\tau(a)$ means the trace of $a$, where $\tau$ is a spectral trace on $\text{Soc}(\mathcal{A})$. The trace on finite rank elements is spectral; in other words, if $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of $a$ each repeated according to its algebraic multiplicity, then $\tau(a) = \sum\limits_{i=1}^{n} \lambda_i$ [19, Page 4]. Since we have for every idempotent $p \in \text{Soc}(\mathcal{A})$ that $\sigma(p) \subseteq \{0, 1\}$, we immediately have that $\tau(p) \in \mathbf{N}$ for every such $p$ (see [19, The trace 2.1]). For detailed introduction, one can refer to [19, Page 4].

**Lemma 4.3** *[19, Corollary 3.15] Suppose $\mathcal{A}$ is a unital semisimple Banach algebra. If $a$ is a Fredholm element of $\mathcal{A}$ with left Barnes idempotent $s$, then $\tau(s) = \text{rank}(s)$ and similarly, for a right Barnes idempotent $t$ we have $\tau(t) = \text{rank}(t)$.*

**Proposition 4.4** *Suppose $a \in \Phi(\mathcal{A})$, where $\Phi(\mathcal{A})$ means the set of all Fredholm elements in $\mathcal{A}$, then $\text{nul}(a) = n(L_a)$ and $\text{def}(a) = d(L_a)$.*

**Proof** Since $a \in \Phi(\mathcal{A})$, then $a$ is left (and right) invertible modulo $\mathrm{Soc}(\mathcal{A})$. From [4, F.1.10], $a$ has a right (and left) Barnes idempotent in $\mathrm{Soc}(\mathcal{A})$. There is no harm in supposing $q$ is a right Barnes idempotent in $\mathrm{Soc}(\mathcal{A})$ of $a$. Since $\mathcal{A}$ is a primitive C*-algebra, associated with [4, F.2.6], one can get that

$$\ker(L_a) = R(a) \bigcap \mathcal{A}p = q\mathcal{A} \bigcap \mathcal{A}p = q\mathcal{A}p,$$

where $p$ is the minimal idempotent in $\mathcal{A}$. It follows that

$$n(L_a) = \mathrm{rank}(L_q) = \mathrm{rank}(q) = \tau(q),$$

where $\tau(q)$ means the trace of $q$. Applying [19, Theorem 3.14], $\tau(q) = \theta(R(a)) = \mathrm{nul}(a)$. Hence, $\mathrm{nul}(a) = n(L_a)$.

Next we prove $\mathrm{def}(a) = d(L_a)$. Since $a$ is right invertible modulo $\mathrm{Soc}(\mathcal{A})$, it has a left Barnes idempotent $l$ in $\mathrm{Soc}(\mathcal{A})$ [4, Theorem F.1.10]. Because $\mathcal{A}$ is primitive, one can prove that

$$L_a(\mathcal{A}p) = a\mathcal{A}p = (e-l)\mathcal{A}p$$

and

$$\mathcal{A}p/L_a(\mathcal{A}p) = \mathcal{A}p/(e-l)\mathcal{A}p = l\mathcal{A}p.$$

So it can be obtained that $d(L_a) = \mathrm{rank}(L_l) = \mathrm{rank}(l)$. According to Lemma 4.3, it follows that $\mathrm{rank}(l) = \tau(l)$. Associated with [19, Theorem 3.14], this indicates that $\mathrm{rank}(l) = \tau(l) = \theta(L(a)) = \mathrm{def}(a) = d(L_a)$. This completes the proof. □

The following generalizes the generalized Weyl's (Browder's) theorem for an operator $T \in B(X)$ to a general situation and shows that for an element $a$ in a primitive C*-algebra, if $a$ satisfies the generalized Weyl's theorem, then $a$ satisfies the generalized Browder's theorem.

**Definition 4.5** *Suppose $\mathcal{A}$ is a unital semisimple Banach algebra. We say that the generalized Weyl's theorem holds for $a \in \mathcal{A}$ if $\sigma(a)\backslash\sigma_{BW}(a) = \pi_0(a)$, and the generalized Browder's theorem holds for $a$ if $\sigma(a)\backslash\sigma_{BW}(a) = P_0(a)$. Here $\pi_0(a) = \{\lambda \in \mathrm{iso}\sigma(a) : \mathrm{nul}(a-\lambda) \neq 0\}$, and $P_0(a) = \{\lambda : \lambda$ is a pole of the resolvent of $a\}$.*

Recall that the left regular representation of the primitive C*-algebra $\mathcal{A}$ on the Banach space $\mathcal{A}p$ is defined by

$$\Gamma : \mathcal{A} \to B(\mathcal{A}p) \text{ such that } \Gamma(a) = L_a,$$

where $p$ is the minimal idempotent element in $\mathcal{A}$. According to [1, Page 903], it follows that $\Gamma$ is an isometric irreducible $*$-representation. Next, an indispensable lemma is provided.

