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# Structure of annihilators of powers

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**Abstract:** We study the following two conditions in rings: (i) the right annihilator of some power of any element is an ideal, and (ii) the right annihilator of any nonzero element $a$ contains an ideal generated by some power of any right zero-divisor of the element $a$. We investigate the structure of rings in relation to these conditions; especially, a ring with the condition (ii) is called *right APIP*. These conditions are shown to be not right-left symmetric. For a prime two-sided APIP ring $R$ we prove that every element of $R$ is either nilpotent or regular, and that if $R$ is of bounded index of nilpotency then $R$ is a domain. We also provide several interesting examples which delimit the classes of rings related to these properties.

**Key words:** Annihilator, right APIP ring, K-ring, nilpotent element, prime ring, Köthe's conjecture, matrix ring

## 1. Introduction

It has enriched many parts in noncommutative ring theory to study the structures of powers of noncentral elements. As an important case, Jacobson investigated the structure of rings with the property that some power of each element is central, and such a ring is called a K-ring which was introduced by Kaplansky (see [14, Chapter 10, Section 1] for details). In this article, we continue the study of powers of elements, concentrating upon two kinds of generalized conditions of K-rings which are related to one-sided annihilator of powers of elements.

Throughout this note every ring is an associative ring with identity unless otherwise stated. Let $R$ be a ring. $N_*(R)$, $N^*(R)$ and $N(R)$ stand for the prime radical, the upper nilradical (i.e. the sum of nil ideals) and the set of all nilpotent elements in $R$, respectively. Note $N_*(R) \subseteq N^*(R) \subseteq N(R)$. The polynomial ring with an indeterminate $x$ over $R$ is denoted by $R[x]$. For $S \subseteq R$, the left (resp., right) annihilator of $S$ in $R$ is denoted by $l_R(S)$ (resp., $r_R(S)$); and if $S = \{a\}$ then we write $l_R(a)$ (resp., $r_R(a)$). A left (resp., right) annihilator ideal means an ideal of the form $l_R(S)$ (resp., $r_R(S)$). When left and right annihilators coincide (e.g., semiprime rings), we call annihilator ideal for them. $\mathbb{Z}$ and $\mathbb{Z}_n$ mean the ring of integers and the ring of integers modulo $n$, respectively. Let $M_n(R)$ (resp., $T_n(R)$) be the $n$ by $n$ ($n \geq 2$) full (resp., upper triangular) matrix ring over $R$, and write $D_n(R) = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$ and $V_n(R) = \{(a_{ij}) \in D_n(R) \mid a_{1j} = a_{2,j+1} = \cdots = a_{n-(j-1),n}$ for $j = 2, \ldots, n\}$. Use $E_{ij}$ for the matrix with

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$(i,j)$-entry 1 and zeros elsewhere.

By the left version of [23, Lemma 2.3.2], we have that for a ring $R$ with the ascending chain condition on left annihilators, each maximal left annihilator has the form $l_R(a)$ with $a \in R$ and, given $b \in R$, there exists $k \geq 1$ such that $l_R(b^h) = l_R(b^k)$ for all $h \geq k$. In this article we concentrate our attention on the cases that such left annihilators are ideals and, as a generalization of this case, that whenever $ab = 0$ for $a, b \in R$ we have that $r_R(a)$ contains $Rb^nR$ for some $n \geq 1$. In fact, the class of rings satisfying these conditions are quite large as we see in the arguments of the sections below.

## 2. Annihilator ideals of powers

In this section, we first study the structure of rings in relation to the one-sided annihilators of powers of elements. Notice that every annihilator of some power of each element in K-rings is an ideal. We concentrate on this property of K-rings in generalized situations, and consider the condition that the one-sided annihilator of some power of any element is an ideal. For a ring $R$, define

(†) For any $r \in R$, there exists $n = n(r) \geq 1$ such that $l_R(r^n)$ is an ideal of $R$;

(‡) For any $s \in R$, there exists $m = m(s) \geq 1$ such that $r_R(s^m)$ is an ideal of $R$.

K-rings clearly satisfy the conditions (†) and (‡), but each converse need not hold by considering the Hamilton quaternions over the field of real numbers or the rings below. The conditions (†) and (‡) for a ring do not imply each other by the following example. Moreover these two conditions clearly pass to subrings.

**Example 2.1** (1) Let $A = \mathbb{Z}_2\langle a, b, c\rangle$ be the free algebra with noncommuting indeterminates $a, b, c$ over $\mathbb{Z}_2$. Let $I$ be the ideal of $A$ generated by $a^2, c^2, ab, ba, bc$, and $ca$. Note that $I$ is homogeneous. Set $R = A/I$ and let $a, b, c$ coincide with their images in $R$ for simplicity. Then every element $r$ in $R$ is of the form

$$\alpha_0 + \alpha_1 a + \alpha_2 f(b) + \alpha_3 c + \alpha_4 ac + \alpha_5 acg(b) + \alpha_6 ch(b),$$

where $\alpha_i \in \mathbb{Z}_2$ and $f(b), g(b), h(b) \in bR[b]$. We note that $R$ does not satisfy the condition (†), since $a \in l_R(b^n)$ for any $n \geq 1$ and $ac \notin l_R(b^n)$.

We will show that for any $r \in R\backslash\{0\}$, $r_R(r^4)$ is an ideal of $R$. Let $r = \alpha_0 + \alpha_1 a + \alpha_2 f(b) + \alpha_3 c + \alpha_4 ac + \alpha_5 acg(b) + \alpha_6 ch(b) \in R$, where $\alpha_i \in \mathbb{Z}_2$ and $f(b), g(b), h(b) \in bR[b]$.

`Claim 1.` $r^4 = \alpha_0 + r' f(b)$ for some $r' \in R$.

**Proof** We have

$$r^2 = \alpha_0 + \alpha_1 a(\alpha_3 c + \alpha_6 ch(b)) + \alpha_2(\alpha_2 f(b) + \alpha_3 c + \alpha_4 ac + \alpha_5 acg(b) + \alpha_6 ch(b))f(b) \text{ and}$$

$$\begin{aligned} r^4 &= r^2 r^2 \\ &= \alpha_0 r^2 + (r^2 - \alpha_0)\alpha_0 + [\alpha_1 a(\alpha_3 c + \alpha_6 ch(b)) + \alpha_2(\alpha_2 f(b) + \alpha_3 c + \alpha_4 ac + \alpha_5 acg(b) + \alpha_6 ch(b))f(b)]^2 \\ &= \alpha_0 + [\alpha_2\alpha_1 a(\alpha_3 c + \alpha_6 ch(b)) + \alpha_2(\alpha_2 f(b) + \alpha_3 c + \alpha_4 ac + \alpha_5 acg(b) + \alpha_6 ch(b))f(b)]f(b) \\ &= \alpha_0 + r' f(b), \end{aligned}$$

where $r' = \alpha_2\alpha_1 a(\alpha_3 c + \alpha_6 ch(b)) + \alpha_2(\alpha_2 f(b) + \alpha_3 c + \alpha_4 ac + \alpha_5 acg(b) + \alpha_6 ch(b))f(b) \in R.$ □

`Claim 2.` For any $r \in R$, $r_R(r^4)$ is an ideal of $R$.

**Proof** If $r' = 0$ in `Claim 1`, then $r^4 = \alpha_0$ and so we are done. Assume that $r' \neq 0$ and let $s \in r_R(r^4)$ for $s = \beta_0 + \beta_1 a + \beta_2 u(b) + \beta_3 c + \beta_4 ac + \beta_5 acv(b) + \beta_6 cw(b) \in R \backslash \{0\}$ with $\beta_i \in \mathbb{Z}_2$ and $u(b), v(b), w(b) \in bR[b]$. Then both $r^4$ and $s$ have zero constant terms, by a similar argument to be noted in the case of $rs = 0$ in Example 3.14. So $r^4 = r' f(b)$ for some $r' \in R$ by `Claim 1`, and $s = \beta_1 a + \beta_2 u(b) + \beta_3 c + \beta_4 ac + \beta_5 acv(b) + \beta_6 cw(b)$. Then

$$
\begin{aligned}
0 = r^4 s &= r' f(b)[\beta_1 a + \beta_2 u(b) + \beta_3 c + \beta_4 ac + \beta_5 acv(b) + \beta_6 cw(b)] \\
&= \beta_2 r' f(b) u(b),
\end{aligned}
$$

and it implies that $\beta_2 = 0$. Thus $s = \beta_1 a + \beta_3 c + \beta_4 ac + \beta_5 acv(b) + \beta_6 cw(b)$. For any $z = \delta_0 + \delta_1 a + \delta_2 f_1(b) + \delta_3 c + \delta_4 ac + \delta_5 acg_1(b) + \delta_6 ch_1(b) \in R$ where $\delta_i \in \mathbb{Z}_2$ and $f_1(b), g_1(b), h_1(b) \in bR[b]$, we obtain

$$
\begin{aligned}
r^4 zs &= [\delta_0 r' f(b) + \delta_2 r' f(b) f_1(b)][\beta_1 a + \beta_3 c + \beta_4 ac + \beta_5 acv(b) + \beta_6 cw(b)] \\
&= 0.
\end{aligned}
$$

This entails that $Rs \subseteq r_R(r^4)$, and thus $r_R(r^4)$ is an ideal of $R$. □

Consequently, $R$ satisfies the condition (‡), but does not satisfy the condition (†).

(2) Let $R^{op}$ be the opposite ring of the ring $R$ in (1). Then $R^{op}$ satisfies the condition (†) but does not satisfy the condition (‡).

Let $R$ be a semiprime ring and $I$ be an ideal of $R$. Then $r_R(I) = l_R(I)$ clearly. Assume that $I$ is a left annihilator in $R$. Then, from the computation that $I = l_R(r_R(I)) = r_R(l_R(I))$, $I$ is also a right annihilator in $R$. From this argument we see the following.

