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On the restricted graded Jacobson radical of rings of Morita context

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Abstract: The class of rings $\mathcal{J} = \{A | (A, \circ) \text{ forms a group}\}$ forms a radical class and it is called the Jacobson radical class. For any ring A, the Jacobson radical $\mathcal{J}(A)$ of A is defined as the largest ideal of A which belongs to $\mathcal{J}$. In fact, the Jacobson radical is one of the most important radical classes since it is used widely in another branch of abstract algebra, for example, to construct a two-sided brace. On the other hand, for every ring of Morita context $T = \begin{pmatrix} R & V \\ W & S \end{pmatrix}$, we will show directly by the structure of the Jacobson radical of rings that the Jacobson radical $\mathcal{J}(T) = \begin{pmatrix} \mathcal{J}(R) & V_0 \\ W_0 & \mathcal{J}(S) \end{pmatrix}$, where $\mathcal{J}(R)$ and $\mathcal{J}(S)$ are the Jacobson radicals of R and S, respectively, $V_0 = \{v \in V | vW \subseteq \mathcal{J}(R)\}$ and $W_0 = \{w \in W | wV \subseteq \mathcal{J}(S)\}$. This clearly shows that the Jacobson radical is an $N-$radical. Furthermore, we show that this property is also valid for the restricted $G-$graded Jacobson radical of graded ring of Morita context.

Key words: Jacobson radical, graded Jacobson radical, $N-$radical, Morita context

1. Introduction

In this paper, for any ring A, the notation $I \lhd A$ (respectively, $L \lhd_l A$) is used to denote I ideal of A (respectively, L left ideal of A). Moreover, we consider the definition of radical class of rings in the sense of Kurosh and Amitsur. Let γ be a class of rings. The class γ is called a radical class if it satisfies the following conditions:

1. the class γ contains all homomorphic images of a ring in γ,
2. for every ring A, $\gamma(A) \in \gamma$ where $\gamma(A) = \Sigma(I \lhd A | I \in \gamma)$, and
3. $\gamma(A/\gamma(A)) = 0$ for every ring A.

The ideal $\gamma(A)$ of A is called the $\gamma-$radical of A. A ring A is called a $\gamma-$ring if $A \in \gamma$, that is, $\gamma(A) = A$. Some conditions which are equivalent to the definition of radical class of rings can be found in [3].

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Define a class of rings $\mathcal{J} = \{A \text{ is a ring} | (A, \circ) \text{ is a group}\}$, where $a \circ b = a + b - ab$ for every $a, b \in A$. The class $\mathcal{J}$ of rings forms a radical class. This radical class is called the Jacobson radical class. The detail of further properties of the Jacobson radical class can be found in [3]. The Jacobson radical class is one of the most important and one of the most famous radical classes since the Jacobson radical class can be widely used in another branch of abstract algebra, for example, in a brace constructions in the determining the set-theoretic solution of Yang-Baxter equation [15]. The definition and concept of a brace can be found in [10]. For any ring A, the set $\mathcal{J}(A) = \sum\{I | I \text{ is an ideal of } A \text{ and } I \in \mathcal{J}\}$ is the largest ideal of A in $\mathcal{J}$. It follows from [14] that the Jacobson radical $\mathcal{J}(A)$ of a ring A is a two-sided brace.

On the other hand, we will use definition of graded ring in [2]. Let G be a group. A ring A is called $G-$graded ring if $A = \bigoplus_{j \in G} A_j$ and $A_j A_g \subseteq A_{gj} \forall j, g \in G$. Moreover, if A is a $G-$graded ring, A^u denotes the underlying ungraded ring [2]. If S is a subset the $G-$graded ring A and $H \subseteq G, J_H = \bigoplus\{I \cap A_h | h \in H\}$. Some further properties and implementation of graded rings can be accessed in [4–6, 9] and [13].