**Lemma 4.6** *An element $a \in \mathcal{A}$ is a Drazin invertible element if and only if $p_l(a) < \infty$ and $q_l(a) < \infty$.*

**Proof** Suppose that $p_l(a) < \infty$ and $q_l(a) < \infty$, in other words, $\alpha(L_a) < \infty$ and $\beta(L_a) < \infty$. Hence, $L_a$ is a Drazin invertible operator [16, Page 1425]. Since $\mathcal{A}$ is a primitive C*-algebra, it follows that $\Gamma(\mathcal{A})$ is Drazin inverse closed in $B(\mathcal{A}p)$ from Example 3.5 in [10], where $p$ is the minimal idempotent in $\mathcal{A}$. Therefore, there exists $L_b \in \Gamma(\mathcal{A})$ such that $L_b$ is a Drazin inverse of $L_a$, where $b \in \mathcal{A}$. That is to say,

$$L_bL_aL_b = L_b, L_aL_b = L_bL_a$$

and

$$(L_a)^k L_b L_a = (L_a)^k = L_{a^k} = L_{a^k} L_b L_a.$$

Note that the left regular representation $\Gamma$ is faithful since $\mathcal{A}$ is a primitive C*-algebra [1, Page 903], which implies that

$$bab = b, ab = ba \text{ and } a^k ba = a^k,$$

Hence, $a$ is a Drazin invertible element.

Conversely, if $a \in \mathcal{A}$ is a Drazin invertible element, then there exist $b \in \mathcal{A}$ and $k \in \mathbf{N}$ such that $bab = b, ab = ba, a^k ba = a^k$. It follows that

$$L_b L_a L_b = L_b, L_a L_b = L_b L_a$$

and

$$L_{a^k} L_b L_a = L_{a^k} = (L_a)^k = (L_a)^k L_b L_a,$$

which implies that $L_a$ is a Drazin invertible operator in $B(\mathcal{A}p)$. Applying [16, Page 1425], one can get that $\alpha(L_a) < \infty$ and $\beta(L_a) < \infty$. One can see $p_l(a) < \infty$ and $q_l(a) < \infty$ from the definitions of the ascent and descent of an element. This completes the proof. $\square$

Next, the characterizations of the generalized Browder's theorem will be presented.

**Theorem 4.7** *Suppose $a \in \mathcal{A}$, then the following statements are equivalent:*

*(1) $a$ satisfies the generalized Browder's theorem.*

*(2) $\sigma_{BW}(a) = \sigma_{BB}(a)$.*

*(3) $\sigma(a) = \sigma_{BW}(a) \bigcup \pi_0(a)$, where $\pi_0(a) = \{\lambda \in \mathrm{iso}\sigma(a) : \mathrm{nul}(a-\lambda) > 0\}$.*

*(4) $\mathrm{acc}\sigma(a) \subseteq \sigma_{BW}(a)$.*

*(5) $\sigma(a) \backslash \sigma_{BW}(a) \subseteq \pi_0(a)$.*

**Proof** $(1) \Rightarrow (2)$. It suffices to prove that $\sigma_{BB}(a) \subseteq \sigma_{BW}(a)$. Suppose $a-\lambda$ is a B-Weyl element, it is not harmful to suppose $\lambda \in \sigma(a) \setminus \sigma_{BW}(a)$. Since $a$ satisfies the generalized Browder's theorem, then one can get

$$p_l(a-\lambda) < \infty \text{ and } q_l(a-\lambda) < \infty,$$

which means that $a-\lambda$ is a Drazin invertible element in $\mathcal{A}$ by Lemma 4.6. Therefore, for any $b \in \mathrm{Soc}(\mathcal{A})$ with $(a-\lambda)b = b(a-\lambda)$, $a-\lambda+b$ is also Drazin invertible in $\mathcal{A}$. In other words, $\lambda \notin \sigma_D(a+b)$. Applying [23, Proposition 3.3.3], it follows that $\lambda \notin \sigma_{BB}(a)$, which implies that $a-\lambda$ is a B-Browder element. Consequently, $\sigma_{BW}(a) = \sigma_{BB}(a)$.

$(2) \Rightarrow (1)$. Firstly, we show $\sigma(a) \setminus \sigma_{BW}(a) \subseteq P_0(a)$, where

$$P_0(a) = \{\lambda : \lambda \textit{ is a pole of the resolvent of } a\}.$$

Assume $\lambda \in \sigma(a) \setminus \sigma_{BW}(a)$, which implies that $a-\lambda$ is a B-Weyl element, and furthermore a B-Browder element. According to [23, Proposition 3.3.3], there exists an element

$$b \in \mathrm{Soc}(\mathcal{A}) \text{ with } (a-\lambda)b = b(a-\lambda)$$

such that $a-\lambda+b$ is Drazin invertible in $\mathcal{A}$. Hence, $a-\lambda$ is a Drazin invertible element in $\mathcal{A}$. It follows that $\lambda$ is a pole of the resolvent of $a$, and that $\lambda \in P_0(a)$.

