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On central polynomials and codimension growth

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Abstract: Let A be an associative algebra over a field of characteristic zero. A central polynomial is a polynomial of the free associative algebra that takes central values of A. In this survey, we present some recent results about the exponential growth of the central codimension sequence and the proper central codimension sequence in the setting of algebras with involution and algebras graded by a finite group.

Key words: Polynomial identity, central polynomials, exponent, codimension growth

1. Introduction

Let A be an associative algebra over a field F of characteristic zero and let $F\langle X\rangle$ be the free associative algebra freely generated over F by the countable set X of variables. A polynomial $f(x_1,\dots,x_n)\in F\langle X\rangle$ is called central polynomial of A if for all $a_1,\dots,a_n\in A$, $f(a_1,\dots,a_n)\in Z(A)$, the center of A. If f takes only the zero value, then it is called polynomial identity, and A is called PI-algebra; otherwise, f is said to be a proper central polynomial.

The interest around central polynomials arose after a conjecture by Kaplansky (see [21]) asserting that the algebra of $n\times n$ matrices $M_n(F)$ has proper central polynomials. Such a conjecture was proved independently by Formanek in [13] and Razmyslov in [24]. In general, the existence of central proper polynomials is not guaranteed even if the algebra has a nontrivial center; for instance, in [19, Lemma 1], it was proved that the block upper triangular matrix algebra has no proper central polynomials.

Concerning matrix algebras, another interesting line of research studies the minimal degree of proper central polynomials of $M_n(F)$. In analogy with the ordinary case, where a celebrated theorem by Amitsur and Levitzki state that the minimal degree for a standard polynomial being an identity of $M_n(F)$ is $2n$, and there are no other identities of less degree; in [10–12] it was investigated the degree of proper central polynomials. In particular, if d_n is such a minimal degree, then in that papers the authors constructed proper central polynomials of $M_n(F)$ such that $d_n=\frac{n^2+3n-2}{2}$ for $n=2,3$ or 4 and $d_n\geq\frac{n^2+3n-2}{2}$ for all $n\geq 5$, strengthening a conjecture by Formanek stating that d_n equals $\frac{n^2+3n-2}{2}$ for all $n\geq 2$.

More generally, if A is an F-algebra, then it is very hard to determine the T-space of central polynomials of A (see for instance [5, 6]). Thus, motivated by an idea of Regev (see [25]), here we are interested in a quantitative approach that may lead us to understand at least how many central polynomials a given algebra

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has.

The idea is the following. Let us consider the space of multilinear polynomials P_n and let us attach to it three numerical sequences: the codimension sequence $c_n(A)$, as the dimension of P_n modulo the polynomial identities of A; the central codimension sequence $c_n^z(A)$, as the dimension of P_n modulo the central polynomials of A; the proper central codimension sequence $\delta_n(A)$, as the dimension of the central polynomials modulo the identities of A. It is clear that the asymptotic behavior of $c_n^z(A)$ and $\delta_n(A)$ give information about the number of central polynomials and proper central polynomials of A,, respectively. Moreover, for all $n \geq 1$

$$c_n(A) = c_n^z(A) + \delta_n(A).$$

In this survey, we present the outcomes of [22, 23] where the exponential rates of the growth of the central and the proper central codimension sequences were captured, in the setting of algebras with involution and group graded algebras. Such results extend the ones of [19, 20] concerning central polynomials for ordinary algebras.

2. A general setting

Throughout the paper, F will denote a field of characteristic zero. In what follows, we firstly introduce the basic definitions and settings for PI-algebras with involution, secondly we will do the same for algebras graded by a finite abelian group G.

Let A be an associative F-algebra with involution $*$, i.e., a linear map $* : A \to A$ such that $*^2 = *$ and $(ab)^* = b^*a^*$, for all $a, b \in A$. We write $A = A^+ \oplus A^-$, where $A^+ = \{a \in A \mid a^* = a\}$ and $A^- = \{a \in A \mid a^* = -a\}$ denote the sets of symmetric and skew elements of A, respectively.