2. Properties of the Jacobson radical class

In this work, we start this section by following the definition of hypernilpotent radical, supernilpotent radical, left strong radical, left hereditary radical which are described in [3]. As a fundamental concept for this section we consider the prime radical β which is precisely the upper radical $\mathcal{U}(\pi)$ of the class π of all prime rings [3]. A radical class γ is called a hypernilpotent radical if γ contains the prime radical β. On the other hand, a radical class γ which is hypernilpotent and for every ring $A \in \gamma$, then every nonzero ideal I of A belongs to γ, is called a supernilpotent radical class. Furthermore, a radical class γ is a left strong radical if $L \in \gamma$ implies $L \subseteq \gamma(A)$ for every $L \trianglelefteq_l A$. Moreover, a radical class γ is a left hereditary radical class if $A \in \gamma$ implies $L \in \gamma$ for every $L \trianglelefteq_l A$. A radical class γ is called an $N-$radical where γ is supernilpotent, left strong, and left hereditary. Hence, it follows from the definition of $N-$radical that every $N-$radical is a supernilpotent radical. However the converse is not generally true because there are some examples of supernilpotent radical which are not $N-$radical. For example, the Brown-McCoy radical class $\mathcal{G}$ [3].

On the other hand, a ring A is called a zero-ring if $A^2 = \sum_{finite\,sum} ab = 0$ for every $a, b \in A$. It follows from Example 2.2.2 in [3] that the lower radical class $\mathcal{LZ}$ of the class $\mathcal{Z}$ of all zero-rings contains the prime radical β. We will use this notion to prove the following proposition.

Proposition 2.1 *[3] The Jacobson radical class $\mathcal{J}$ is a supernilpotent radical.*

Proof Let $A \in \mathcal{J}$. Then $(A, \circ)$ forms a group. Now suppose I to be a nonzero ideal of A. It is clear that $(I, \circ)$ also forms a group. Thus $I \in \mathcal{J}$ which implies $\mathcal{J}$ is hereditary. Let R be any zero-ring and let a be any member of R. Then $a \circ (-a) = a - a - (a(-a)) = 0$ which implies the zero-ring $R \in \mathcal{J}$. Hence, the Jacobson radical class $\mathcal{J}$ contains the class $\mathcal{Z}$ of all zero-rings. It follows from the construction of lower radical class $\mathcal{LZ}$ of the class $\mathcal{Z}$ of all zero-rings, the Jacobson radical class $\mathcal{J}$ contains the lower radical class $\mathcal{LZ}$ of the class $\mathcal{Z}$ of all zero-rings. Therefore, the Jacobson radical class $\mathcal{J}$ contains the prime radical β. Therefore, we can infer that the Jacobson radical class $\mathcal{J}$ is a supernilpotent radical. □

We will be explaining the further property of Jacobson radical related to a ring of Morita context. Hence, we start from the definition of a ring of Morita context.

Definition 2.2 *[3] A quadruple (R, A, B, S) is called a Morita context if the set*

$$T = \begin{pmatrix} R & A \\ B & S \end{pmatrix}$$

of 2×2 matrices forms a ring under matrix addition and multiplication where R and S are rings, A and B are $R-S-$ bimodule and $S-R-$ bimodule, respectively.

Some studies on ring of Morita context can be seen in [1, 11, 16] and [12]. Furthermore, the relationship between radical class of rings and the concept of the ring of Morita context starts from the following definition.

Definition 2.3 *[3] A radical class γ of rings is called a normal radical if every Morita context (R, A, B, S) satisfies $A\gamma(R)B \subseteq \gamma(S)$. Furthermore, a supernilpotent normal radical is called an $N-$radical.*

Some properties of $N-$radicals were presented in the work of Sands [11]. In Theorem 1 [11], Sands showed that if γ is an $N-$radical, then for every Morita context (R, A, B, S)

$$\gamma(T) = \gamma \begin{pmatrix} R & A \\ B & S \end{pmatrix} = \begin{pmatrix} \gamma(R) & A_0 \\ B_0 & \gamma(S) \end{pmatrix}$$

where $A_0 = \{a \in A | aB \subseteq \gamma(R)\}$ and $B_0 = \{b \in B | bA \subseteq \gamma(S)\}$. Furthermore, the converse of Theorem 1 [11] is also true as shown in the work of Jaegermann [7]. We rewrite the theorem as reminder.