Conversely, if $\lambda \in P_0(a)$, then $\lambda$ is a pole of the resolvent of $a$. From [27, Page 303], it follows that $p_l(a-\lambda) < \infty$ and $q_l(a-\lambda) < \infty$, which implies that $a-\lambda$ is Drazin invertible but not invertible in $\mathcal{A}$ according to Lemma 4.6. From [22, Theorem 4.2], one can show that $\lambda \in \mathrm{iso}\sigma(a)$. It follows from [23] that $a-\lambda$ is a B-Weyl element. Finally, one can get that $\lambda \in \sigma(a) \setminus \sigma_{BW}(a)$.

(2) $\Rightarrow$ (3). It suffices to prove that $\sigma(a) \subseteq \sigma_{BW}(a) \bigcup \pi_0(a)$. If $\lambda \in \sigma(a) \setminus \sigma_{BW}(a)$, then $a-\lambda$ is a B-Weyl element. Associated with the condition (2), it follows that $a-\lambda$ is a B-Browder element, which implies $\lambda \notin \sigma_{BB}(a)$. Therefore, there exists $b \in \mathrm{Soc}(\mathcal{A})$ with $(a-\lambda)b = b(a-\lambda)$ such that $a-\lambda+b$ is Drazin invertible in $\mathcal{A}$. This indicates that $a-\lambda$ is Drazin invertible in $\mathcal{A}$. Furthermore, one can obtain that $\lambda \in \sigma(a) \setminus \sigma_D(a)$, which shows that

$$\lambda \in \mathrm{iso}\sigma(a) \text{ and } \mathrm{nul}(a-\lambda) > 0.$$

Otherwise, if $\mathrm{nul}(a-\lambda) = 0$, then it is an invertible element in $\mathcal{A}$. It is a contradiction with the fact that $\lambda \in \sigma(a)$. Hence, $\lambda \in \pi_0(a)$. Consequently, $\sigma(a) = \sigma_{BW}(a) \bigcup \pi_0(a)$.

(3) $\Rightarrow$ (2). It suffices to prove that $a-\lambda$ is a B-Browder element provided that $a-\lambda$ is a B-Weyl element.

Suppose that $a-\lambda$ is a B-Weyl element, it is not harmful to suppose that $\lambda \in \sigma(a) \setminus \sigma_{BW}(a)$. Then one can get that $\lambda \in \mathrm{iso}\sigma(a)$ and $\mathrm{nul}(a-\lambda) > 0$. Applying [23, Proposition 3.2.2], it follows that $a-\lambda$ is Drazin invertible in $\mathcal{A}$. Therefore, for any $b \in \mathrm{Soc}(\mathcal{A})$ with $(a-\lambda)b = b(a-\lambda)$, we have the conclusion that $a-\lambda+b$ is Drazin invertible in $\mathcal{A}$, which implies that

$$\lambda \notin \sigma_D(a+b).$$

It is not difficult to check that $0 \notin \sigma_{BB}(a-\lambda)$. It follows that $a-\lambda$ is a B-Browder element. This leads to the conclusion that $\sigma_{BW}(a) = \sigma_{BB}(a)$.

(1) $\Leftrightarrow$ (4). Suppose $\lambda \notin \sigma_{BW}(a)$, which infers that $a-\lambda$ is a B-Weyl element. It is not harmful to suppose that $\lambda \in \sigma(a) \setminus \sigma_{BW}(a)$, one can get that $\lambda \in \mathrm{iso}\sigma(a)$ and $\mathrm{nul}(a-\lambda) > 0$ since $a$ satisfies the generalized Browder's theorem. Therefore, $\lambda \notin \mathrm{acc}\sigma(a)$.

Conversely, if $\lambda \in \sigma(a) \setminus \sigma_{BW}(a)$, then $\lambda \in \mathrm{iso}\sigma(a)$. Since $a-\lambda$ is a B-Weyl element, it follows that $a-\lambda$ is Drazin invertible in $\mathcal{A}$ from [23, Proposition 3.2.2]. Therefore, $\lambda \in P_0(a)$. For the converse direction, if $\lambda \in P_0(a)$, then $\lambda$ is a pole of the resolvent of $a$. From [27, Page 303], $p_l(a-\lambda) < \infty$ and $q_l(a-\lambda) < \infty$, which implies that $a-\lambda$ is Drazin invertible but not invertible in $\mathcal{A}$ according to Lemma 4.6. From [22, Theorem 4.2], it can be proved that $\lambda \in \mathrm{iso}\sigma(a)$. Applying [23, Proposition 3.2.2], $a-\lambda$ is a B-Weyl element. That is to say, $P_0(a) \subseteq \sigma(a) \setminus \sigma_{BW}(a)$. Consequently, $a$ satisfies the generalized Browder's theorem.