**Proposition 2.2** (1) *Let $R$ be a ring and $r \in R$. Then $R$ satisfies the condition (†) (resp., (‡)) if and only if there exists $n \geq 1$ such that $l_R(r^n) = l_R(Rr^nR)$ (resp., $r_R(r^n) = r_R(Rr^nR)$).*
(2) *Let $R$ be a semiprime ring. Then the conditions (†) and (‡) are equivalent.*

**Proof** (1) is clear from definition, and (2) is proved by (1) and the argument above. □

Note that the ring $R$ in Example 2.1(1) is not semiprime; in fact, $RaR$ is a nonzero nilpotent ideal of $R$.

Due to Bell [1], a ring $R$ (possibly without identity) is said to satisfy insertion-of-factors-property (simply called IFP) if $ab = 0$ for $a, b \in R$ implies $aRb = 0$. It is clear that a ring $R$ is IFP if and only if $l_R(a)$ is an ideal of $R$ if and only if $r_R(a)$ is an ideal of $R$. The concepts of a K-ring and an IFP ring are independent of each other. In fact, there exists a K-ring but not IFP by [21, Theorem 2.3] and [17, Example 1.3]; and there exists an IFP ring but not a K-ring by the existence of reduced ring which has an element $a$ such that $a^n$ is noncentral for all $n \geq 1$ (for example, subrings of Hamilton quaternions over the field of real numbers).

IFP rings obviously satisfy the conditions (†) and (‡), but not conversely by the next example. Recall that a ring (possibly without identity) is usually called abelian if every idempotent is central. IFP rings are clearly abelian.

**Example 2.3** Recall from [11] that a ring $S$ is called generalized right p.p. if for any $r \in S$ the right annihilator of $r^n$ is generated by an idempotent for some $n = n(r) \geq 1$. Left cases may be defined analogously. A ring is called a generalized p.p. ring if it is both generalized left and right p.p.

Let $R = D_n(S)$ over a generalized p.p. abelian ring $S$ and $n \geq 4$. For $A = \sum_{i=1}^n aE_{ii} \in R$, by the proof of [11, Proposition 3], there exist positive integers $n, m$ such that $l_R(A^n) = RE$ and $r_R(A^m) = FR$, where $E$ and $F$ are central idempotents in $R$. Thus $R$ satisfies the conditions both (†) and (‡), but $R$ is not IFP by [17, Example 1.3].

As generalizations of the conditions (†) and (‡), a ring $R$ shall be called right APIP if the right annihilator of any nonzero element $a$ in $R$ contains the principal ideal of $R$ generated by some power of any right zero-divisor of $a$, equivalently, $ab = 0$ for $a, b \in R$ implies $Rb^mR \subseteq r_R(a)$ (i.e. $aRb^m = 0$) for some $m \geq 1$. The left APIP can be defined by symmetry. The APIP condition is not left-right symmetric by Example 2.1 (see also [3, Example 2.5]). So, a ring is called an APIP ring if it is both left and right APIP.

All rings satisfying the condition (†) (resp., (‡)) are right (resp., left) APIP clearly. But, in the following example, we construct a right APIP ring which does not satisfy the condition (†).

**Example 2.4** Let $A = F\langle a_1, a_2, \ldots, b, c\rangle$ be the free algebra with noncommuting indeterminates $a_1, a_2, \ldots, b, c$ over an infinite field $F$. Write $B = \{f \in A \mid \text{the constant term of } f \text{ is zero}\}$. Consider the ideal $I$ of $A$ generated by the following elements:

$$a_ia_j,\ a_iea_j,\ a_ic^i,\ a_iec^{2i}, ba_i,\ b^2,\ ca_i,\ cb,$$

where $i, j \geq 1$ and $e \in B$. Note that $I$ is homogeneous. Set $R = A/I$ and let $a_1, a_2, \ldots, b, c$ coincide with their images in $R$ for simplicity. By the construction of $R$, we have

$$a_iRa_j = 0, a_iRc^{2i} = 0, bRa_i = 0, bRb = 0, cRa_i = 0, \text{ and } cRb = 0;$$

and we also get that every element $r \in R$ is of the form

$$r = \alpha + \sum_{i=1}^{s} \alpha_i a_i + \beta b + \sum_{j=1}^{t} \gamma_j c^j + \sum_{k=1}^{u} \delta_k bc^k + \sum_{i=2}^{v}\sum_{j=1}^{i-1} \epsilon_{i,j} a_i c^{i-j} + \sum_{i=1}^{w}\sum_{j=1}^{2i-1} \eta_{i,j} a_i bc^{2i-j},$$

where $\alpha, \alpha_i, \beta, \gamma_j, \delta_k, \epsilon_{i,j}, \eta_{i,j} \in F$.

`Claim 1.` $R$ is a right APIP ring.

**Proof** Suppose that $rr' = 0$ for some $r, r' \in R$, where

$$r' = \alpha' + \sum_{i'=1}^{s'} \alpha'_{i'} a_{i'} + \beta' b + \sum_{j'=1}^{t'} \gamma'_{j'} c^{j'} + \sum_{k'=1}^{u'} \delta'_k bc^{k'} + \sum_{i'=2}^{v'}\sum_{j'=1}^{i'-1} \epsilon'_{i',j'} a_{i'} c^{i'-j'} + \sum_{i'=1}^{w'}\sum_{j'=1}^{2i'-1} \eta'_{i',j'} a_{i'} bc^{2i'-j'}.$$

Then clearly $\alpha = \alpha' = 0$. By the construction of $R$ and $rr' = 0$, we have

$$\left(\sum_{i=1}^{s} \alpha_i a_i\right)(\beta' b) + \left(\sum_{i=1}^{s} \alpha_i a_i\right)\left(\sum_{j'=1}^{t'} \gamma'_{j'} c^{j'}\right) + \left(\sum_{i=1}^{s} \alpha_i a_i\right)\left(\sum_{k'=1}^{u'} \delta'_{k'} b c^{k'}\right) + \beta b \left(\sum_{j'=1}^{t'} \gamma'_{j'} c^{j'}\right)$$
$$+ \left(\sum_{j=1}^{t} \gamma_j c^j\right)\left(\sum_{j'=1}^{t'} \gamma'_{j'} c^{j'}\right) + \left(\sum_{k=1}^{u} \delta_k b c^k\right)\left(\sum_{j'=1}^{t'} \gamma'_{j'} c^{j'}\right) + \left(\sum_{i=2}^{v}\sum_{j=1}^{i-1} \epsilon_{i,j} a_i c^{i-j}\right)\left(\sum_{j'=1}^{t'} \gamma'_{j'} c^{j'}\right)$$
$$+ \left(\sum_{i=1}^{w}\sum_{j=1}^{2i-1} \eta_{i,j} a_i b c^{2i-j}\right)\left(\sum_{j'=1}^{t'} \gamma'_{j'} c^{j'}\right) = 0.$$

Therefore we have the following cases.

`Case 1.` $\beta b \neq 0$.

From $rr' = 0$, we have $\sum_{j'=1}^{t'} \gamma'_{j'} c^{j'} = 0$, and so

$$\left(\sum_{i=1}^{s} \alpha_i a_i\right)(\beta' b) + \left(\sum_{i=1}^{s} \alpha_i a_i\right)\left(\sum_{k'=1}^{u'} \delta'_{k'} b c^{k'}\right) = 0.$$

If $\sum_{i=1}^{s} \alpha_i a_i \neq 0$, then $\beta' b = \sum_{k'=1}^{u'} \delta'_{k'} b c^{k'} = 0$. Thus

$$r' = \sum_{i=1}^{s} \alpha'_{i'} a_{i'} + \sum_{i'=2}^{v'}\sum_{j'=1}^{i'-1} \epsilon'_{i',j'} a_{i'} c^{i'-j'} + \sum_{i'=1}^{w'}\sum_{j'=1}^{2i'-1} \eta'_{i',j'} a_{i'} b c^{2i'-j'}.$$

Since $a_i R a_j = b R a_j = c R a_j = 0$, we obtain that $rRr' = 0$.
If $\sum_{i=1}^{s} \alpha_i a_i = 0$, then

$$r' = \sum_{i'=1}^{s'} \alpha'_{i'} a_{i'} + \beta' b + \sum_{k'=1}^{u'} \delta'_k b c^{k'} + \sum_{i'=2}^{v'}\sum_{j'=1}^{i'-1} \epsilon'_{i',j'} a_{i'} c^{i'-j'} + \sum_{i'=1}^{w'}\sum_{j'=1}^{2i'-1} \eta'_{i',j'} a_{i'} b c^{2i'-j'}.$$

Since $bRa_j = bRb = cRa_j = cRb = 0$ and $r$ has no the term $\sum_{i=1}^{s} \alpha_i a_i$, we also get that $rRr' = 0$.

`Case 2.` $\beta b = 0$ and $\sum_{i=1}^{s} \alpha_i a_i \neq 0$.

Then $r = \sum_{i=1}^{s} \alpha_i a_i + \sum_{j=1}^{t} \gamma_j c^j + \sum_{k=1}^{u} \delta_k b c^k + \sum_{i=2}^{v} \sum_{j=1}^{i-1} \epsilon_{i,j} a_i c^{i-j} + \sum_{i=1}^{w} \sum_{j=1}^{2i-1} \eta_{i,j} a_i b c^{2i-j}$.
From $\sum_{i=1}^{s} \alpha_i a_i \neq 0$ and $rr' = 0$, we have that $\beta' b = 0$, and thus

$$r' = \sum_{i'=1}^{s'} \alpha'_{i'} a_{i'} + \sum_{j'=1}^{t'} \gamma'_{j'} c^{j'} + \sum_{k'=1}^{u'} \delta'_k b c^{k'} + \sum_{i'=2}^{v'}\sum_{j'=1}^{i'-1} \epsilon'_{i',j'} a_{i'} c^{i'-j'} + \sum_{i'=1}^{w'}\sum_{j'=1}^{2i'-1} \eta'_{i',j'} a_{i'} b c^{2i'-j'}.$$

Then we have the following subcases.