Theorem 2.4 *[7] Let γ be a radical class of rings. The following conditions are equivalent.*

1. *γ is an $N-$radical*

2. *For every Morita context (R, A, B, S)*

$$\gamma(T) = \gamma \begin{pmatrix} R & A \\ B & S \end{pmatrix} = \begin{pmatrix} \gamma(R) & A_0 \\ B_0 & \gamma(S) \end{pmatrix}$$

where $A_0 = \{a \in A | aB \subseteq \gamma(R)\}$ and $B_0 = \{b \in B | bA \subseteq \gamma(S)\}$.

Proof Please see the proof of Theorem 3 in [7] or the proof of Theorem 3.18.14 in [3]. □

Theorem 2.4 is a necessary and sufficient condition for every radical class γ of rings to be $N-$radical that was discovered by Jaegermann in his paper [7]. On the other hand, some examples of $N-$radicals can be found in [11]. Let γ be any $N-$radical class. It follows from Theorem 1 in [11] that for every ring of Morita context $T = \begin{pmatrix} R & V \\ W & S \end{pmatrix}$, the radical $\gamma(T) = \begin{pmatrix} \gamma(R) & V_0 \\ W_0 & \gamma(S) \end{pmatrix}$, where $\gamma(R)$ and $\gamma(S)$ are the largest ideals of R and S which are contained in the radical class γ, respectively, $V_0 = \{v \in V | vW \subseteq \gamma(R)\}$ and $W_0 = \{w \in W | wV \subseteq \gamma(S)\}$. Moreover, Sands in his paper [11] also showed that the Jacobson radical is an $N-$radical in the point of view radical of rings. In this paper, for simplicity, we implement Theorem 2.4 to show that the Jacobson radical is an $N-$radical. Precisely, in the next theorem, we will show that the Jacobson radical class $\mathcal{J}$ also satisfies the condition explained in Theorem 2.4 number 2 which clearly implies that the Jacobson radical class $\mathcal{J}$ is an $N-$radical.

Theorem 2.5 *[11] Let R and S be rings with identity and Let $T = \begin{pmatrix} R & V \\ W & S \end{pmatrix}$ be any ring of Morita context. Then $\mathcal{J}(T) = \begin{pmatrix} \mathcal{J}(R) & V_0 \\ W_0 & \mathcal{J}(S) \end{pmatrix}$, where $\mathcal{J}(R)$ and $\mathcal{J}(S)$ are the Jacobson radicals of R and S, respectively, $V_0 = \{v \in V | vW \subseteq \mathcal{J}(R)\}$ and $W_0 = \{w \in W | wV \subseteq \mathcal{J}(S)\}$.*

Proof We will prove Theorem 2.5 by classifying maximal right ideals into three different types.

(a). Let I be a maximal right ideal of R. Put $M = \begin{pmatrix} I & V_i \\ W & S \end{pmatrix}$, where $V_i = \{v \in V | vW \subseteq I\}$, a right $S-$submodule of V. It is easy to see that M is a right ideal of T. We claim M is a maximal right ideal. We assume, on the contrary, that it is not maximal. Let N be a right ideal of T with $T \supseteq N \supset M$ and $n = \begin{pmatrix} a & v \\ w & s \end{pmatrix} \in N - M$, that is, either $a \notin I$ or $v \notin V_i$. If $a \notin I$, then $1 = ar + i$ for some $r \in R$ and $i \in I$ and so $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & v \\ w & s \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} i & 0 \\ -wr & 0 \end{pmatrix} \in N$. Hence $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in N$ and $T = N$ follows.