Let $F\langle X, *\rangle = F\langle x_1, x_1^*, x_2, x_2^*, \dots\rangle$ be the free associative algebra with involution on a countable set $X = \{x_1, x_2, \dots\}$ over F. It is useful to regard $F\langle X, *\rangle$ as generated by the symmetric variables and by the skew variables , i.e., $F\langle X, *\rangle = F\langle x_1^+, x_1^-, x_2^+, x_2^-, \dots\rangle$, where $x_i^+ = x_i + x_i^*$, $x_i^- = x_i - x_i^*$, $i \geq 1$. Recall that a polynomial $f(x_1^+, \dots, x_r^+, x_1^-, \dots, x_s^-) \in F\langle X, *\rangle$ is a $*$-polynomial identity of A (or simply a $*$-identity), and we write $f \equiv 0$, if $f(a_1^+, \dots, a_r^+, a_1^-, \dots, a_s^-) = 0$ for all $a_1^+, \dots, a_r^+ \in A^+$, $a_1^-, \dots, a_s^- \in A^-$. The set $\mathrm{Id}^*(A)$ of all $*$-identities of A is a T_*-ideal of $F\langle X, *\rangle$, i.e., an ideal invariant under all endomorphisms of the free algebra commuting with the involution $*$.

Let us denote by P_n^* the space of multilinear polynomials of degree n in $x_1^+, x_1^-, \dots, x_n^+, x_n^-$, i.e.,

$$P_n^* = \mathrm{span}_F\Big\{w_{\sigma(1)} \cdots w_{\sigma(n)} \mid \sigma \in S_n, \quad w_i \in \left\{x_i^+, x_i^-\right\},\ i = 1, \dots, n\Big\};$$

Hence, for every $1 \leq i \leq n$ either x_i^+ or x_i^- appears in every monomial of P_n^* at degree 1 (but not both). Since, in characteristic zero, every $*$-identity is equivalent to a system of multilinear $*$-identities, the study of $\mathrm{Id}^*(A)$ is equivalent to the study of $P_n^* \cap \mathrm{Id}^*(A)$, for all $n \geq 1$. Thus, we construct the quotient space

$$P_n^*(A) = \frac{P_n^*}{P_n^* \cap \mathrm{Id}^*(A)}$$

and we call its dimension, $c_n^*(A) = \dim P_n^*(A)$, $n \geq 1$ the nth $*$-codimension of A.

In [16], it was proved that for a $*$-algebra A satisfying a nontrivial polynomial identity, $c_n^*(A)$ is exponentially bounded, and the limit

$$\exp^*(A) = \lim_{n \to +\infty} \sqrt[n]{c_n^*(A)}$$

exists, and it is a nonnegative integer called the PI $*$-exponent of A. The proof of such result uses a powerful tool called the Grassmann envelope described as follows.

First, we recall that an algebra B is a superalgebra if B is graded by $\mathbb{Z}_2$, the cyclic group of order 2, i.e., $B = B_0 \oplus B_1$, as vector space, such that $B_0B_0 + B_1B_1 \subseteq B_0$ and $B_0B_1 + B_1B_0 \subseteq B_1$. The elements of B_0 and B_1 are called homogeneous elements of degree 0 and 1, respectively.

Moreover, let E denote the infinite Grassmann algebra generated by the elements $1, e_1, e_2, \dots$ subject to the relations $e_ie_j = -e_je_i$, for all i, j. Let $E = E_0 \oplus E_1$ be its natural $\mathbb{Z}_2$-grading, where

$$E_0 = \mathrm{span}_F\big\{e_{i_1} \dots e_{i_{2k}} \mid 1 \leq i_1 < \dots < i_{2k},\ k \geq 0\big\}$$

and

$$E_1 = \mathrm{span}_F\big\{e_{i_1} \dots e_{i_{2k+1}} \mid 1 \leq i_1 < \dots < i_{2k+1},\ k \geq 0\big\}.$$

If $B = B_0 \oplus B_1$ is a superalgebra, then the algebra

$$E(B) = E_0 \otimes B_0 \oplus E_1 \otimes B_1$$

is called the Grassmann envelope of B.