`Subcase 2-1.` $\sum_{j'=1}^{t'} \gamma'_{j'} c^{j'} \neq 0$ and $\sum_{k'=1}^{u'} \delta'_k b c^{k'} \neq 0$.

Then $\sum_{j=1}^{t} \gamma_j c^j = 0$ and there exist the smallest positive integers $p, q$ such that $\gamma'_1 = \cdots = \gamma'_{p-1} = 0, \gamma'_p \neq 0$ and $\delta'_1 = \cdots = \delta'_{q-1} = 0, \delta'_q \neq 0$, respectively. Since $(\sum_{i=1}^{s} \alpha_i a_i) \left(\sum_{j'=p}^{t'} \gamma'_{j'} c^{j'}\right) = 0$ and $(\sum_{i=1}^{s} \alpha_i a_i) \left(\sum_{k'=q}^{u'} \delta'_k b c^{k'}\right) = 0$, we note that $p \geq s$ and $q \geq 2s$. From $a_i R a_j = a_i R c^{2i} = c R a_j = c R b = 0$, we obtain $rR(r')^2 = 0$.

`Subcase 2-2.` $\sum_{j'=1}^{t'} \gamma'_{j'} c^{j'} \neq 0$ and $\sum_{k'=1}^{u'} \delta'_k b c^{k'} = 0$.

Then $\sum_{j=1}^{t} \gamma_j c^j = 0$ and there exists the smallest positive integer $p$ such that $\gamma'_1 = \cdots = \gamma'_{p-1} = 0, \gamma'_p \neq 0$. Since $(\sum_{i=1}^{s} \alpha_i a_i) \left(\sum_{j'=p}^{t'} \gamma'_{j'} c^{j'}\right) = 0$, we have that $p \geq s$. From $a_i R a_j = a_i R c^{2i} = c R a_j = c R b = 0$, we obtain $rR(r')^2 = 0$.

`Subcase 2-3.` $\sum_{j'=1}^{t'} \gamma'_{j'} c^{j'} = 0$ and $\sum_{k'=1}^{u'} \delta'_k b c^{k'} \neq 0$.

Then there exists the smallest positive integer $q$ such that $\delta'_1 = \cdots = \delta'_{q-1} = 0, \delta'_q \neq 0$. Since $(\sum_{i=1}^{s} \alpha_i a_i) \left(\sum_{k'=q}^{u'} \delta'_k b c^{k'}\right) = 0$, we get that $q \geq 2s$. From $a_i R a_j = a_i R c^{2i} = c R a_j = c R b = 0$, we obtain $rRr' = 0$.

`Subcase 2-4.` $\sum_{j'=1}^{t'} \gamma'_{j'} c^{j'} = 0$ and $\sum_{k'=1}^{u'} \delta'_k b c^{k'} = 0$.

From $a_i R a_j = c R a_j = c R b = 0$, we also obtain $rRr' = 0$.

`Case 3.` $\beta b = 0$ and $\sum_{i=1}^{s} \alpha_i a_i = 0$.

Then $r = \sum_{j=1}^{t} \gamma_j c^j + \sum_{k=1}^{u} \delta_k b c^k + \sum_{i=2}^{v} \sum_{j=1}^{i-1} \epsilon_{i,j} a_i c^{i-j} + \sum_{i=1}^{w} \sum_{j=1}^{2i-1} \eta_{i,j} a_i b c^{2i-j}$.

`Subcase 3-1.` $\sum_{j=1}^{t} \gamma_j c^j \neq 0$.

Then $\sum_{j'=1}^{t'} \gamma'_{j'} c^{j'} = 0$, and so $r' = \sum_{i'=1}^{s'} \alpha'_{i'} a_{i'} + \beta' b + \sum_{k'=1}^{u'} \delta'_k b c^{k'} + \sum_{i'=2}^{v'} \sum_{j'=1}^{i'-1} \epsilon'_{i',j'} a_{i'} c^{i'-j'}$ $+ \sum_{i'=1}^{w'} \sum_{j'=1}^{2i'-1} \eta'_{i',j'} a_{i'} b c^{2i'-j'}$. Since $cRa_j = cRb = 0$, we obtain that $rRr' = 0$.

`Subcase 3-2.` $\sum_{j=1}^{t} \gamma_j c^j = 0$.

If $\sum_{j'=1}^{t'} \gamma'_{j'} c^{j'} \neq 0$, then $\sum_{k=1}^{u} \delta_k b c^k = 0$, and there exists the smallest positive integer $p'$ such that $\gamma'_1 = \cdots = \gamma'_{p'-1} = 0$ and $\gamma'_{p'} \neq 0$. Since

$$\left(\sum_{i=2}^{v} \sum_{j=1}^{i-1} \epsilon_{i,j} a_i c^{i-j}\right) \left(\sum_{j'=p'}^{t'} \gamma'_{j'} c^{j'}\right) = \left(\sum_{i=1}^{w} \sum_{j=1}^{2i-1} \eta_{i,j} a_i b c^{2i-j}\right) \left(\sum_{j'=p'}^{t'} \gamma'_{j'} c^{j'}\right) = 0,$$

we notice that $p' \geq v - 1$ and $p' \geq 2w - 1$. By the construction of $R$, we obtain that $rR(r')^2 = 0$. Finally, if $\sum_{j'=1}^{t'} \gamma'_{j'} c^{j'} = 0$, then from $cRa = cRb = 0$, we obtain $rRr' = 0$.

Consequently, we complete the proof that $R$ is right APIP. □

`Claim 2.` $R$ does not satisfy the condition (†).

**Proof** By the construction of $R$, $a_i \in l_R(c^i)$ for each $i \geq 1$, but $a_i b \notin l_R(c^i)$. This implies that for every $i$,

$l_R(c^i)$ is not an (right) ideal of $R$, as desired. □

In the following we argue about relations between the right APIP condition and the condition (†) when given rings satisfy the ascending chain condition for left annihilators.

**Theorem 2.5** (1) *Let $R$ be a right APIP ring such that $N = \{n(u,v) \geq 1 \mid uRv^{n(u,v)} = 0$ for some $u, v \in R\}$ is bounded above. Then $R$ satisfies the condition (†).*

(2) *Let $R$ be a ring that satisfies the ascending chain condition for left annihilators. If $R$ is right APIP then $R$ satisfies the condition (†).*

**Proof** (1) Let $n_0$ be the least upper bound of $N$. Assume that there exists $a \in R$ such that $l_R(a^k)$ is not two-sided for all $k \geq 1$. Then there exist $b, c \in R$ such that $ba^{n_0} = 0$ but $bca^{n_0} \neq 0$, i.e. $bRa^{n_0} \neq 0$. Since $R$ is right APIP, $bR(a^{n_0})^{n_1} = 0$ for some $n_1 \geq 1$. But $n_0 n_1 \in N$, so that $n_0 n_1$ must equal to $n_0$ because $n_0 n_1 \leq n_0$. From this we obtain $bRa^{n_0} = 0$, a contradiction. Therefore $R$ satisfies the condition (†).

(2) Assume that there exists $a \in R$ such that $l_R(a^k)$ is not two-sided for all $k \geq 1$. Then there exist $b_1, c_1 \in R$ such that $b_1 a = 0$ but $b_1 c_1 a \neq 0$, i.e. $b_1 Ra \neq 0$. Since $R$ is right APIP, $b_1 R a^{n_1} = 0$ for some $n_1 \geq 1$. But $l_R(a^{n_1})$ is not two-sided, there exist $b_2, c_2 \in R$ such that $b_2 a^{n_1} = 0$ but $b_2 c_2 a^{n_1} \neq 0$, i.e. $b_2 R a^{n_1} \neq 0$. Since $R$ is right APIP, $b_2 R a^{n_1 n_2} = 0$ for some $n_2 \geq 1$. Proceeding in this manner, we get an ascending chain

$$l_R(a) \subset l_R(a^{n_1}) \subset l_R(a^{n_1 n_2}) \subset \cdots \subset l_R(a^{n_1 \cdots n_t}) \subset l_R(a^{n_1 \cdots n_t n_{t+1}}) \subset \cdots,$$

where $t \geq 1$. Write $l_R(a^{p_0}) = l_R(a)$ and $l_R(a^{p_t}) = l_R(a^{n_1 \cdots n_t})$. By hypothesis, $l_R(a^{p_s}) = l_R(a^{p_{s+1}}) = l_R(a^{p_{s+2}}) = \cdots$ for some $s \geq 0$. But, by assumption, there exists $b_{s+1} \in R$ such that $b_{s+1} a^{p_s} = 0$, $b_{s+1} R a^{p_s} \neq 0$ and $b_{s+1} R a^{p_{s+1}} = 0$. Since $l_R(a^{p_s}) = l_R(a^{p_{s+1}})$, we see that $b_{s+1} R \subseteq l_R(a^{p_{s+1}}) = l_R(a^{p_s})$, entailing $b_{s+1} R a^{p_s} = 0$, contrary to $b_{s+1} R a^{p_s} \neq 0$. Therefore $R$ satisfies the condition (†). □

The IFP condition does not pass to polynomial rings by [12, Example 2]. But, we have the APIP condition for linear polynomials.