If $v \notin V_i$, then there is a $w \in W$ with $vw \in I$ and so $1 = vwr + j$ for some $r \in R$ and $j \in I$. Thus $\begin{pmatrix} a & v \\ w & s \end{pmatrix}\begin{pmatrix} 0 & 0 \\ wr & 0 \end{pmatrix} = \begin{pmatrix} vwr & 0 \\ swr & 0 \end{pmatrix} \in N$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} vwr & 0 \\ swr & 0 \end{pmatrix} + \begin{pmatrix} j & 0 \\ -swr & 0 \end{pmatrix} \in N$. Hence $N = T$, showing M is a maximal right ideal of T.

(b). Let J be a maximal right ideal of S and $N = \begin{pmatrix} R & V \\ W_j & J \end{pmatrix}$, where $W_j = \{w \in W | wV \subseteq J\}$. Then N is a maximal right ideal of T as shown in (a). Since $\cap V_i = \cap_i \{v \in V | vW \subseteq I_i\} = \{v \in V | vW \subseteq \mathcal{J}(R)\} = V_0$, where I_i runs over all maximal right ideals of R and $\cap W_j = \cap_j \{w \in W | wV \subseteq Jj\} = \{w \in W | wV \subseteq \mathcal{J}(S)\} = W_0$, where J_j runs over all maximal right ideals of S, we have $\begin{pmatrix} \mathcal{J}(R) & V_0 \\ W_0 & \mathcal{J}(R) \end{pmatrix} \supseteq \mathcal{J}(T)$.

(c). Let M be a maximal right ideal of T which is a different type from ones in (a) and (b). Then we will prove that $M \supseteq \begin{pmatrix} \mathcal{J}(R) & V_0 \\ W_0 & \mathcal{J}(R) \end{pmatrix}$. First we will show that there is an element in M whose $(1,1)-$component is 1. Put $I_0 = \{r \in R | \begin{pmatrix} r & v \\ w & s \end{pmatrix} \in M\}$, a right ideal of R. If $I_0 \neq R$, then there is a maximal right ideal I with $I \supseteq I_0$. By (a) $N = \begin{pmatrix} I & V_i \\ W & S \end{pmatrix}$ is a maximal right ideal which is different from M. Therefore, $M + N = T$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r & v \\ w & s \end{pmatrix} + \begin{pmatrix} i & v_1 \\ w_1 & s_1 \end{pmatrix}$ for some $\begin{pmatrix} r & v \\ w & s \end{pmatrix} \in M$ and $\begin{pmatrix} i & v_1 \\ w_1 & s_1 \end{pmatrix} \in N$. Thus $r + i = 1$ and $r, i \in I$, a contradiction. Hence $I_0 = R$. Next we will prove either $M \supseteq \begin{pmatrix} 0 & V_0 \\ 0 & 0 \end{pmatrix}$ or $M \supseteq \begin{pmatrix} 0 & 0 \\ W_0 & 0 \end{pmatrix}$. Suppose that M does not contain $\begin{pmatrix} 0 & V_0 \\ 0 & 0 \end{pmatrix}$. Since $\begin{pmatrix} 0 & V_0 \\ 0 & 0 \end{pmatrix} T = \begin{pmatrix} V_0W & V_0 \\ 0 & 0 \end{pmatrix}$, we have $M + \begin{pmatrix} V_0W & V_0 \\ 0 & 0 \end{pmatrix} = T$. So $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} r & v \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} r_1 & v_0 \\ 0 & 0 \end{pmatrix}$ for some $\begin{pmatrix} r & v \\ 0 & 0 \end{pmatrix} \in M$ and $\begin{pmatrix} r_1 & v_0 \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} V_0W & V_0 \\ 0 & 0 \end{pmatrix}$. Since $r_1 \in V_0W \subseteq \mathcal{J}(R), r$ is a unit in R. Thus $M \ni \begin{pmatrix} r & v \\ 0 & 0 \end{pmatrix}\begin{pmatrix} r^{-1} & 0 \\ 0 & 0 \end{pmatrix} =$