Now recall that a superinvolution on a superalgebra B is a graded linear map $\sharp : B \to B$ such that $(a^\sharp)^\sharp = a$ for all $a \in B$ and $(ab)^\sharp = (-1)^{|a||b|}b^\sharp a^\sharp$, for every $a, b \in B_0 \cup B_1$ of homogeneous degree $|a|$ and $|b|$, respectively. Since $\mathrm{char} F = 0$, we can write $B = B_0^+ \oplus B_0^- \oplus B_1^+ \oplus B_1^-$, where for $i = 0, 1$, $B_i^+ = \{a \in B_i | a^\sharp = a\}$ and $B_i^- = \{a \in B_i | a^\sharp = -a\}$ denote the sets of homogeneous symmetric and skew elements of B_i, respectively. Notice that the Grassmann envelope $E(B)$ can be regarded as an algebra with involution $* : E(B) \to E(B)$ such that $(a \otimes g)^* = a^\sharp \otimes g^\star$, where $\star : E \to E$ is the superinvolution defined by $e_i^\star = -e_i$, for $i \geq 1$.

In [4], it was proved the following theorem.

Theorem 2.1 ([4], Theorem 4) *If A is an algebra with involution satisfying a nontrivial $*$-identity, then there exists a finite dimensional superalgebra with superinvolution B such that $Id^*(A) = Id^*(E(B))$.*

Since there is a one-to-one correspondence between T_*-ideals and $*$-varieties, it is convenient to translate all the objects we have defined above in the language of $*$-varieties. Recall that, roughly speaking, a $*$-variety is the class of all $*$-algebras sharing the same T_*-ideal of identities. Thus, if $\mathcal{V} = \mathrm{var}^*(A)$ is the $*$-variety generated by A,, then we set $\mathrm{Id}^*(\mathcal{V}) = \mathrm{Id}^*(A)$, $c_n^*(\mathcal{V}) = c_n^*(A)$ and so on.

Now, we focus our attention to group graded algebras. To this end, let G be a finite abelian group and let A be a G-graded algebra, i.e., $A = \bigoplus_{g \in G} A_g$, where the A_g's are subspaces of A with the property that $A_gA_h \subseteq A_{gh}$ for all $g, h \in G$. We say that the element a has homogeneous degree g if $a \in A_g$.

As we did for $*$-algebras, we have to introduce now the analogous definitions and objects in the graded case.

By taking into account the free algebra $F\langle X\rangle$, one can define on such an algebra a G-grading in a natural way: write $X = \bigcup_{g \in G} X_g$, where $X_g = \{x_{1,g}, x_{2,g}, \dots\}$ are disjoint sets and the elements of X_g have homogeneous degree g. If we denote by $\mathcal{F}_g$ the subspace of $F\langle X\rangle$ spanned by all monomials in the variables of X having homogeneous degree g,, then $F\langle X\rangle = \bigoplus_{g \in G} \mathcal{F}_g$ is a G-graded algebra called the free associative G-graded algebra of countable rank over F. We shall denote it by $F\langle X, G\rangle$.

From now on, let $G = \{g_1, \dots, g_s\}$. A G-graded polynomial or simply a G-polynomial.

$$f = f(x_{1,g_1}, \dots, x_{t_1,g_1}, \dots, x_{1,g_s}, \dots, x_{t_s,g_s})$$

of $F\langle X, G\rangle$ is a G-graded identity (or simply graded identity) of A, and we write $f \equiv 0$, if

$$f(a_{1,g_1}, \dots, a_{t_1,g_1}, \dots, a_{1,g_s}, \dots, a_{t_s,g_s}) = 0$$

for all $a_{i,g_i} \in A_{g_i}$, $t_i \geq 0$, for all $1 \leq i \leq s$.

Let $\mathrm{Id}^G(A) = \{f \in F\langle X, G\rangle \mid f \equiv 0 \text{ on } A\}$ be the ideal of graded identities of A. It is easily seen that $\mathrm{Id}^G(A)$ is a T_G-ideal, i.e., it is an ideal invariant under all graded endomorphisms of $F\langle X, G\rangle$.

Notice that if for some $i \geq 1$ we set $x_i = x_{i,g_1} + \cdots + x_{i,g_s}$, then $F\langle X\rangle$ is naturally embedded into $F\langle X, G\rangle$ so that we can look at the (ordinary) identities of A as a special kind of graded identities.