**Remark 2.6** Let $R$ be an IFP ring and suppose that $f(x)g(x) = 0$ for $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = b_0 + b_1 x$ in $R[x]$. We claim that $f(x)R[x]g(x)^{2(m+1)} = 0$. Since $f(x)g(x) = 0$, we have $a_i b_0^{i+1} = 0$ and $a_i b_1^{m-i+1} = 0$ for each $i \in \{0, 1, \ldots, m\}$, by [25, Lemma 1]. Consider $h(x) = g(x)^{2(m+1)}$. Then, in each coefficient of $h(x)$, either $b_0$ occurs at least $m+1$ times or $b_1$ occurs at least $m+1$ times. Thus $a_i R h(x) = 0$ for all $i$ by IFP condition of $R$, from which we see $f(x)Rg(x)^{2(m+1)} = 0$. This result is equivalent to $f(x)R[x]g(x)^{2(m+1)} = 0$.

For a ring $R$, consider the condition: for any $0 \neq r \in R$ there exists $n = n(r) \geq 1$ such that $r^n \neq 0$ and $l_R(r^n)$ is an ideal of $R$, (†)′ say. Rings with the condition (†)′ clearly satisfy the condition (†), but the converse does not hold in general as follows.

**Example 2.7** (1) Consider $R = D_n(A)$ over any ring $R$ for $n \geq 4$. Since $E_{13} \notin l_R(E_{34}) = AE_{12} + AE_{14} + AE_{24} + AE_{34} + \cdots + AE_{1n} + AE_{2n} + \cdots + AE_{(n-1)n}$ is not two-sided and $E_{34}^2 = 0$, $R$ does not satisfy the condition (†)′.

(2) Consider $R = D_n(B)$ over a domain $B$ for $n \geq 2$. Then, for any $M \in R$, either $l_R(M) = 0$ or $M \in N(R)$, hence $R$ satisfies the condition $(\dagger)$.

(3) Consider $R = D_3(B)$ over a domain $B$ and $0 \neq M \in R$. Then either $l_R(M) = 0$ or $l_R(M)$ is one of the following ideals: $BE_{12} + BE_{13} + BE_{23}$ and $BE_{13} + BE_{23}$ which are both two-sided. Thus $R$ satisfies the condition $(\dagger)'$.

A ring (possibly without identity) is usually said to be reduced if it has no nonzero nilpotent elements. It is easy to show that reduced rings are IFP.

**Remark 2.8** Let $R$ satisfy both the condition $(\dagger)'$ and the ascending chain condition for left annihilators. Then we have the following assertions:

(1) Each maximal left annihilator has the form $l_R(a)$ for some $a \in R$ that is an ideal of $R$.

(2) If $R$ is semiprime, then

(i) every minimal prime ideal of $R$ is a maximal left annihilator, and it has the form $l_R(a)$ for some $a \in R$ that is an ideal of $R$.

(ii) $R$ is a subdirect product of a finite number of prime factor rings which satisfy the condition $(\dagger)$.

**Proof** (1) Since $R$ satisfies the ascending chain condition for left annihilators, it is clear that each maximal left annihilator has the form $l_R(a)$ for some $0 \neq a \in R$. Since $R$ satisfies the condition $(\dagger)'$, $a^m \neq 0$ and $l_R(a^m)$ is two-sided for some $m \geq 1$, from which we see that $l_R(a) = l_R(a^m)$ by the maximality of $l_R(a)$. Thus $l_R(a)$ is an ideal of $R$.

(2) Suppose that $R$ is a semiprime ring that satisfies both the condition $(\dagger)'$ and the ascending chain condition for left annihilators. Then $R$ is clearly reduced and satisfies the condition $(\dagger)$. By [2, Lemma 1.16], $R$ has only a finite number of minimal prime ideals, $P_1, \ldots, P_n$ say, such that every $P_i$ is an annihilator ideal. Note $P_1 \cap \cdots \cap P_n = 0$.

(i) Write $P_i = l_R(S)$ with $S \subseteq R \backslash \{0\}$ and assume on the contrary that $l_R(S)$ is not maximal. Then there exists a maximal left annihilator $A_i$ such that $P_i \subsetneq A_i$. But $A_i = l_R(b)$, an ideal of $R$, for some $b \in R\backslash\{0\}$ by Theorem 2.5(2). Then $l_R(S) = l_R(S \cup \{b\})$, so that we may let $P_i = l_R(S \cup \{b\})$. Write $T = S \cup \{b\}$. Take $x \in A_i$ such that $x \notin P_i$. Then $xRT = 0$, but since $P_i$ is prime, we have $T \subseteq P_i = l_R(T)$. This yields $TRT = 0$. But $R$ is semiprime, entailing $RTR = 0$, contrary to $T \subseteq R\backslash\{0\}$. Therefore $P_i$ is maximal and has the form $l_R(a)$, an ideal of $R$, for some $a \in R$ by (1).

(ii) Since $R$ is reduced and has only a finite number of minimal prime ideals by the above, $R$ is a subdirect product of a finite number of domains by [27, Proposition 1.11(b)]. □

## 3. Structures and relations

In this section we study various useful properties of right APIP rings. We first investigate the structure of right APIP rings in relation to prime ideals, nilpotent elements and right annihilators, and observe the relationship between related rings. It is easily proved that semiprime IFP rings are reduced, but Example 3.15 to follow shows that there exists a right APIP prime ring that is not reduced.

**Proposition 3.1** (1) *Let $R$ be a right APIP ring. If $R$ is prime, then we have the following:*

(i) $N(R) = \bigcup\{r_R(A) \mid A \subseteq R\backslash\{0\}\}$.

(ii) $N(R) = \{br \mid b \in N(R)$ *and* $r \in R\} = \bigcup_{b\in N(R)} bR$.

(iii) $bR$ *is nil for all* $b \in N(R)$.

(iv) *Suppose* $ab = 0$ *for* $a, b \in R\backslash\{0\}$. *Then* $b \in N(R)$, *and either* $a \in N(R)$ *or* $ba^k \neq 0$ *for all* $k \geq 1$.

(v) *If* $ab \in N(R)$ *for* $a, b \in R\backslash\{0\}$, *then* $a \in N(R)$ *or* $b \in N(R)$.

(2) *In a prime APIP ring, every one-sided zero-divisor is a nilpotent element.*

**Proof** (1) (i) Let $0 \neq a \in R$ and $b \in r_R(a)$ (i.e. $ab = 0$). Since $R$ is right APIP, $aRb^n = 0$ for some $n \geq 1$. If $R$ is prime, then $a \neq 0$ implies $b^n = 0$, that is, $b \in N(R)$. Thus $r_R(a) \subseteq N(R)$. Next let $A \subseteq R\backslash\{0\}$. Since $r_R(A) = \cap_{a\in A} r_R(a)$, we get $r_R(A) \subseteq N(R)$, and it follows that the union of $r_R(A)$'s is contained in $N(R)$. The converse inclusion is clear from the fact that $bR \subseteq r_R(b^{n-1})$ for each $0 \neq b \in N(R)$, where $b^n = 0$ and $b^{n-1} \neq 0$ for some $n \geq 1$.

(ii) and (iii) are immediate consequences of (i).

(iv) The first result is obtained from (i). There exists $n \geq 1$ such that $b^n = 0$ and $b^{n-1} \neq 0$. Assume $a \notin N(R)$. Consider $b^{n-1}ba^k = 0$ where $k$ is any positive integer. If $ba^k = 0$ then $bR(a^k)^m = 0$ for some $m \geq 1$ since $R$ is right APIP. So $a^{km} = 0$ because $R$ is prime, contrary to the assumption. Thus $ba^k \neq 0$ for all $k \geq 1$.

(v) Suppose that $ab \in N(R)$ for $a, b \in R\backslash\{0\}$, then $ba \in N(R)$. Assume that $ab = 0$ and $ba = 0$. Then $a, b \in N(R)$ by (iv). Assume $ab \neq 0$ or $ba \neq 0$. Let $ab \neq 0$. Then there exists $k \geq 1$ such that $(ab)^k = 0$ and $(ab)^{k-1} \neq 0$. If $(ab)^{k-1}a \neq 0$ then $b \in N(R)$ by (iv). If $(ab)^{k-1}a = 0$ then $a \in N(R)$ by (iv). The proof for the case of $ba \neq 0$ can be done by a similar manner.

(2) This is clear from (1)–(iv). □

Notice that the ring $R$ in Example 3.15 to follow is an example of Proposition 3.1. Furthermore, in Proposition 3.1(2), the condition "prime" is not superfluous as can be seen by $(1, 0)(0, 1) = 0$ in the reduced ring $R \times R$, where $R$ is a reduced ring.

Following Marks [22], a ring $R$ is called *NI* if $N(R) = N^*(R)$. It is obvious that a ring $R$ is NI if and only if $R/N^*(R)$ is reduced. IFP rings are easily shown to be NI, but the NI ring $T_2(\mathbb{Z}_2)$ is not right APIP by Proposition 3.11(1) below, since it is not abelian. Recall that an element $u$ of a ring $R$ is right regular if $ur = 0$ for $r \in R$ implies $r = 0$. The left regular is defined similarly, and *regular* means both right and left regular (hence not a one-sided zero-divisor). Denote the set of all regular elements in $R$ by $C(R)$. Recall that a ring $R$ is said to be of bounded index of nilpotency if there exists $n \geq 1$ such that $a^n = 0$ for all $a \in N(R)$.

**Theorem 3.2** *Let $R$ be a prime APIP ring. Then we have the following.*

(1) *Every element of $R$ is either nilpotent or regular.*

(2) *If $R$ is of bounded index of nilpotency then $R$ is a domain.*

(3) *If $R$ is an NI ring then $R/N^*(R)$ is a domain.*

**Proof** (1) Let $a \in R\backslash N(R)$. Then $a \in C(R)$ by Proposition 3.1(2).

(2) We first claim $N(R) = 0$. Assume $N(R) \neq 0$ and let $0 \neq a \in N(R)$. Then $aR$ is nil by Proposition 3.1(3). If $R$ is of bounded index of nilpotency, then $aR$ contains a nonzero nilpotent ideal $I$ of $R$ by Levitzki [10, Lemma 1.1] or Klein [19, Lemma 5]. But since $R$ is prime, we have $I = 0$, a contradiction. Thus $N(R) = 0$, from which we see that $R$ is a domain by (1).