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Similarly if M does not contain $\begin{pmatrix} 0 & 0 \\ W_0 & 0 \end{pmatrix}$, then $M \ni \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Hence we have proved either $M \supseteq \begin{pmatrix} 0 & V_0 \\ 0 & 0 \end{pmatrix}$ or $M \supseteq \begin{pmatrix} 0 & 0 \\ W_0 & 0 \end{pmatrix}$. In case $M \supseteq \begin{pmatrix} 0 & V_0 \\ 0 & 0 \end{pmatrix}$, we will prove $M \supseteq \begin{pmatrix} \mathcal{J}(R) & 0 \\ 0 & 0 \end{pmatrix}$. We assume, on the contrary, that M does not contain $\begin{pmatrix} \mathcal{J}(R) & 0 \\ 0 & 0 \end{pmatrix}$. Then $T = M + \begin{pmatrix} \mathcal{J}(R) & \mathcal{J}(R)V \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r_1 & v_1 \\ w_1 & s_1 \end{pmatrix} + \begin{pmatrix} r & v_0 \\ 0 & 0 \end{pmatrix}$, where $\begin{pmatrix} r_1 & v_1 \\ w_1 & s_1 \end{pmatrix} \in M$ and $\begin{pmatrix} r & v_0 \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} \mathcal{J}(R) & \mathcal{J}(R)V \\ 0 & 0 \end{pmatrix}$, that is, r_1 is a unit in $R, w_1 = 0, s_1 = 1$, and $v_1 = -v_0 \in V_0$. Thus $M \ni \begin{pmatrix} r_1 & -v_0 \\ 0 & 1 \end{pmatrix}$ and so $\begin{pmatrix} r_1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r_1 & -v_0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & v_0 \\ 0 & 0 \end{pmatrix} \in M$. Hence $M \ni \begin{pmatrix} r_1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r_1^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, a contradiction. Therefore $\begin{pmatrix} \mathcal{J}(R) & 0 \\ 0 & 0 \end{pmatrix} \subseteq M$ and so $\begin{pmatrix} \mathcal{J}(R) & V_0 \\ 0 & 0 \end{pmatrix} \subseteq M$. If $M \supseteq \begin{pmatrix} 0 & 0 \\ W_0 & 0 \end{pmatrix}$, then we have $M \supseteq \begin{pmatrix} 0 & 0 \\ W_0 & \mathcal{J}(S) \end{pmatrix}$ similarly. Hence $M \supseteq \begin{pmatrix} \mathcal{J}(R) & V_0 \\ W_0 & J(S) \end{pmatrix}$ as desired. If M does not contain $\begin{pmatrix} 0 & 0 \\ W_0 & 0 \end{pmatrix}$, then $M \ni \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ as it has been pointed out and so $M \supseteq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ W & S \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ W & S \end{pmatrix}$. Thus $M \supseteq \begin{pmatrix} \mathcal{J}(R) & V_0 \\ W & S \end{pmatrix}$. For $\begin{pmatrix} 1 & v \\ w & s \end{pmatrix} \in M, \begin{pmatrix} 1 & v \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & v \\ w & s \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -w & -s \end{pmatrix} \in M$ and $M \ni \begin{pmatrix} 1 & v \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Hence $M = T$, a contradiction.

Therefore $\mathcal{J}(T) = \begin{pmatrix} \mathcal{J}(R) & V_0 \\ W_0 & \mathcal{J}(S) \end{pmatrix}$ by (a), (b), and (c) which completes the proof. □

As a consequence of Theorem 2.5, we have the following corollary.

Corollary 2.6 *The Jacobson radical class $\mathcal{J}$ is an $N-$radical.*

Proof Let R and S be ring with identity. It follows from Theorem 2.5 that for every ring of Morita context $T = \begin{pmatrix} R & A \\ B & S \end{pmatrix}$, the Jacobson radical $\mathcal{J}(T) = \begin{pmatrix} \mathcal{J}(R) & V_0 \\ W_0 & \mathcal{J}(S) \end{pmatrix}$, where $\mathcal{J}(R)$ and $\mathcal{J}(S)$ are the Jacobson radicals of R and S, respectively, $V_0 = \{v \in V | vW \subseteq \mathcal{J}(R)\}$ and $W_0 = \{w \in W | wV \subseteq \mathcal{J}(S)\}$. In more general case, when R and S do not necessarily have the identity, this property is also held by applying Theorem 3.8.10 in [3] since the Jacobson radical class $\mathcal{J}$ is a normal radical as shown in the Example 3.18.6 in [3]. Hence, it follows from Theorem 10 in [7] and it is confirmed by Theorem 3.18.14 in [3] that the Jacobson radical $\mathcal{J}$ is an $N-$radical. □