Since $\mathrm{char} F = 0$, then $\mathrm{Id}^G(A)$ is determined by the multilinear G-polynomials it contains. Thus, for all $n \geq 1$,, one can define

$$P_n^G = \mathrm{span}_F\{x_{\sigma(1),g_{i_1}} \cdots x_{\sigma(n),g_{i_n}} \mid \sigma \in S_n, g_{i_1}, \dots, g_{i_n} \in G\}$$

as the space of multilinear G-polynomials in the graded variables $x_{1,g_{i_1}}, \dots, x_{n,g_{i_n}}$, $g_{i_j} \in G$. The T_G-ideal $\mathrm{Id}^G(A)$ is determined by the sequence of subspaces $P_n^G \cap \mathrm{Id}^G(A)$, $n \geq 1$, and we can construct the quotient space

$$P_n^G(A) = \frac{P_n^G}{P_n^G \cap \mathrm{Id}^G(A)}.$$

The nonnegative integer

$$c_n^G(A) = \dim_F P_n^G(A),\ n \geq 1,$$

is called the nth codimension of A. In [17], it was proved that if A satisfies a non-trivial ordinary polynomial identity, then such a sequence is exponentially bounded. Moreover, in [2, 3, 15] the authors captured this exponential growth by proving the existence, and the integrability of the limit

$$\exp^G(A) = \lim_{n\to+\infty} \sqrt[n]{c_n^G(A)},$$

called the G-exponent of A. We highlight that such a result was achieved for any finite group.

As in the case of $*$-algebras, one can define the Grassmann envelope for graded algebras. In particular, let $B = \bigoplus_{(g,i)\in G\times\mathbb{Z}_2} B_{(g,i)}$ be a $G \times \mathbb{Z}_2$-graded algebra, then B has an induced $\mathbb{Z}_2$-grading, $B = B_0 \oplus B_1$, where $B_0 = \bigoplus_{g\in G} B_{(g,0)}$ and $B_1 = \bigoplus_{g\in G} B_{(g,1)}$.

Therefore the Grassmann envelope of B

$$E(B) = B_0 \otimes E_0 \oplus B_1 \otimes E_1$$

has an induced G-grading by setting $E(B)_g = (B_{(g,0)} \otimes E_0) \oplus (B_{(g,1)} \otimes E_1)$ for all $g \in G$.

Moreover, we have the following theorem.

Theorem 2.2 (**[1, 26]**) *Let A be a G-graded algebra satisfying a non-trivial ordinary polynomial identity. Then, there exists a finite dimensional $G \times \mathbb{Z}_2$-graded algebra B such that $Id^G(A) = Id^G(E(B))$.*

3. On central polynomials

This section is devoted to the definition of the main object of the survey.

Let A be an algebra with involution $*$ and let $f(x_1^+,\dots,x_r^+,x_1^-,\dots,x_s^-) \in F\langle X,*\rangle$. Then, f is a central $*$-polynomial of A if $f(a_1^+,\dots,a_r^+,a_1^-,\dots,a_s^-) \in Z(A)$ for all $a_1^+,\dots,a_r^+ \in A^+$ and $a_1^-,\dots,a_s^- \in A^-$.

It is clear that all the $*$-polynomial identities of A are in particular central $*$-polynomials; on the other hand, if a central $*$-polynomial takes at least a non-zero value in $Z(A)$, then it is called proper central $*$-polynomial.

Let $\mathrm{Id}^{*,z}(A)$ denote the set of central $*$-polynomials of A. Notice that $\mathrm{Id}^{*,z}(A)$ is a T-space of $F\langle X,*\rangle$, i.e., a vector space invariant under all endomorphisms of the free algebra commuting with the involution $*$.

If we set

$$P_n^{*,z}(A) = \frac{P_n^*}{P_n^* \cap \mathrm{Id}^{*,z}(A)},$$

, then the quotient space

$$\Delta_n^*(A) = \frac{P_n^* \cap \mathrm{Id}^{*,z}(A)}{P_n^* \cap \mathrm{Id}^*(A)}$$

corresponds to the space of proper central $*$-polynomials in n fixed variables, and we call $c_n^{*,z}(A) = \dim_F P_n^{*,z}(A)$ and $\delta_n^*(A) = \dim_F \Delta_n^*(A)$, $n = 1,2,\dots$ the sequences of central $*$-codimensions and proper central $*$-codimensions of A, respectively.