(3) This is clear from (1) when $R$ is NI. □

The condition "of bounded index of nilpotency" in Theorem 3.2(2) is not superfluous by the prime APIP ring $R$ in Example 3.15 below that is neither reduced nor bounded index of nilpotency. K-rings are clearly APIP, and so we obtain the following by Proposition 3.1 and Theorem 3.2.

**Corollary 3.3** (1) [14, Lemma 10.1.2] *If $R$ is a prime K-ring then every (one-sided) zero-divisor in $R$ is nilpotent.*

(2) [21, Proposition 3.2(1)] *If $R$ is a prime K-ring then every element of $R$ is either nilpotent or regular.*

(3) [21, Proposition 3.2(2)] *If $R$ is a prime K-ring then $R/N^*(R)$ is a commutative domain.*

**Proof** (1) By Proposition 3.1(2). (2) By Theorem 3.2(1).

(3) Every K-ring is NI by Proposition 3.1(1-(iii)) and the argument in the proof of [14, Lemma 10.1.3] which shows that $N(R)$ is closed under addition. Whence we obtain the result by Theorem 3.2(3) and [14, Theorem 2]. □

Recall that the Köthe's conjecture means that nil one-sided ideals are contained in the upper nilradical in any ring; equivalently, the sum of two nil right (left) ideals in any ring is nil. Notice that it is well-known that Köthe's conjecture holds for NI rings.

**Proposition 3.4** *If the Köthe's conjecture holds, then every right APIP prime ring is NI.*

**Proof** Let $R$ be a right APIP prime ring. Then $N(R) = \cup_{b \in N(R)} bR$ by Proposition 3.1(1-(iii)). Assume that the Köthe's conjecture holds. Then every $bR$ belongs to $N^*(R)$, so that $N(R) = N^*(R)$. Thus $R$ is NI. □

One may ask whether the class of right APIP rings is closed under prime factor rings. The answer is negative by the following.

**Example 3.5** Let $R$ be the Hamilton quaternions $\mathbb{H}(\mathbb{Z})$ over $\mathbb{Z}$. Then $R$ is clearly a domain (hence APIP). Let $p$ be an odd prime integer and consider the prime ideal $pR = \mathbb{H}(p\mathbb{Z})$ of $R$. Then $R/pR$ is isomorphic to $M_2(\mathbb{Z}_p)$ by the argument in [8, Exercise 2A], but $M_2(\mathbb{Z}_p)$ is not right APIP as can be seen by the argument that $E_{11}E_{22} = 0$ and $E_{12} \in E_{11}RE_{22} = E_{11}RE_{22}^n \neq 0$ for all $n \geq 1$. Thus $R/pR$ is not right APIP.

We see conditions under which right APIP condition passes to factor rings, vice versa.

**Proposition 3.6** (1) *Let $R$ be a right APIP ring and $A$ be a finite subset of $R$ such that $r_R(A)$ is an ideal of $R$. Then $R/r_R(A)$ is a right APIP ring.*

(2) *Let $R$ be a ring and $I$ a proper ideal of $R$. If $R/I$ is right APIP and $I$ is a reduced ring without identity, then $R$ is right APIP.*

**Proof** (1) Let $A = \{a_1, \ldots, a_k\}$ be a subset of $R$ and consider the factor ring $R/r_R(A)$. Suppose that $bc \in r_R(A)$ for $b, c \in R$. Then $a_ibc = 0$ for all $i$. Since $R$ is right APIP, we have that for each $i$, there exists $n_i \geq 1$ such that $a_ibRc^{n_i} = 0$, from which we infer that $a_ibRc^n = 0$ for all $i$, where $n$ is greatest in $\{n_1, \ldots, n_k\}$. This implies $AbRc^n = 0$, that is, $bRc^n \subseteq r_R(A)$. Therefore $R/r_R(A)$ is right APIP.
(2) It is a similar computation to the proof of [4, Proposition 1.12]. □

The condition "$I$ is a reduced ring" in the Proposition 3.6(2) cannot be weakened by the condition "$I$ is an IFP ring" as follows.

**Example 3.7** Consider a ring $R = T_2(F)$ where $F$ is a field, which is not right APIP as noted above. The only nonzero proper ideals of $R$ are $I_1 = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}, I_2 = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$ and $I_3 = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Then $R/I_1$ and $R/I_2$ are isomorphic to $F$ and $R/I_3 = \left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} + I_3 \mid a, c \in F \right\}$ is a reduced ring, and hence each $R/I_i$ (for $i = 1, 2, 3$) is right APIP. Notice that each $I_i$ is IFP, but not reduced.

**Proposition 3.8** *Let $\{R_\gamma \mid \gamma \in \Gamma\}$ be a family of rings.*

(1) *Let $\Gamma$ is a finite set. Then the direct product $R = \prod_{\gamma\in\Gamma} R_\gamma$ of $R_\gamma$'s is right APIP if and only if $R_\gamma$ is right APIP for every $\gamma \in \Gamma$.*

(2) *Let $\Gamma$ is a infinite set. If the direct product $R = \prod_{\gamma\in\Gamma} R_\gamma$ of $R_\gamma$'s is right APIP, then $R_\gamma$ is right APIP for every $\gamma \in \Gamma$.*

**Proof** This is similar to the proof of [4, Proposition 2.3]. □

**Corollary 3.9** *Let $R$ be a ring and $e^2 = e$ be central. Then $R$ is right APIP if and only if both $eR$ and $(1-e)R$ are right APIP.*

**Proof** It comes from Proposition 3.11(2) to follow and Proposition 3.8, since $R = eR \oplus (1-e)R$ for $e^2 = e \in R$. □

Due to Feller [6], a ring (possibly without identity) is called right duo if every right ideal is two-sided. Left duo rings are defined similarly. A ring is called duo if it is both left and right duo. It is clear that every one-sided duo ring is IFP. Following Yao [28], a ring $R$ is called weakly right duo if for each $a$ in $R$ there exists a positive integer $n$ such that $a^nR$ is a two-sided ideal of $R$. Weakly left duo rings are defined similarly. A ring is called weakly duo if it is both weakly left and right duo. A ring $R$ is called right $\pi$-duo [16] if for any $a \in R$ there exists $n \geq 1$ such that $Ra^n \subseteq aR$. Left $\pi$-duo rings are defined similarly. A ring is called $\pi$-duo if it is both left and right $\pi$-duo. Note that commutative rings are duo, right duo rings are weakly right duo, and weakly right duo rings are right $\pi$-duo, but not conversely in each case. Note that right $\pi$-duo rings are abelian by [16, Proposition 1.9(4)].

IFP rings are clearly APIP, and right $\pi$-duo rings are right APIP by [16, Proposition 1.9(3)], but each converse does not hold in general as seen in the next example.

**Example 3.10** (1) Consider $R = D_4(S)$ where $S$ is a division ring. Then $R$ is not IFP by [17, Example 1.3]. Let $A, B \in R\backslash\{0\}$ and suppose $AB = 0$. Then the diagonal entries of $A$ and $B$ are zero because matrices with nonzero diagonals are invertible. Hence $A^3 = B^3 = 0$, so that $A^3RB = 0$ and $ARB^3 = 0$. Thus $R$ is APIP.

(2) We refer to [23, Theorem 1.3.5, Corollary 2.1.14 and Theorem 2.1.15]. Let $F\langle a,b\rangle$ be the free algebra with noncommuting indeterminates $a, b$ over a field $F$ of characteristic zero. The first Weyl algebra $A_1(F) \cong \frac{F\langle a,b\rangle}{(ba-ab-1)}$, $R$ say, is a domain whose invertible elements are nonzero in $F$, where $(ba - ab - 1)$ is the ideal of $F\langle a,b\rangle$ generated by $ba - ab - 1$. Then $R$ is right APIP, but it is neither left nor right $\pi$-duo by [16, Example 1.11(2)].

McCoy [24] called a ring $R$ *$\pi$-regular* if for each $a \in R$ there exists a positive integer $n = n(a)$ and $b \in R$ such that $a^n = a^n b a^n$. Note that $\pi$-regular rings need not be right APIP as can be seen by nonabelian semisimple Artinian rings.

**Proposition 3.11** (1) *Right APIP rings are abelian.*

(2) *The class of right APIP rings is closed under subrings.*

(3) *Let $R$ be a right APIP ring of bounded index of nilpotency. Then $R$ is prime if and only if $R$ is a domain.*

(4) *Every abelian $\pi$-regular ring is APIP.*

**Proof** (1) This comes from the fact that right APIP rings are $\pi$-IFP, and (2) is obvious.

(3) Let $R$ be a prime ring. Then $aR$ is nil for each $a \in N(R)$ by Proposition 3.1(1-(iii)), since $R$ is prime right APIP. But $R$ is of bounded index of nilpotency by hypothesis, and so either $aR = 0$ or $aR$ contains a nonzero nilpotent ideal of $R$ by Levitzki [10, Lemma 1.1] or Klein [19, Lemma 5]. Since $R$ is prime, we have $aR = 0$, entailing $a = 0$. Thus $R$ is reduced, and so $R$ is a domain. The converse is evident.

(4) Let $R$ be an abelian $\pi$-regular ring and suppose that $ab = 0$ for $a, b \in R$. Since $R$ is $\pi$-regular, $b^h = b^h c b^h$ for some $h \geq 1$ and $c \in R$. Note $b^h c, cb^h \in I(R)$. Since $R$ is abelian, we see $aRb^h = aRb^h cb^h = ab^h cRb^h = 0$, and hence $R$ is right APIP. $R$ can be shown to be left APIP analogously. □

In fact, the proof of Proposition 3.11(1) is easy as can be seen by the argument that for any idempotent $e$ in $R$, $e(1-e) = 0$ implies $eR(1-e) = eR(1-e)^n = 0$ for all $n \geq 1$. Using Proposition 3.11(1), we claim that both $M_n(A)$ and $T_n(A)$ cannot be right APIP for any ring $A$ and $n \geq 2$, since they are nonabelian.