3. Graded Jacobson radical

In this section, we construct the restricted graded Jacobson radical class and determine what actually the restricted graded Jacobson radical of a graded ring of Morita context is. Let G be a group. Define a class of rings $\mathcal{J}^G = \{A \text{ is a } G-\text{graded ring } | A^u \in \mathcal{J}\}$, where $\mathcal{J}$ is the Jacobson radical class. Let γ be a radical class. In general radical class, we can define the restricted graded radical class $\gamma^G = \{A \text{ is a } G-\text{graded ring } | A^u \in \gamma\}$ of γ to the category of $G-$graded rings. In general radical classes, some properties of the γ^G were described in [2]. We call $\mathcal{J}^G$ as $G-$graded Jacobson radical class. In this section, we provide some further properties of the restriction $\mathcal{J}^G = \{A$ is a $G-$graded ring $|A^u \in \mathcal{J}\}$ of the Jacobson radical $\mathcal{J}$. Set $G = \mathbb{Z}$, in the next example, we provide an example of $G-$graded Jacobson ring which implies $\mathcal{J}^G$ is a nonempty set.

Example 3.1 *Consider the ring* $J = \{\frac{2x}{2y+1}|gcd(2x, 2y+1) = 1, x, y \in \mathbb{Z}\}$*. Furthermore, the set* $M_2(J) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} |a, b, c, d \in J\}$ *of all* $2\times 2-$*matrices over* J *is a* $\mathbb{Z}_4-$*graded Jacobson radical ring.*

Proof It follows from Example 3.2.13 (ii) in [3] that the ring J is a Jacobson radical ring which means that $J \in \mathcal{J}$. It is clear that $M_2(J)$ is a ring under addition and multiplication matrix operation. It follows from Example 4.9.8 in [3] that the Jacobson radical class $\mathcal{J}$ is matric-extensible. Hence, if A is a ring in $\mathcal{J}$, then the ring $M_n(A)$ of all $n \times n-$matrices over A is also contained in $\mathcal{J}$. As a consequence of the matrix-extensible property of the Jacobson radical class $\mathcal{J}$ the Jacobson radical class $\mathcal{J}$ contains $M_2(J)$. Now, consider the set $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ of all integers modulo 4. Define

$$T_0 = \{\begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} |a_{11}, a_{22} \in J\}$$

$$T_1 = T_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$T_2 = \{\begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix} |a_{12}, a_{21} \in J\}.$$

It is clear that $M_2(J) = T_0 \oplus T_1 \oplus T_2 \oplus T_3$. In other words, $M_2(J) = \oplus_{g\in\mathbb{Z}_4} T_g$. Moreover, $T_gT_h \subseteq T_{gh}$ for every $g, h \in \mathbb{Z}_4$. This means that the ring $M_2(J)$ is a $\mathbb{Z}_4-$graded ring which proves that $M_2(J) \in \mathcal{J}^{\mathbb{Z}_4}$. In other words, the set $M_2(J) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} |a, b, c, d \in J\}$ of all $2 \times 2-$matrices over J is a $\mathbb{Z}_4-$graded Jacobson radical ring. □

On the other hand, we shall follow the definition of a graded supernilpotent radical from [2] below. Let G be a group. Define a class β_G of rings as $\beta_G = \{A$ is a $G-$rings $|\beta_G(A) = A\}$, where $\beta_G(A)$ is the intersection of all the graded prime ideals of a $G-$graded ring A. We use the definition of a graded supernilpotent radical as follows.