(4) $\Leftrightarrow$ (5). Suppose $\lambda \in \sigma(a) \setminus \sigma_{BW}(a)$, then $\lambda \notin \mathrm{acc}\sigma(a)$ from the condition (4), and thus $\lambda \in \mathrm{iso}\sigma(a)$. Therefore, $a-\lambda$ is Drazin invertible in $\mathcal{A}$, which implies that $\mathrm{nul}(a-\lambda) > 0$. Otherwise, if $\mathrm{nul}(a-\lambda) = 0$, it is an invertible element in $\mathcal{A}$, which contradicts with the fact that $\lambda \in \sigma(a)$. Hence, $\lambda \in \pi_0(a)$.

Conversely, if $\sigma(a) \setminus \sigma_{BW}(a) \subseteq \pi_0(a)$ holds. Let $\lambda \in \sigma(a) \setminus \sigma_{BW}(a)$, then $\lambda \in \pi_0(a)$. One can see that $\lambda \in \mathrm{iso}\sigma(a)$, which indicates that

$$\lambda \in \sigma(a) \setminus \mathrm{acc}\sigma(a).$$

Therefore, $\mathrm{acc}\sigma(a) \subseteq \sigma_{BW}(a)$. This completes the proof. $\square$

**Example 4.8** *(1)Recall that an element $a$ of a unital Banach algebra is said to be algebraic if there exists a polynomials $\alpha$ with complex coefficients such that $\alpha(a) = 0$. If $a \in \mathcal{A}$ is an algebraic element, then it satisfies the generalized Browder's theorem from Theorem 4.7. Indeed, it is enough to note that $\sigma(a) = \{0\}$ and $\mathrm{acc}\sigma(a) = \emptyset$.*

*(2) If $a \in \mathcal{A}$ is a quasinilpotent element, then it satisfies the generalized Browder's theorem from Theorem 4.7 due to $\sigma(a) = \{0\}$ and $\mathrm{acc}\sigma(a) = \emptyset$.*

For $a \in \mathcal{A}$, we will give two corollaries applying Theorem 4.7, one corollary discusses whether $a^*$ satisfies the generalized Browder's theorem if $a$ satisfies the generalized Browder's theorem, and the other characterizes the generalized Browder's theorem of $f(a)$, where $f \in Hol(a)$.

**Corollary 4.9** *Suppose $a \in \mathcal{A}$, then it satisfies the generalized Browder's theorem if and only if $a^*$ satisfies the generalized Browder's theorem.*

**Proof** From the Theorem 2.11 in [16], it follows that

$$\sigma(L_a) = \sigma((L_a)^*) \text{ and } \sigma_{BW}(L_a) = \sigma_{BW}((L_a)^*).$$

This leads to the conclusion that

$$\begin{aligned} a \ is\ an\ invertible\ element \ &\Leftrightarrow\ L_a \ is\ an\ invertible\ operator\ in\ \Gamma(\mathcal{A}) \\ &\Leftrightarrow\ (L_a)^* \ is\ an\ invertible\ operator\ in\ \Gamma(\mathcal{A}) \\ &\Leftrightarrow\ L_{a^*} \ is\ an\ invertible\ operator\ in\ \Gamma(\mathcal{A}) \\ &\Leftrightarrow\ a^* \ is\ an\ invertible\ element\ in\ \mathcal{A}. \end{aligned}$$

Since $\mathcal{A}$ is a primitive C*-algebra, one has

$$\begin{aligned} a\ is\ a\ B-Weyl\ element \ &\Leftrightarrow\ L_a\ is\ a\ B-Weyl\ operator \\ &\Leftrightarrow\ (L_a)^*\ is\ a\ B-Weyl\ operator\ on\ \mathcal{A}p \\ &\Leftrightarrow\ L_{a^*}\ is\ a\ B-Weyl\ operator\ on\ \mathcal{A}p \\ &\Leftrightarrow\ a^*\ is\ a\ B-Weyl\ element\ in\ \mathcal{A}. \end{aligned}$$

Therefore, $\sigma(a) = \sigma(a^*)$ and $\sigma_{BW}(a) = \sigma_{BW}(a^*)$, which implies that

$$\mathrm{acc}\sigma(a) \subseteq \sigma_{BW}(a) \Leftrightarrow \mathrm{acc}\sigma(a^*) \subseteq \sigma_{BW}(a^*).$$

Applying Theorem 4.3, one can get that the element $a$ satisfies the generalized Browder's theorem if and only if the element $a^*$ satisfies the generalized Browder's theorem. □

**Corollary 4.10** *If $a \in \mathcal{A}$ satisfies the generalized Browder's theorem, then the following statements are equivalent:*

*(1) $f(a)$ satisfies the generalized Browder's theorem for any $f \in Hol(a)$.*

*(2) for any $\lambda, \mu \in \rho_{BF}(a)$, $i(a-\lambda) \cdot i(a-\mu) \geq 0$.*

*(3) $\sigma_{BW}(f(a)) = f(\sigma_{BW}(a))$ for any $f \in Hol(a)$.*

**Proof** From Theorem 3.5, it is clear that $(2) \Leftrightarrow (3)$. It suffices to prove $(3) \Leftrightarrow (1)$.

(3) $\Rightarrow$ (1). Suppose $\sigma_{BW}(f(a)) = f(\sigma_{BW}(a))$. From Theorem 4.7(2) and Proposition 3.2, it follows that

$$\sigma_{BB}(f(a)) = f(\sigma_{BB}(a)) = f(\sigma_{BW}(a)) = \sigma_{BW}(f(a)).$$

Applying Theorem 4.7(2), one can get that $f(a)$ satisfies the generalized Browder's theorem for any $f \in Hol(a)$.