An ideal $I$ of a ring $R$ is usually said to be idempotent-lifting if idempotents in $R/I$ can be lifted to $R$. Nil ideals are idempotent-lifting by [20, Proposition 3.3.6]. Related to Proposition 3.11(1), we let $R$ be an abelian $\pi$-regular ring. Then $R/J(R)$ is also abelian $\pi$-regular since $J(R)$ is nil and hence $J(R)$ is idempotent-lifting by [20, Proposition 3.3.6]. By the same manner, $R/N$ is also abelian $\pi$-regular for any nil ideal $N$ of $R$, and hence $R/N$ is APIP and, especially, $R/N_*(R)$ is APIP. This fact is compared with Example 3.5.

The condition "bounded index of nilpotency" in Proposition 3.11(3) is not superfluous by the APIP prime ring $R$ in Example 3.15 below that is neither reduced nor bounded index of nilpotency.

The following example illuminates that for $r \in R$, the power $n$ of $l_R(r^n)$ in a right APIP ring $R$ depends on $r$.

**Example 3.12** Let $A = \mathbb{Z}_2\langle a, b, c\rangle$ be the free algebra with noncommuting indeterminates $a, b, c$ over $\mathbb{Z}_2$. Let $I$ be the ideal of $A$ generated by $a^2, b^3, c^2, ab, ba, bc$ and $ca$. Note that $I$ is homogeneous. Set $R = A/I$ and let $a, b, c$ coincide with their images in $R$ for simplicity. Then every element $r$ in $R$ is of the form $\alpha_0 + \alpha_1 a + \alpha_2 b + \alpha_3 b^2 + \alpha_4 c + \alpha_5 ac + \alpha_6 acb + \alpha_7 acb^2 + \alpha_8 cb + \alpha_9 cb^2$, where $\alpha_i \in \mathbb{Z}_2$. Note that $a \in l_R(b^2)$, but $acb^2 \neq 0$ and so $ac \notin l_R(b^2)$. Thus $l_R(b^2)$ is not a right ideal of $R$.

We now show that $R$ is right APIP. Suppose that $rs = 0$, for any $r = \alpha_0 + \alpha_1 a + \alpha_2 b + \alpha_3 b^2 + \alpha_4 c + \alpha_5 ac + \alpha_6 acb + \alpha_7 acb^2 + \alpha_8 cb + \alpha_9 cb^2$, $s = \beta_0 + \beta_1 a + \beta_2 b + \beta_3 b^2 + \beta_4 c + \beta_5 ac + \beta_6 acb + \beta_7 acb^2 + \beta_8 cb + \beta_9 cb^2 \in R\backslash\{0\}$, where $\alpha_i, \beta_j \in \mathbb{Z}_2$. By a similar computation to be noted in the case of $rs = 0$ in Example 3.14 to follow, both $r$ and $s$ have zero constant terms. Thus

$$s = \beta_1 a + \beta_2 b + \beta_3 b^2 + \beta_4 c + \beta_5 ac + \beta_6 acb + \beta_7 acb^2 + \beta_8 cb + \beta_9 cb^2 \text{ and}$$

$$s^2 = \beta_1\beta_4 ac + (\beta_1\beta_8 + \beta_5\beta_2)acb + (\beta_1\beta_9 + \beta_5\beta_3 + \beta_6\beta_2)acb^2 + \beta_2^2 b^2 + \beta_4\beta_2 cb + (\beta_4\beta_3 + \beta_8\beta_2)cb^2,$$

so let $s^2 = \gamma_1 ac + \gamma_2 acb + \gamma_3 acb^2 + \gamma_4 b^2 + \gamma_5 cb + \gamma_6 cb^2$, where $\gamma_i \in \mathbb{Z}_2$. Then

$$s^4 = s^2 s^2 = \gamma_1\gamma_4 acb^2 \text{ and}$$

$$s^5 = s^4 s = (\gamma_1\gamma_4 acb^2)(\beta_1 a + \beta_2 b + \beta_3 b^2 + \beta_4 c + \beta_5 ac + \beta_6 acb + \beta_7 acb^2 + \beta_8 cb + \beta_9 cb^2) = 0,$$

by the construction of $R$. Thus $rRs^5 = 0$, and so $R$ is right APIP.

**Remark 3.13** (1) According to [5], a right ideal $I$ of a ring $R$ is called a generalized weak ideal (simply, a *GW*-ideal) in case that for any $a \in I$, there exists a positive integer $n$ such that $Ra^n \subseteq I$, and $R$ is called right generalized weak zero insertive (simply, *right GWZI*), if $r_R(a)$ is a GW-ideal of $R$ for any $a \in R$. In [3], it is shown that a ring $R$ is right GWZI if and only if $R$ is right APIP.

(2) As another generalization of IFP rings, a ring $R$ is called $\pi$*-IFP* [4, Definition 1.7] (also, [3] and [26]) if $ab = 0$ for $a, b \in R$ implies $a^m R b^n = 0$ for some $m, n \geq 1$. It is clear that a ring $R$ is $\pi$-IFP if and only if $a^n R b^n = 0$ for some $n \geq 1$ whenever $ab = 0$ for $a, b \in R$. By [4, Lemma 1.8], $\pi$-IFP rings are abelian.

In the remainder of this section, we study the relationships among right APIP rings and related rings. We first note that right APIP rings are clearly $\pi$-IFP, but not conversely by the following.

**Example 3.14** Let $R$ be the ring in Example 2.1(1). We will show that $R$ is $\pi$-IFP. Suppose that $rs = 0$, where $r = \alpha_0 + \alpha_1 a + \alpha_2 f(b) + \alpha_3 c + \alpha_4 ac + \alpha_5 acg(b) + \alpha_6 ch(b)$ and $s = \beta_0 + \beta_1 a + \beta_2 u(b) + \beta_3 c + \beta_4 ac + \beta_5 acv(b) + \beta_6 cw(b)$ in $R\backslash\{0\}$ with $\alpha_i, \beta_j \in \mathbb{Z}_2$ and $f(b), g(b), h(b), u(b), v(b), w(b) \in bR[b]$. From $0 = rs$, we have $\alpha_0\beta_0 = 0$, and it implies that $(\alpha_0 = 0, \beta_0 = 1), (\alpha_0 = 1, \beta_0 = 0)$ or $(\alpha_0 = 0, \beta_0 = 0)$.

If $\alpha_0 = 0$ and $\beta_0 = 1$, then, by the construction of $R$,

$$\begin{aligned} 0 = rs = {} & \alpha_1 a + \alpha_1\beta_3 ac + \alpha_1\beta_6 acw(b) + \alpha_2 f(b) + \alpha_2\beta_2 f(b)u(b) + \alpha_3 c + \alpha_3\beta_2 cu(b) \\ & + \alpha_4 ac + \alpha_4\beta_2 acu(b) + \alpha_5 acg(b) + \alpha_5\beta_2 acg(b)u(b) + \alpha_6 ch(b) + \alpha_6\beta_2 ch(b)u(b) \end{aligned}$$

implies that $\alpha_1 = 0$ and $\alpha_3 = 0$, hence $\alpha_4 = 0$; and, since the degree of $f(b)$ is less than one of $f(b)u(b)$, $\alpha_2 = 0$. By the same reason, $\alpha_5 = \alpha_6 = 0$, a contradiction to $r \neq 0$.

In case of $\alpha_0 = 1$ and $\beta_0 = 0$, we have $s = 0$ by the same argument as above, a contradiction. Therefore $\alpha_0 = 0$ and $\beta_0 = 0$, and so, from $rs = 0$, we get that

$$\alpha_1\beta_3 ac + \alpha_1\beta_6 acw(b) + \alpha_2\beta_2 f(b)u(b) + \alpha_3\beta_2 cu(b) + \alpha_4\beta_2 acu(b) + \alpha_5\beta_2 acg(b)u(b) + \alpha_6\beta_2 ch(b)u(b) = 0,$$

from which we have that

$$\alpha_1\beta_3 = 0, \alpha_2\beta_2 = 0, \alpha_3\beta_2 = 0, \alpha_6\beta_2 = 0, \text{ and } \alpha_1\beta_6 acw(b) + \alpha_4\beta_2 acu(b) + \alpha_5\beta_2 acg(b)u(b) = 0. \tag{3.1}$$

From $\alpha_1\beta_3 = 0$ and $\alpha_2\beta_2 = 0$ in Equation (3.1), we have the following 9 cases.

We first consider the case of $\alpha_1 = 0$ and $\beta_3 = 1$.

(1-i) $\alpha_2 = 0$, $\beta_2 = 1$: Then $\alpha_3 = 0$ and $\alpha_6 = 0$; and moreover $\alpha_4 acu(b) + \alpha_5 acg(b)u(b) = 0$ implies $\alpha_4 = 0$ and $\alpha_5 = 0$ since the degree of $u(b)$ is less than one of $g(b)u(b)$. Hence $r = 0$, a contradiction to $r \neq 0$.

(1-ii) $\alpha_2 = 1$, $\beta_2 = 0$: Then $s = \beta_1 a + c + \beta_4 ac + \beta_5 acv(b) + \beta_6 cw(b)$. So $s^3 = 0$, and it implies that $r^n Rs^3 = 0$ for any $n \geq 1$.

(1-iii) $\alpha_2 = 0$, $\beta_2 = 0$: We have the same $s$ as the case of (1-ii). Thus $r^n Rs^3 = 0$ for any $n \geq 1$.

We next consider the case of $\alpha_1 = 1$ and $\beta_3 = 0$.