As we recalled in the Introduction, such two sequences are connected to the sequence of $*$-codimensions by the relation

$$c_n^*(A) = c_n^{*,z}(A) + \delta_n^*(A),$$

for all $n \geq 1$. Thus, the main interest in central polynomials and their codimension sequences lies in the fact that one can gather information about $c_n^*(A)$ by knowing how $c_n^{*,z}(A)$ and $\delta_n^*(A)$ behave and vice-versa. Since $c_n^*(A)$ is exponentially bounded, provided that A is a PI-algebra, then also $c_n^{*,z}(A)$ and $\delta_n^*(A)$ are exponentially bounded and one can define

$$\exp^{*,z}(A) = \lim_{n\to+\infty} \sqrt[n]{c_n^{*,z}(A)}$$

$$\exp^{*,\delta}(A) = \lim_{n\to+\infty} \sqrt[n]{\delta_n^*(A)}$$

as the central $*$-exponent and the proper central $*$-exponent, respectively. In the next section we present some results proved in [23] concerning such two limits.

As we did for $*$-algebras, we can consider the analogous definitions in the setting of G-graded algebras. Thus, $\mathrm{Id}^{G,z}(A)$ denotes the T-space of central G-polynomials and $P_n^{G,z}(A)$ and $\Delta_n^G(A)$ are the space of multilinear G-polynomials reduced modulo the graded identities of A and the space of proper central G-polynomials, respectively. We define the central G-codimension sequence $c_n^{G,z}(A)$ and the proper central G-codimension sequence $\delta_n^G(A)$, and we study the two limits

$$\exp^{G,z}(A) = \lim_{n\to+\infty} \sqrt[n]{c_n^{G,z}(A)}$$

$$\exp^{G,\delta}(A) = \lim_{n\to+\infty} \sqrt[n]{\delta_n^G(A)},$$

called the central G-exponent and the proper central G-exponent, respectively. The last section of the paper is devoted to such limits where, according to [22], their existence and integrability are established.

4. Algebras with involution

Let A be a PI-algebra with involution $*$ over a field of characteristic zero. Then, according to Theorem 2.1, there exists a finite dimensional superalgebra $B = B_0 \oplus B_1$ with superinvolution such that $\mathrm{Id}^*(A) = \mathrm{Id}^*(E(B))$.

Notice that, since the codimension sequences do not change by extension of the ground field, we may assume that F is algebraically closed. Hence, B has a Wedderburn–Malcev decomposition, i.e.

$$B = B_1 \oplus \cdots \oplus B_m + J,$$

where $B_1, \ldots, B_m$ are simple superalgebras with superinvolution and $J = J(B)$ is the Jacobson radical of B.

We give the following definitions.

Definition 4.1 *The superalgebra B is called reduced if for some permutation $(i_1, \ldots, i_m)$ of $(1, \ldots, m)$, $B_{i_1} J B_{i_2} J \cdots J B_{i_m} \neq 0$.*

Definition 4.2 *A semisimple subalgebra $B' = B_{i_1} \oplus \cdots \oplus B_{i_k}$ of B, where $i_1, \ldots i_k \in \{1, \ldots, m\}$ are distinct, is centrally admissible in $E(B)$ if there exists a multilinear proper central $*$-polynomial $f(x_1^+, \ldots, x_r^+, x_1^-, \ldots, x_s^-)$ of $E(B)$ with $r + s \geq k$ such that*

$$f(a_1^+, \ldots, a_{k_1}^+, b_1^+, \ldots, b_{r-k_1}^+, a_1^-, \ldots, a_{k_2}^-, b_1^-, \ldots, b_{s-k_2}^-) \neq 0$$

for some $a_1^+ \in E(B_{i_1})^+, \ldots, a_{k_1}^+ \in E(B_{i_{k_1}})^+, a_1^- \in E(B_{i_{k_1+1}})^-, \ldots, a_{k_2}^- \in E(B_{i_k})^-$, $b_1^+, \ldots, b_{r-k_1}^+ \in E(B)^+$, $b_1^-, \ldots b_{s-k_2}^- \in E(B)^-$, $k_1 + k_2 = k$.

Remark that if the semisimple subalgebra B' of B is centrally admissible in $E(B)$ of maximal dimension, then $\widehat{B} = B' + J$ is reduced. In fact, without loss of generality let us assume that $B' = B_1 \oplus \cdots \oplus B_k$ and let $f = f(x_1^+, \ldots, x_r^+, x_1^-, \ldots, x_s^-)$ be a proper multilinear central polynomial as in the previous definition. Since $E(B_i)E(B_j) = 0$ for any $i \neq j$, then $E(B_{i_1})E(J)E(B_{i_2})E(J)\ldots E(J)E(B_{i_m}) \neq 0$ for some permutation $(i_1, \ldots, i_k)$ of $(1, \ldots, k)$. Thus, it follows that $B_{i_1} J B_{i_2} J \ldots J B_{i_k} \neq 0$, i.e., the superalgebra with superinvolution $\widehat{B} = B' + J$ is reduced.