Definition 3.2 *A graded radical* ξ *is a graded supernilpotent radical class if* ξ *is a graded hereditary radical and* $\beta_G \subseteq \xi$

We therefore have the following property for the class $\mathcal{J}^G$.

Theorem 3.3 *The class* $\mathcal{J}^G = \{A$ *is a* $G-$*graded ring*$|A^u \in \mathcal{J}\}$ *of rings forms a graded supernilpotent radical class.*

Proof It is clearly shown from Proposition 2 in [2] that $\mathcal{J}^G$ is a graded supernilpotent radical class since the Jacobson radical $\mathcal{J}$ is a supernilpotent radical class. □

Furthermore, Morita context has an important role on radical theory of rings. In fact, the left hereditariness and the left strongness of radical class can be determined by using Morita context. Moreover, the normality of a radical class can be identified by using Morita context. In the previous section, we have discussed about the ring of Morita context. In this section, we rewrite the concept of graded Morita context which was explained in [2]. We will use the concept of graded Morita context to define a graded normal radical and a graded $N-$radical.

Definition 3.4 *[2] Let G be a group, let R and S be $G-$graded rings, and let A and B are $G-$graded $R-S-$bimodule and $G-$graded $S-R-$bimodule, respectively. The quadruple (R,A,B,S) is a graded Morita context if $A_gB_h \subseteq R_{gh}$ and $B_hA_g \subseteq S_{hg}$ for all $g,h \in G$. Furthermore, a graded radical ξ is a graded normal radical if every graded Morita context $(R,A,B,S), A\xi(S)B \subseteq R$ (equivalently, $B\xi(R)A \subseteq S$).*

Now we define a graded $N-$radical.

Definition 3.5 *A graded radical ξ is a graded $N-$radical if ξ is a graded supernilpotent radical and ξ is a graded normal radical.*

We continue to explain the following property.

Theorem 3.6 *The graded Jacobson radical $\mathcal{J}^G$ is a graded $N-$radical.*

Proof It follows from Theorem 3.3 that $\mathcal{J}^G$ is a graded supernilpotent radical. Moreover, it follows from Corollary 2.6 and Theorem 4 in [2] that $\mathcal{J}^G$ is a graded normal radical. Hence, $\mathcal{J}^G$ is a graded $N-$radical. □

It follows from the construction of the restricted graded Jacobson radical $\mathcal{J}^G$ for a fixed group G, a natural question which asked whether the graded version of Theorem 2.5 holds. We will explain the answer of this question as the final result of this paper.

In a commutative case, it follows from Lemma 1.74. in [8] that the graded Jacobson radical of any graded ring is precisely the intersection of all graded maximal ideals. On the other hand, it follows from Proposition 2 in [2] that $\mathcal{J}^G(A) = (\mathcal{J}(A))_G$ for every ring A, where $(\mathcal{J}(A))_G = \oplus\{\mathcal{J}(A) \cap A_g | g \in G\}$. We will show that $\mathcal{J}^G(A) = (\mathcal{J}(A))_G = \oplus_{g\in G}\{\mathcal{J}(A) \cap A_g | g \in G\}$ is precisely the intersection of all $G-$graded maximal ideals of A. Hence, we explain the following lemma.

Lemma 3.7 *Let A be a commutative ring with identity. Then $\mathcal{J}^G(A) = (\mathcal{J}(A))_G = \oplus_{g\in G}\{\mathcal{J}(A) \cap A_g | g \in G\}$ is precisely the intersection of all $G-$graded maximal ideals of A.*

Proof Now let $\{I_\lambda | \lambda \in \Lambda\}$ be the collection of all maximal ideals of A. Furthermore, we denote the $G-$graded maximal ideal of A as $(I_\lambda)_G$ and it follows from [2] that $(I_\lambda)_G = \oplus_{g\in G} I_\lambda \cap A_g$. Moreover, the intersection of all $G-$graded maximal ideals of A can be denoted by $\cap(I_\lambda)_G$. Furthermore,

$$\cap(I_\lambda)_G = \cap_{\lambda\in\Lambda}(\oplus_{g\in G} I_\lambda \cap A_g) = \oplus_{g\in G}\{\cap_{\lambda\in\Lambda}(I_\lambda) \cap A_g | g \in G\} = \oplus_{g\in G}\{\mathcal{J}(A) \cap A_g | g \in G\}.$$