(1) $\Rightarrow$ (3). Since $a$ satisfies the generalized Browder's theorem, applying Theorem 4.7(2) and Proposition 3.2, one can obtain that $\sigma_{BW}(f(a)) = \sigma_{BB}(f(a)) = f(\sigma_{BB}(a)) = f(\sigma_{BW}(a))$. This completes the proof. $\square$

In what follows, we will discuss the relations between the generalized Weyl's theorem and the generalized Browder's theorem by virtue of the following fact: if $a \in \mathcal{A}$, then $P_0(a) \subseteq \pi_0(a)$. Indeed, it is not harmful to suppose that $0 \in P_0(a)$, then $p_l(a) < \infty$, $q_l(a) < \infty$. One can prove that $0 \in \mathrm{iso}\sigma(a)$. Note that in this case $a$ is a Drazin invertible element in $\mathcal{A}$ by Lemma 4.6, which implies that $\mathrm{nul}(a) > 0$. Otherwise, if $\mathrm{nul}(a) = 0$, then it is an invertible element in $\mathcal{A}$. It is a contradiction with the fact that $0 \in \sigma(a)$. Therefore, $\mathrm{nul}(a) > 0$. In other words, $0 \in \pi_0(a)$.

**Proposition 4.11** *Suppose $a \in \mathcal{A}$ satisfies the generalized Weyl's theorem, then it satisfies the generalized Browder's theorem.*

**Proof** Firstly, the relation "$\sigma(a) \backslash \sigma_{BW}(a) \subseteq P_0(a)$" will be proved. If $\lambda \in \sigma(a)$ and $a - \lambda$ is a B-Weyl element. One can get that $\lambda \in \pi_0(a)$ since $a$ satisfies the generalized Weyl's theorem. That is to say, $\lambda \in \mathrm{iso}\sigma(a)$ and $\mathrm{nul}(a - \lambda) > 0$. Associated with [23, Proposition 3.2.2], it follows that $a - \lambda$ is Drazin invertible in $\mathcal{A}$, then $\lambda$ is a pole of the resolvent of $a$, which implies that $\lambda \in P_0(a)$.

Subsequently, the relation "$P_0(a) \subseteq \sigma(a) \setminus \sigma_{BW}(a)$" will be examined. Suppose $\lambda \in P_0(a)$, then $\lambda$ is a pole of the resolvent of $a$. From Lemma 4.6, $a - \lambda$ is Drazin invertible in $\mathcal{A}$. Therefore, $0 \in \mathrm{iso}\sigma(a)$ [22, Theorem 4.2]. One can assert that $\mathrm{nul}(a - \lambda) > 0$. Indeed, if $\mathrm{nul}(a - \lambda) = 0$, then it is an invertible element in $\mathcal{A}$. It is a contradiction with the fact that $\lambda \in \sigma(a)$. Thus, $\mathrm{nul}(a - \lambda) > 0$, which means $\lambda \in \pi_0(a)$. Since $a$ satisfies the generalized Weyl's theorem, then $\lambda \in \sigma(a) \setminus \sigma_{BW}(a)$. This completes the proof. $\square$

The following example shows that the converse of Proposition 4.11 is not true.

**Example 4.12** *Let $T_1 \in B(l^2)$ be given by*

$$T_1(x_1, x_2, x_3, \cdots) = (0, \frac{x_1}{2}, \frac{x_2}{3}, \frac{x_3}{4}, \cdots).$$

*Let $T_2 = 0$ and $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$. One can calculate that $\sigma(T) = \sigma_{BW}(T) = \pi_0(T) = \{0\}$ and $P_0(T) = \emptyset$. Therefore, $T$ satisfies the generalized Browder's theorem, but it does not satisfy the generalized Weyl's theorem.*

Next, we give necessary and sufficient conditions such that $a \in \mathcal{A}$ satisfies the generalized Weyl's theorem when it satisfies the generalized Browder's theorem.

**Proposition 4.13** *If $a \in \mathcal{A}$ satisfies the generalized Browder's theorem, then the following statements are equivalent:*

*(1) $a$ satisfies the generalized Weyl's theorem.*

*(2) $\sigma_{BW}(a) \bigcap \pi_0(a) = \emptyset$.*

*(3) $P_0(a) = \pi_0(a)$.*

**Proof** (1) $\Rightarrow$ (2). If $a$ satisfies the generalized Weyl's theorem, then $\sigma(a)\setminus\sigma_{BW}(a)=\pi_0(a)$. It can be easily got that $\sigma_{BW}(a)\bigcap\pi_0(a)=\emptyset$.