The importance of centrally admissible subalgebras is highlighted by the following theorem.

Theorem 4.3 ([23], Theorem 4) *Let A be a $*$-algebra and let B be a finite dimensional superalgebra with superinvolution such that $Id^*(A) = Id^*(E(B))$. If there exist centrally admissible subalgebras in $E(B)$, then for n large enough,*

$$C_1 n^{a_1} d^n \leq \delta_n^*(A) \leq C_2 n^{a_2} d^n,$$

for some constants $C_1 > 0$, C_2, a_1, a_2, where d is the maximal dimension of a centrally admissible subalgebra in $E(B)$.

In case $E(B)$ has proper central $*$-polynomials but has not centrally admissible subalgebras, then the following Proposition holds.

Proposition 4.4 ([23], Proposition 1) *If $E(B)$ has proper central $*$-polynomials but has no centrally admissible subalgebras, then $\delta_n^*(E(B)) = 0$, for n large enough.*

By putting together the previous results, we get the claim about the proper central $*$-exponent.

Corollary 4.5 ([23], Corollary 1) *If A is a $*$-algebra over a field of characteristic zero, then the proper central $*$-exponent $\exp^{*,\delta}(A)$ exists and is a nonnegative integer. Moreover, $\exp^{*,\delta}(A) \leq \exp^*(A)$.*

Concerning the central $*$-exponent, it turns out that its computation is easier than the one of the proper central $*$-exponent. It is worth mentioning that a crucial role in such computation is played by the so-called minimal $*$-varieties of exponential growth of the codimensions, i.e., a $*$-variety $\mathcal{V}$ such that $c_n^*(\mathcal{V}) \approx d^n$ and for any proper subvariety $\mathcal{U} \subsetneq \mathcal{V}$, $c_n^*(\mathcal{U}) \approx t^n$ with $t < d$. Minimal varieties were completely described in [8, 9, 18], in the setting of ordinary polynomial identities, identities with involution, and identities with graded involution. Moreover, in [7], the authors studied minimal varieties for algebras graded by the cyclic group $\mathbb{Z}_p$.

In [23, Section 6] some results are presented in the setting of superalgebras with superinvolution. For instance, on one hand, it was proved that if B is not simple as superalgebra with superinvolution and generates a minimal variety, then $E(B)$ has no proper central $*$-polynomials (see [23, Lemma 10]); on the other hand, any $*$-variety contains a minimal $*$-variety of the same $*$-exponent. Based on such statements, the following theorem can be proved.

Theorem 4.6 ([23], Theorem 6) *Let A be a $*$-algebra over a field of characteristic zero such that $\exp^*(A) \geq 2$. Then, either $c_n^{*,z}(A) = 0$ for all $n \geq 0$ or*

$$C_1 n^{t_1} \exp^*(A)^n \leq c_n^{*,z}(A) \leq C_2 n^{t_2} \exp^*(A)^n,$$

for some constants $C_1 > 0$, C_2, t_1, t_2.

It turns out that the central $*$-exponent exists, and we are able to compare it with $\exp^*(A)$.

Corollary 4.7 ([23], Theorem 7) *Let A be any $*$-algebra over a field of characteristic zero, then its central exponent $\exp^{*,z}(A)$ exists. Moreover, if $\exp^*(A) \geq 3$, then $\exp^{*,z}(A) = \exp^*(A)$. When $\exp^*(A) \leq 2$, then $\exp^{*,z}(A) = \exp^*(A)$ or 0.*

5. Group graded algebras

Let G be a finite abelian group and let A be a G-graded algebra, i.e., $A = \bigoplus_{g \in G} A_g$, where the A_g's are vector subspaces such that $A_g A_h \subseteq A_{gh}$ for all $g, h \in G$. Theorem 2.2 ensures us that there exists a finite dimensional $G \times \mathbb{Z}_2$-graded algebra B such that $\mathrm{Id}^G(A) = \mathrm{Id}^G(E(B))$.