(2-i) $\alpha_2 = 0$, $\beta_2 = 1$: Then $\alpha_3 = 0$, $\alpha_6 = 0$ and $\beta_6 acw(b) + \alpha_4 acu(b) + \alpha_5 acg(b)u(b) = 0$. Consider $\beta_6 acw(b) + \alpha_4 acu(b) + \alpha_5 acg(b)u(b) = 0$. If $\beta_6 = 1$, then $\alpha_4 = 0$ since the degree of $u(b)$ is less than one of $g(b)u(b)$. Thus $\beta_6 = 1 = \alpha_5$, i.e. $w(b) = g(b)u(b)$. Then $r = a + acg(b)$ and $s = \beta_1 a + u(b) + \beta_4 ac + \beta_5 acv(b) + cw(b)$; if $\beta_6 = 0$, then $\alpha_4 = 0$ and $\alpha_5 = 0$ by the same reason as above. So $r = a$ and $s = \beta_1 a + u(b) + \beta_4 ac + \beta_5 acv(b)$. In both cases, we have $r^2 = 0$, and it implies that $r^2 Rs^n = 0$ for any $n \geq 1$.

(2-ii) $\alpha_2 = 1$, $\beta_2 = 0$: Then $\beta_6 acw(b) = 0$ implies $\beta_6 = 0$, and so $s = \beta_1 a + \beta_4 ac + \beta_5 acv(b)$. So $s^2 = 0$, and it implies that $s^2 = 0$. Thus $r^n Rs^2 = 0$ for any $n \geq 1$.

(2-iii) $\alpha_2 = 0$, $\beta_2 = 0$: Then $\beta_6 = 0$ from $\beta_6 acw(b) = 0$, and so we get the same $s$ as the case of (2-ii). Thus $r^n Rs^2 = 0$ for any $n \geq 1$.

We finally consider the case of $\alpha_1 = 0$ and $\beta_3 = 0$.

(3-i) $\alpha_2 = 0$, $\beta_2 = 1$: Then $\alpha_3 = 0, \alpha_6 = 0$, and $\alpha_4 acu(b) + \alpha_5 acg(b)u(b) = 0$. Then $\alpha_4 acu(b) + \alpha_5 acg(b)u(b) = 0$ entails $\alpha_4 = 0$ and $\alpha_5 = 0$ by the same reason as above. Thus $r = 0$, a contradiction.

(3-ii) $\alpha_2 = 1$, $\beta_2 = 0$: Then $s = \beta_1 a + \beta_4 ac + \beta_5 acv(b) + \beta_6 cw(b)$. So $s^3 = 0$, and it implies that $r^n Rs^3 = 0$ for any $n \geq 1$.

(3-iii) $\alpha_2 = 0$, $\beta_2 = 0$: Then we obtain the same $s$ as the case of (3-ii). Thus $r^n Rs^3 = 0$ for any $n \geq 1$.

Consequently, $R$ is a $\pi$-IFP ring by above. But $0 \neq acb^n \in aRb^n$ for any $n \geq 1$, even if $ab = 0$ (for example, the case (2-i)). Thus $R$ is not right APIP.

Note that the concept of a right APIP ring is not only seated between IFP rings and $\pi$-IFP rings, but seated between right $\pi$-duo rings and $\pi$-IFP rings.

The following diagram shows all implications among the concepts above.

$$\begin{tikzcd}
\bullet\ \text{reduced ring} \arrow[r] & \text{IFP ring} \arrow[r] & \boxed{\text{right APIP ring}} \arrow[r] & \pi\text{-IFP ring} \arrow[d] \\
\bullet\ \text{right duo ring} \arrow[r] \arrow[ur] & \text{weakly right duo ring} \arrow[r] & \text{right } \pi\text{-duo ring} \arrow[r] \arrow[u] & \text{abelian ring}
\end{tikzcd}$$

Recall that a ring $R$ is said to be von Neumann regular [7] if for each $a \in R$ there exists $b \in R$ such that $a = aba$. An abelian von Neumann regular ring is both reduced and duo by [7, Theorem 3.2] and [15, Lemma 2.4], respectively. Thus the concepts of all rings above are coincided in a von Neumann regular ring.

Following the literature, a ring is called locally finite if every finite subset generates a finite multiplicative semigroup. Locally finite rings belong to the class of $\pi$-regular rings by [12, Proposition 16]. Furthermore, by [4, Proposition 2.1(1)] and [16, Proposition 1.10], we have that the rings, weakly duo, weakly right duo, right $\pi$-duo, right APIP, $\pi$-IFP, and abelian are equivalent when the rings are locally finite. Thus, it is natural to ask whether a ring $R$ is IFP if $R$ is locally finite and weakly duo. But the answer is negative by the following.

**Example 3.15** We apply the construction and argument in [13, Example 1.2]. Let $R_n = D_{2^n}(\mathbb{Z}_2)$ for $n \geq 1$ with the function $\sigma : R_n \to R_{n+1}$ by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$. Set $R = \bigcup_{n=1}^{\infty} R_n$, noting that $R_n$ can be considered as a subring of $R_{n+1}$ via $\sigma$. Then $R$ is a semiprime ring by [15, Theorem 2.2], and moreover $R$ is prime by applying the proof of [13, Proposition 1.3]. But $R$ is not IFP by applying [17, Example 1.3]. Note that $R$ is evidently locally finite, and every idempotent in $R$ is either zero or the identity matrix by [11, Lemma 2], so $R$ is abelian. Therefore $R$ is weakly duo (if and only if $R$ is APIP) by the argument above. Furthermore, it is easily checked that $R$ is $\pi$-regular, hence $R$ is also APIP by Proposition 3.11(4).

## 4. APIP condition of matrix rings and polynomial rings

In this section we examine several kinds of ring extensions by which the class of right APIP rings is able to be extended, and find conditions under which this work may be possible, if necessary.

**Lemma 4.1** [9, Lemma 2.2(1)] *Let $R$ be a ring and $A = (a_{ij}) \in D_n(R)$ for $n \geq 2$ with $a = a_{ii}$ for all $i$. Then the entries of $A^m$ are in $(RaR)^{m-n+1}$ for any $m \geq n$. In particular, every sum-factor of entries of $A^n$ contains $a$.*

Recall that $D_n(R)$ is IFP over a reduced ring $R$ when $n \leq 3$ by [17, Proposition 1.2]. But $D_n(R)$ is not IFP over any ring $R$ for $n \geq 4$ by [17, Example 1,3]. We here have affirmative results for right APIP rings as follows, from which one can always construct right APIP rings but not IFP, over given any right APIP ring.

**Theorem 4.2** *Let $R$ be a ring and $n \geq 2$. Then the following conditions are equivalent:*

(1) *$R$ is right (resp., left) APIP;*

(2) *$D_n(R)$ is right (resp., left) APIP;*

(3) *$V_n(R)$ is right (resp., left) APIP.*

**Proof** It suffices to prove (1) $\Rightarrow$ (2) by Proposition 3.11(2). Let $R$ be right APIP and suppose that $AB = 0$ for $A = (a_{ij}), B = (b_{ij}) \in D_n(R)\backslash\{0\}$. Then $ab = 0$, where $a = a_{ii}$ and $b = b_{ii}$. Since $R$ is right APIP, $aRb^{h_1} = 0$ for some $h_1 \geq 1$. We will proceed by induction on $i, j$.

Note $ab_{12} + a_{12}b = 0$. Since $aRb^{h_1} = 0$, we get $a_{12}b^{1+h_1} = 0$ by multiplying the preceding equality by $b^{h_1}$ on the right. Since $R$ is right APIP, $a_{12}Rb^{(1+h_1)h_2} = 0$ for some $h_2 \geq 1$. Set $h_3 = (1 + h_1)h_2$. Then we have

$$a_{ij}Rb^{h_3} = 0 \text{ for all } 1 \leq i, j \leq 2. \tag{4.1}$$

Note $ab_{13} + a_{12}b_{23} + a_{13}b = 0$. Multiplying this equality by $b^{h_3}$ on the right, we get $a_{13}b^{1+h_3} = 0$ by the result (4.1). Since $R$ is right APIP, $a_{13}Rb^{(1+h_3)h_4} = 0$ for some $h_4 \geq 1$. Set $h_5 = (1 + h_3)h_4$.

Note $ab_{23} + a_{23}b = 0$. Since $aRb^{h_1} = 0$, we get $a_{23}b^{1+h_1} = 0$ by multiplying the preceding equality by $b^{h_1}$ on the right. Since $R$ is right APIP, $a_{23}Rb^{(1+h_1)h_6} = 0$ for some $h_6 \geq 1$. Set $h_7 = (1 + h_1)h_6$.

Set $h = max\{h_5, h_7\}$. Then we have

$$a_{ij}Rb^h = 0 \text{ for all } 1 \leq i, j \leq 3.$$

Now, suppose by induction that there exists $k \geq 1$ such that

$$a_{ij}Rb^k = 0 \text{ for all } 1 \leq i, j \leq n - 1. \tag{4.2}$$

Note $a_{11}b_{1n} + a_{12}b_{2n} + \cdots + a_{1(n-1)}b_{(n-1)n} + a_{1n}b_{nn} = 0$. Multiplying this equality by $b^k$ on the right, we get $a_{1n}b^{1+k} = 0$ by the result (4.2). Since $R$ is right APIP, $a_{1n}Rb^{(1+k)k_1} = 0$ for some $k_1 \geq 1$.

Note $a_{22}b_{2n} + \cdots + a_{2(n-1)}b_{(n-1)n} + a_{2n}b_{nn} = 0$. Multiplying this equality by $b^k$ on the right, we get $a_{2n}b^{1+k} = 0$ by the result (4.2). Since $R$ is right APIP, $a_{2n}Rb^{(1+k)k_2} = 0$ for some $k_2 \geq 1$.