(2) $\Rightarrow$ (3). It suffices to prove $\pi_0(a)\subseteq P_0(a)$. Suppose $\lambda\in\pi_0(a)$, then one can get that $\lambda\in\sigma(a)\setminus\sigma_{BW}(a)$ since $\sigma_{BW}(a)\bigcap\pi_0(a)=\emptyset$. It follows that $a-\lambda$ is a B-Weyl element, which implies that $a-\lambda$ is Drazin invertible in $\mathcal{A}$ [23, Proposition 3.2.2]. Therefore, $\lambda$ is a pole of the resolvent of $a$, which means $\lambda\in P_0(a)$.

(3) $\Rightarrow$ (1). It is clear from Definition 4.5. □

Now we come to the final result of this paper, which gives the perturbations of the generalized Weyl's theorem and the generalized Browder's theorem under the socle of $\mathcal{A}$. To begin with, we need a lemma which investigates the perturbation of the Drazin spectrum under $\mathrm{Soc}(\mathcal{A})$.

**Lemma 4.14** *Suppose $a\in\mathcal{A}$, $k\in\mathrm{Soc}(\mathcal{A})$ with $ak=ka$, then $\sigma_D(a)=\sigma_D(a+k)$.*

**Proof** In order to prove $\sigma_D(a)=\sigma_D(a+k)$, we only need to check that if $0\notin\sigma_D(a)$, then $0\notin\sigma_D(a+k)$. Therefore, it suffices to prove that if $a\in\mathcal{A}$ is Drazin invertible, $k\in\mathrm{Soc}(\mathcal{A})$ with $ak=ka$, then $a+k$ is Drazin invertible. Suppose $a$ is Drazin invertible, then one can show that $L_a$ is a Drazin invertible operator. It follows that $L_k\in F(\mathcal{A}p)$ since $\mathcal{A}$ is a primitive C*-algebra [1, Page 903]. It is clear that $L_aL_k=L_kL_a$ since $ak=ka$. One can get that $L_a+L_k$ is also Drazin invertible by Theorem 2.7 in [8]. In other words, $L_{a+k}$ is a Drazin invertible operator in $B(\mathcal{A}p)$. Therefore, $L_{a+k}$ is a Drazin invertible operator in $\Gamma(\mathcal{A})$ since $\mathcal{A}$ is a primitive C*-algebra [10, Example 3.5]. Since $\mathcal{A}$ is a primitive C*-algebra, then the left regular representation is isometric [1, Page 903], this leads to the conclusion that $a+k$ is a Drazin invertible element. It can be obtained from the above discussion that $\sigma_D(a)=\sigma_D(a+k)$. □

**Theorem 4.15** *Suppose $a\in\mathcal{A}$ satisfies the generalized Weyl's theorem and $s\in\mathrm{Soc}(\mathcal{A})$ with $as=sa$, then $a+s$ satisfies the generalized Weyl's theorem if and only if $\pi_0(a+s)=P_0(a+s)$.*

**Proof** If $a+s$ satisfies the generalized Weyl's theorem, then $a+s$ satisfies the generalized Browder's theorem. From Proposition 4.13, one can get that $\pi_0(a+s)=P_0(a+s)$.

Conversely, if $\pi_0(a+s)=P_0(a+s)$, since $a$ satisfies the generalized Weyl's theorem, then one can conclude that $\sigma_{BW}(a)=\sigma_D(a)$. Indeed, if $a$ is Drazin invertible in $\mathcal{A}$, then $a+s$ is Drazin invertible in $\mathcal{A}$. It follows that $0\in\mathrm{iso}\sigma(a+s)$, which implies that $a+s$ is a B-Weyl element from [23, Proposition 3.2.2]. Applying [9, Proposition 3.3], one can see that $a$ is a B-Weyl element. It is not harmful to suppose that $0\in\sigma(a)\setminus\sigma_{BW}(a)$. Since $a$ satisfies the generalized Weyl's theorem, then $0\in\pi_0(a)$, which means $0\in\mathrm{iso}\sigma(a)$ and $\mathrm{nul}(a)>0$. It follows that $a$ is Drazin invertible in $\mathcal{A}$. According to [9, Proposition 3.3],

$$\sigma_{BW}(a)=\sigma_{BW}(a+s).$$

Associated with Lemma 4.14, $\sigma_D(a)=\sigma_D(a+s)$, which suggests $\sigma_{BW}(a+s)=\sigma_D(a+s)$. One can check that $a+s$ satisfies the generalized Browder's theorem. From Proposition 4.13 and $\pi_0(a+s)=P_0(a+s)$, one can obtain that $a+s$ satisfies the generalized Weyl's theorem. This completes the proof. □

An element $a\in\mathcal{A}$ is called an isoloid element if $\mathrm{nul}(a-\lambda)>0$ for $\lambda\in\mathrm{iso}\sigma(a)$. Next, two applications of Theorem 4.15 will be listed as follows:

**Example 4.16** *If $a\in\mathcal{A}$ is an isoloid element satisfying the generalized Weyl's theorem and $s\in\mathrm{Soc}(\mathcal{A})$ with $as=sa$, then $a+s$ satisfies the generalized Weyl's theorem. Indeed, it suffices to prove $\pi_0(a+s)\subseteq P_0(a+s)$.*

*It is not harmful to suppose that* $0 \in \pi_0(a+s)$*, then* $0 \in \mathrm{iso}\sigma(a+s)$ *and* $\mathrm{nul}(a+s) > 0$*. One can check that* $0 \in \mathrm{iso}\sigma(a)$*, which implies* $\mathrm{nul}(a) > 0$*. Therefore,* $0 \in \pi_0(a)$*. It follows that* $0 \in P_0(a)$ *since* $a$ *satisfies the generalized Weyl's theorem. It can be proved that* $0 \in P_0(a+s)$*. Hence,* $\pi_0(a+s) = P_0(a+s)$*, and consequently,* $a+s$ *satisfies the generalized Weyl's theorem.*

**Example 4.17** *If* $a \in \mathcal{A}$ *is a quasinilpotent element satisfying the generalized Weyl's theorem and* $s \in \mathrm{Soc}(\mathcal{A})$ *with* $as = sa$*, then* $a+s$ *satisfies the generalized Weyl's theorem. In fact, it suffices to prove* $\pi_0(a+s) \subseteq P_0(a+s)$*. One can conclude that* $\sigma_{BW}(a) = \emptyset$ *if* $a$ *is a quasinilpotent element. Indeed, one can get that* $\sigma(L_a) = \{0\}$ *since* $\sigma(a) = \{0\}$*. It follows that* $\sigma_{BW}(L_a) = \emptyset$*. Hence,* $\sigma_{BW}(a) = \emptyset$ *since* $\mathcal{A}$ *is a primitive C*-algebra [10, Theorem 3.8]. Suppose*

$$0 \in \mathrm{iso}\sigma(a+s) \text{ and } \mathrm{nul}(a+s) > 0,$$

*then one can check that* $0 \in \mathrm{iso}\sigma(a)$*. It is easy to calculate that*

$$\sigma(a) \setminus \sigma_{BW}(a) = \{0\}.$$

*Hence,* $\pi_0(a) = \{0\}$ *since* $a$ *satisfies the generalized Weyl's theorem. It can be obtained that* $P_0(a) = \{0\}$*. In other words,* $a$ *is Drazin invertible in* $\mathcal{A}$*, which implies* $a+s$ *is Drazin invertible in* $\mathcal{A}$*. From Lemma 4.6, it follows that* $0 \in P_0(a+s)$*. Consequently, the relation* $\pi_0(a+s) = P_0(a+s)$ *holds. Therefore,* $a+s$ *satisfies the generalized Weyl's theorem.*

Similarly, one has the following result.

**Corollary 4.18** *If* $a \in \mathcal{A}$ *satisfies the generalized Browder's theorem and* $s \in \mathrm{Soc}(\mathcal{A})$ *with* $as = sa$*, then* $a+s$ *satisfies the generalized Browder's theorem.*

**Proof** From the proof of Theorem 4.15, one can see that $\sigma_{BW}(a) = \sigma_{BW}(a+s)$ and $\sigma_D(a) = \sigma_D(a+s)$. It can be proved that $a+s$ satisfies the generalized Browder's theorem applying Theorem 4.7. □

**Example 4.19** *If* $a \in \mathcal{A}$ *is a quasinilpotent element and* $s \in \mathrm{Soc}(\mathcal{A})$ *with* $as = sa$*, then* $a+s$ *satisfies the generalized Browder's theorem. Indeed, from Example 4.8, it follows that* $a$ *satisfies the generalized Browder's theorem. Applying Proposition 4.18, one can get that* $a+s$ *satisfies the generalized Browder's theorem.*

**Acknowledgment**

The author would like to express their heart-felt thanks to the referees for corrections of the manuscript and some valuable comments. This work was supported by the National Natural Science Foundation of China under Grant number 11871303.

## References

[1] Abdellatif B, María B. Linear maps preserving regularity in C*-algebras. Illinois Journal of Mathematics 2009; 53 (3): 899-914. doi: 10.1215/ijm/1286212922

[2] Aiena P. Fredholm and local spectral theory, with applications to multipliers. Dipartimento di Matematica ed Applicazioni, Italy, 2004.

[3] Aiena P. Fredholm and local spectral theory II with application to Weyl-type theorems. Dipartimento di Matematica Università degli Studi di Palermo, Italy, 2018.

[4] Barnes BA, Murphy GJ, Smyth MRF, West TT. Riesz and Fredholm theory in Banach algebras. Research Notes in Mathematics, 1982.