As in the involution case, we assume, as we may, that F is algebraically closed, and we write

$$B = B_1 \oplus \cdots \oplus B_m + J$$

where $B_1, \ldots, B_m$ are $G \times \mathbb{Z}_2$-graded simple algebras and $J = J(B)$ is the Jacobson radical of B. We now give the definition of centrally admissible subalgebra in case of G-graded algebras.

Definition 5.1 *A semisimple $G \times \mathbb{Z}_2$-graded subalgebra $B' = B_{i_1} \oplus \cdots \oplus B_{i_k}$, where $i_1, \dots, i_k \in \{1, \dots, m\}$ are all distinct, is called centrally admissible; in $E(B)$, there exists a proper central G-polynomial*

$$f(x_{1,g_{j_1}}, \dots, x_{r,g_{j_r}})$$

of $E(B)$ such that $r \geq k$ and $f(a_1, \dots, a_r) \neq 0$, for some homogeneous elements $a_1 \in E(B_{i_1})_{g_{j_1}}, \dots, a_k \in E(B_{i_k})_{g_{j_k}}$, $a_{k+1} \in E(B)_{g_{j_{k+1}}}, \dots, a_r \in E(B)_{g_{j_r}}$.

Notice that if B' is centrally admissible in $E(B)$ of maximal dimension, then $\widehat{B} = B' + J$ is reduced.

The following theorem proves the existence and the integrability of the proper central G-exponent.

Theorem 5.2 ([22], Theorem 3) *Let A be a G-graded algebra over a field F of characteristic zero and let B a finite dimensional $G \times \mathbb{Z}_2$-graded algebra such that $Id^G(A) = Id^G(E(B))$. If there exists a centrally admissible $G \times \mathbb{Z}_2$-subalgebra in $E(B)$, then, for n large enough, there exist constants $C_1, > 0$ C_2, a_1, a_2 such that*

$$C_1 n^{a_1} d^n \leq \delta_n^G(A) \leq C_2 n^{a_2} d^n,$$

where d is the maximal dimension of a centrally admissible $G \times \mathbb{Z}_2$-subalgebra in $E(B)$. Thus, the proper central G-exponent $\exp^{G,\delta}(A)$ exists and is a nonnegative integer.

It is worth mentioning that, unlike the involution case, the upper bound of the proper central G-codimension sequence was proved by using pure combinatorial methods. Such a technique takes into account the generic elements of an algebra and was firstly introduced by Procesi. On the other hand, the lower bound is built up through the representation theory of symmetric groups, and its computation requires quite a few technical results involving hook-shaped partitions of integers and their connection with cocharacters of G-graded algebras and the dimensions of some suitable irreducible modules. In general, the lower bound of the codimension sequence of PI-algebras in a given setting, such as algebras with involution, superinvolution, with derivations and so on, was always obtained employing the representation theory of groups. The only example of lower bound computed with combinatorial methods concerns algebras with trace, and it can be found in [14].

In analogy with the involution case, Proposition 2 of [22] states that if there are no centrally admissible subalgebras in $E(B)$, then $\delta_n^G(A) = 0$, for n large enough. Therefore,

Corollary 5.3 ([22], Corollary 1) *If A is a G-graded algebra over a field of characteristic zero, then the proper central G-exponent $\exp^{G,\delta}(A)$ exists and is a nonnegative integer. Moreover, $\exp^{G,\delta}(A) \leq \exp^G(A)$.*

By means of minimal varieties of G-graded algebras, the result about the central G-exponent is achieved.

Theorem 5.4 ([22], Theorem 5) *Let A be a G-graded algebra over a field of characteristic zero such that $\exp^G(A) \geq 2$. Then, either $c_n^{G,z}(A) = 0$ for all $n \geq 0$, or*

$$C_1 n^{t_1} \exp^G(A)^n \leq c_n^{G,z}(A) \leq C_2 n^{t_2} \exp^G(A)^n$$

for some constants $C_1 > 0$, C_2, t_1, t_2.

Corollary 5.5 ([22], Theorem 6) *Let A be any G-graded algebra over a field of characteristic zero. Then, its central G-exponent $\exp^{G,z}(A)$ exists. Moreover, either $\exp^{G,z}(A) = \exp^G(A)$ or $\exp^{G,z}(A) = 0$.*

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