Proceeding in this manner, we can obtain $k_s \geq 1$ for each $s$ with $1 \leq s \leq n-1$ such that $a_{sn}Rb^{(1+k)k_s} = 0$, from the equality $a_{ss}b_{sn} + \cdots + a_{s(n-1)}b_{(n-1)n} + a_{sn}b_{nn} = 0$.

Set $l = max\{(1 + k)k_1, \ldots, (1 + k)k_{n-1}\}$. Then we now have

$$a_{ij}Rb^l = 0 \ \text{ for all } \ 1 \leq i, j \leq n. \tag{4.3}$$

Next consider $(B^l)^n$. Then every entry of $B^{ln}$ is contained in $Rb^lR$ by Lemma 4.1, noting that each diagonal of $B^l$ is $b^l$. Thus, by (4.3), we see $AD_n(R)B^{ln} = 0$ because every entry of matrices in $AD_n(R)B^{lk}$ belongs to $\sum_{i,j=1}^{n} a_{ij}Rb^lR$. Therefore $D_n(R)$ is right APIP. The proof for the left case is obtained similarly. □

In the following, we apply Theorem 4.2 to provide a method of constructing right APIP rings but not IFP, from given any ring. The center of a ring $S$ is denoted by $Z(S)$.

**Example 4.3** (1) Let $A$ be any ring and consider $R = Z(A)$. Then $D_n(R)$ for $n \geq 2$ is APIP.

(2) Let $A$ be any ring and $M$ be a maximal ideal of $R$. Consider $R = Z(A/M)$, a field. Then $D_n(R)$ for $n \geq 2$ is APIP.

(3) Let $A$ be any ring and $\{M_i \mid i = 1, \ldots, k\})$ be a set of maximal ideals of $A$. Set $R_i = Z(A/M_i)$. Then $\prod_{i=1}^{k} D_{n_i}(R_i)$ for $n_i \geq 2$ is APIP by (2) and Proposition 3.8(1).

The ring below shows that the right APIP condition does not go up to polynomial rings.

**Example 4.4** We apply the example in [18, Example 1.10] here. Let $A = \mathbb{Z}_2\langle a_0, a_1, a_2, b_0, b_1, b_2, b_3\rangle$ be the free algebra with noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, b_3$ over $\mathbb{Z}_2$. Let $I$ be the ideal of $A$ generated by

$$a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0, a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0, a_1b_3 + a_2b_2 + a_3b_1,$$

$$a_2b_3 + a_3b_2, a_3b_3, a_0a_j(0 \leq j \leq 3), a_3a_j(0 \leq j \leq 3), a_1a_j + a_2a_j(0 \leq j \leq 3),$$
$$b_ib_0(0 \leq i \leq 3), b_ib_3(0 \leq i \leq 3), b_ib_1 + b_ib_2(0 \leq i \leq 3), b_ia_j(0 \leq i, j \leq 3).$$

Note that $I$ is homogeneous. Set $R = A/I$ and let $a_0, a_1, a_2b_0, b_1, b_2, b_3$ coincide with their images in $R$ for simplicity. Then by [18, Example 1.10], $R$ is IFP and hence it is right APIP.

Now we show that $R[x]$ is not right APIP.

`Claim 1.` For any $n \geq 1$, $(a_1b_1)(b_0 + b_1x + b_2x^2 + b_3x^3)^n = a_1b_1^{n+1}x^n + \cdots + a_1b_1b_2^nx^{2n}$.

**Proof** By the construction of $R$,

$$\begin{aligned} a_1b_1(b_0 + b_1x + b_2x^2 + b_3x^3)^n &= a_1b_1(b_1x + b_2x^2)^n \\ &= a_1b_1(b_1^nx^n + \cdots + b_2^nx^{2n}) \\ &= a_1b_1^{n+1}x^n + \cdots + a_1b_1b_2^nx^{2n} \end{aligned}$$

for any $n \geq 1$. □

`Claim 2.` $R[x]$ is not right APIP.

**Proof** Consider $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$, $g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 \in R[x]$. Then $f(x)g(x) = 0$ by the construction of $R$. Assume that $R[x]$ is right APIP. Then there exists $k \geq 1$ such that $f(x)Rg(x)^k = 0$. Then $f(x)a_1b_1g(x)^k = 0$ for $a_1b_1 \in R$. But, by `Claim 1` and the construction of $R$,

$$\begin{aligned} f(x)a_1b_1g(x)^k &= (a_0 + a_1x + a_2x^2 + a_3x^3)a_1b_1(b_0 + b_1x + b_2x^2 + b_3x^3)^k \\ &= (a_1x + a_2x^2)a_1b_1(b_1x + b_2x^2)^k \\ &= (a_1x + a_2x^2)(a_1b_1^{n+1}x^n + \cdots + a_1b_1b_2^nx^{2n}) \\ &= a_1^2b_1^{k+1}x^{k+1} + \cdots + a_2a_1b_1b_2^kx^{2k+2} \neq 0, \end{aligned}$$

a contradiction. Thus $R[x]$ is not right APIP. □

**Remark 4.5** Example 4.4 illuminates that it is a counterexample of Question (2) in [4, p. 539], i.e. $R[x]$ is not $\pi$-IFP even if $R$ is an IFP ring. In fact, for $k \geq 2$, $(a_0 + a_1x + a_2x^2 + a_3x^3)^ka_1 = (a_1x + a_2x^2)^ka_1 = (a_1^kx + \cdots + a_2^kx^{2k})a_1$ by the construction of $R$. Thus $f(x)g(x) = 0$ for $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$, $g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 \in R[x]$, but

$$\begin{aligned} f(x)^ka_1b_1g(x)^k &= (a_0 + a_1x + a_2x^2 + a_3x^3)^ka_1b_1(b_0 + b_1x + b_2x^2 + b_3x^3)^k \\ &= (a_1x + a_2x^2)^ka_1b_1(b_1x + b_2x^2)^k \\ &= (a_1^kx + \cdots + a_2^kx^{2k})(a_1b_1^{k+1}x^k + \cdots + a_1b_1b_2^kx^{2k}] \\ &= a_1^{k+1}b_1^{k+1}x^{k+1} + \cdots + a_2^ka_1b_1b_2^kx^{4k} \neq 0, \end{aligned}$$

showing that $f(x)^kR[x]g(x)^k \neq 0$. Thus $R[x]$ is not $\pi$-IFP, either.

Notice that this also shows that the right APIP ring property cannot go up to formal power series rings by help of Proposition 3.11(2).

For an algebra $R$ over a commutative ring $S$, the Dorroh extension of $R$ by $S$ is the Abelian group $D = R \oplus S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$, where $r_i \in R$ and $s_i \in S$.

**Theorem 4.6** *Let $R$ be an algebra over a commutative domain $S$. Then $R$ is right APIP if and only if the Dorroh extension $D$ of $R$ by $S$ is right APIP.*

**Proof** Note that $s \in S$ is identified with $s1 \in R$ and so we let $R = \{r + s \mid (r, s) \in D\}$.

Suppose that $R$ is right APIP. Let $\alpha\beta = 0$, where $\alpha = (r_1, s_1), \beta = (r_2, s_2) \in D$. Then $\alpha\beta = 0$ implies $r_1r_2 + s_1r_2 + s_2r_1 = 0$ and $s_1s_2 = 0$. Since $S$ is a domain, $s_1 = 0$ or $s_2 = 0$.

If $s_1 = 0$, then $0 = r_1r_2 + s_2r_1 = r_1(r_2 + s_2)$ and it implies $r_1R(r_2 + s_2)^n = 0$ for some $n \geq 1$, since $R$ is right APIP. Then $r_1(r + s)(r_2 + s_2)^n = 0$ is equivalent to $(r_1, 0)(r, s)(r_2, s_2)^n = 0$ for any $r, s \in R$, and hence $\alpha D\beta^n = 0$.

Similarly, if $s_2 = 0$, then we have $0 = r_1r_2 + s_1r_2 = (r_1 + s_1)r_2$ and it implies $(r_1 + s_1)Rr_2^n = 0$ for some $n \geq 1$ by assumption. Hence $\alpha D\beta^n = 0$.

Consequently $D$ is right APIP. The converse is clear by Proposition 3.11(2). □

**Corollary 4.7** *If $N$ be a nil algebra over a commutative domain $S$, then $R = S + N$ is right APIP.*

**Proof** It follows from Theorem 4.6(1), since $R$ is the Dorroh extension of $N$ by $F$. □

**Proposition 4.8** *Let $R$ be a ring and $\Delta$ be a multiplicatively closed subset of $R$ consisting of central regular elements. Then $R$ is a right APIP ring if and only if $\Delta^{-1}R$ is right APIP.*

**Proof** It is enough to show the necessity by Proposition 3.11(2). Assume that $R$ is right APIP and let $\alpha\beta = 0$ for $\alpha = u^{-1}a, \beta = v^{-1}b \in \Delta^{-1}R$. Then, from $\alpha\beta = 0$, we have $ab = 0$ and it follows by assumption that $aRb^n = 0$ for some $n \geq 1$. Hence, for any $w^{-1}r \in \Delta^{-1}R$, we obtain that

$$0 = u^{-1}w^{-1}(v^{-1})^n arb^n = u^{-1}aw^{-1}r(v^{-1})^n b^n = \alpha w^{-1}r\beta^n,$$

from which we see $\alpha(\Delta^{-1}R)\beta^n = 0$. Therefore $\Delta^{-1}R$ is right APIP. □

**Corollary 4.9** *For a ring $R$, $R[x]$ is right APIP if and only if the ring $R[x; x^{-1}]$ of Laurent polynomials in $x$ is right APIP.*

**Proof** It follows directly from Proposition 4.8. For, letting $\Delta = \{1, x, x^2, \ldots\}$, we have that $\Delta$ is a multiplicatively closed subset of $R[x]$ consisting of central regular elements and $R[x; x^{-1}] = \Delta^{-1}R[x]$. □

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