[5] Barnes BA. A generalized Fredholm theory for certain maps in the regular representation of an algebra. Canadian Journal of Mathematics 2018; 20: 495-504. doi: 10.4153/CJM-1968-048-2

[6] Barnes BA. The Fredholm elements of a ring. Canadian Journal of Mathematics 1969; 21: 84-95. doi: 10.4153/CJM-1969-009-1

[7] Berkani M. Index of B-Fredholm operators and generalization of a Weyl theorem. Proceedings of the American Mathematical Society 2001; 130 (6): 1717-1723. doi: 10.2307/2699766

[8] Berkani M. B-Weyl spectrum and poles of the resolvent. Journal of Mathematical Analysis and Applications 2002; 272: 596-603. doi: 10.1016/S0022-247X(02)00179-8

[9] Berkani M. A trace formula for the index of B-Fredholm operators. Proceedings of the Edinburgh Mathematical Society 2018; 61: 1063-1068. doi: 10.1017/s0013091518000147

[10] Berkani M. B-Fredholm elements in rings and algebras. Publicationes Mathematicae 2018; 92 (1-2): 171-181. doi: 10.5486/PMD.2018.7846

[11] Berkani M. On a class of quasi-Fredholm operators. Integral Equations and Operator Theory 1999; 34 (2): 244-249. doi: 10.1007/BF01236475

[12] Bonsall FF, Duncan J. Numerical ranges II. Cambridge University Press. 1973.

[13] Cao X. Weyl type theorem for bounded linear operators. Doctoral dissertation, Peking University, 2004.

[14] Cao X, Guo M, Meng B. Weyl type theorems for p-hyponormal and M-hyponormal operators. Studia Mathematica 2010; 2: 177-188. doi: 10.4064/sm163-2-5

[15] Carlos Carpintero, D Muñoz, Ennis Rosas, José Eduardo Sanabria, Orlando García. Weyl type theorems and restrictions. Mediterranean Journal of Mathematics 2013; 11 (4): 1215-1228. doi: 10.1007/s00009-013-0369-7

[16] Curto RE, Han YM. Generalized Browder's and Weyl's theorems for Banach space operators. Journal of Mathematical Analysis and Applications 2006; 336 (2): 1424-1442. doi: 10.1016/j.jmaa.2007.03.060

[17] Cvetković MD, Boasso Enrico, Snežana Č Ž. Generalized B-Fredholm Banach algebra elements. Mediterranean Journal of Mathematics 2016; 3729-3746. doi: 10.1007/s00009-016-0711-y

[18] García OJ, Carpintero CR, Rosas ER, Sanabria JE. Semi B-Fredholm and semi B-Weyl spectrum under perturbations. Boletín de la Sociedad Matemática Mexicana 2014; 20 (1): 39-47. doi: 10.1007/s40590-014-0006-x

[19] Grobler JJ, Raubenheimer H, Swartz A. The index for Fredholm elements in a Banach algebra via a trace. Czechoslovak Mathematical Journal 2016; 66 (1): 205-211. doi: 10.1007/s10587-016-0250-5

[20] Harte R. Fredholm, Weyl and Browder theory. Proceedings of the Royal Irish Academy, Section A: Mathematical and Physical Sciences 1985; 85A (2): 151-176.

[21] Harte R. Fredholm theory relative to a Banach algebra homomorphism. Mathematische Zeitschrift 1982; 179: 431-436. doi: 10.1007/BF01215344

[22] Koliha JJ. A generalized Drazin inverse. Glasgow Mathematical Journal 1996; 38: 367-381. doi: 10.1017/S0017089500031803

[23] Kong Y, Jiang L, Ren Y. B-Fredholm elements in primitive C*-algebra. Submit to Results in Mathematics.

[24] Kong Y, Jiang L, Ren Y. The Weyl's theorem and its perturbations in semisimple Banach algebra. Acta Mathematica Sinica, English Series 2021; 37 (5): 675-688. doi: 10.1007/s10114-021-0434-2

[25] Ren Y, Jiang L, Kong Y. Property (We) and topological uniform descent. Bulletin of The Belgian Mathematical Society-Simon Stevin. doi: 10.36045/j.bbms.210413.

[26] Männle D, Schmoeger C. Generalized Fredholm theory in semisimple algebras. Rendiconti di Matematica e delle Sue Applicazioni 1999; 19 (4): 583-613.

[27] Pearlman LD. Riesz points of the spectrum of an element in a semisimple Banach algebra. Transactions of the American Mathematical Society 1974; 193: 303-328. doi: 10.2307/1996917

[28] Rickart C. General theory of Banach algebras. The University Series in Higher Mathematics, D.van Nostrand, New York, 1960

[29] Smyth MRF. Fredholm theory in Banach algebras. Journal of Spectral Theory 1982; 8: 403-414. doi: 10.4064/-8-1-403-414

[30] Taylor AE, Lay DC. Introduction to functional analysis. Robert E. Krieger Publishing Company, Wiley, 1980.

[31] Weyl H. Über beschränkte quadratische Formen, deren Differenz vollsteig ist. Rendiconti del Circolo Matematico di Palermo 1909; 27: 373-392 (in German).

[32] Zguitti H. A note on generalized Weyl's theorem. Journal of Mathematical Analysis and Applications 2006; 316 (1): 373-381. doi: 10.1016/j.jmaa.2005.04.057