Hence, we can infer that $\oplus_{g\in G}\{\mathcal{J}(A) \cap A_g | g \in G\}$ is precisely the intersection of all $G-$graded maximal ideals of A. □

As a final result of this paper, we can show that the graded version of Theorem 2.5 is also valid for the graded Jacobson radical $\mathcal{J}^G$.

Theorem 3.8 *For every graded Morita context* (R, A, B, S), *the graded Jacobson radical* $\mathcal{J}^G(T)$ *of*

$$T = \begin{pmatrix} R & A \\ B & S \end{pmatrix}$$

is given below

$$\mathcal{J}^G(T) = \begin{pmatrix} \mathcal{J}^G(R) & A_0 \\ B_0 & \mathcal{J}^G(S) \end{pmatrix},$$

where $\mathcal{J}^G(R)$ *and* $\mathcal{J}^G(S)$ *are the graded Jacobson radicals of* R *and* S, *respectively,* $A_0 = \{a \in A | aB \subseteq \mathcal{J}^G(R)\}$ *and* $B_0 = \{b \in B | bA \subseteq \mathcal{J}^G(S)\}$.

Proof It follows from Proposition 2 in [2] that $\mathcal{J}^G(T) = (\mathcal{J}(T))_G$, where $(\mathcal{J}(T))_G = \bigoplus\{J(T) \cap T_g | g \in G\}$. We therefore have

$$\begin{aligned}
(\mathcal{J}(T))_G &= \bigoplus \begin{pmatrix} (\mathcal{J}(R))_G & A_0 \cap A_g \\ B_0 \cap B_g & (\mathcal{J}(S))_G \cap S_g \end{pmatrix} \\
(\mathcal{J}(T))_G &= \begin{pmatrix} \oplus\{\mathcal{J}(R) \cap R_g\} & \oplus\{A_0 \cap A_g\} \\ \oplus\{B_0 \cap B_g\} & \oplus\{\mathcal{J}(S) \cap S_g\} \end{pmatrix} \\
(\mathcal{J}(T))_G &= \begin{pmatrix} (\mathcal{J}(R)_G & A_0 \\ B_0 & (\mathcal{J}(S))_G \end{pmatrix} \\
(\mathcal{J}(T))_G &= \begin{pmatrix} \mathcal{J}^G(R) & A_0 \\ B_0 & \mathcal{J}^G(S) \end{pmatrix}
\end{aligned}$$

Hence,

$$\mathcal{J}^G(T) = \begin{pmatrix} \mathcal{J}^G(R) & A_0 \\ B_0 & \mathcal{J}^G(S) \end{pmatrix},$$

where $\mathcal{J}^G(R)$ and $\mathcal{J}^G(S)$ are the graded Jacobson radicals of R and S, respectively, $A_0 = \{a \in A | aB \subseteq \mathcal{J}^G(R)\}$ and $B_0 = \{b \in B | bA \subseteq \mathcal{J}^G(S)\}$ which complete the proof. □

4. Conclusion

The Jacobson radical is an $N-$radical since for every ring of Morita context $T = \begin{pmatrix} R & V \\ W & S \end{pmatrix}$, the Jacobson radical $\mathcal{J}(T) = \begin{pmatrix} \mathcal{J}(R) & V_0 \\ W_0 & \mathcal{J}(S) \end{pmatrix}$, where $\mathcal{J}(R)$ and $\mathcal{J}(S)$ are the Jacobson radicals of R and S, respectively, $V_0 = \{v \in V | vW \subseteq \mathcal{J}(R)\}$ and $W_0 = \{w \in W | wV \subseteq \mathcal{J}(S)\}$. Moreover, for any group G, this property is also valid for the restricted $G-$graded Jacobson radical of graded ring of Morita context.

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