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On (a, d)-edge local antimagic coloring number of graphs

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Abstract: For any graph $G = (V, E)$, the order and size of G are p and q. A bijection l from $V(G)$ to $\{1, 2, .., p\}$ is called (a, d)-edge local antimagic labeling if for any two adjacent edges are not received the same edge-weight (color) and the set of all edge-weights are formed an arithmetic progression $\{a, a+d, a+2d, \dots, a+(c-1)d\}$, for some integers $a, d > 0$ and c is the number of distinct colors used in the proper coloring. An edge-weight (color) $w(uv)$ is the sum of two end vertices labels, $w(uv) = f(u) + f(v), uv \in E(G)$. The (a, d)-edge local antimagic coloring number is the least color (edge-weight) used in any (a, d)-edge local antimagic labeling. In the present study, we introduce a new type of labeling and a parameter, also we obtain the (a, d)-edge local antimagic coloring number for paths and wheel graph $W_n, n = 3, 4, 5$. Moreover, we obtain an upper bound of the (a, d)-edge local antimagic coloring number for wheel $W_n, n \geq 6$.

Key words: (a, d)-edge local antimagic labeling, (a, d)-edge local antimagic coloring number, paths and wheel graph

1. Introduction

A graph $G = (V, E)$ is a finite and undirected graph without loops and multiple edges. Let p and q be the number of vertices and edges of G. For graph-theoretic terminology, we refer to Chartrand and Lesniak [2].

Hartsfield and Ringel [5] introduced an antimagic labeling in 1990. An antimagic labeling is a bijection from the set of edges to $\{1, 2, ..., q\}$ such that all the vertex weights are distinct, where a weight of the vertex v is $w(v) = \sum_{e \in E(v)} f(e)$, and $E(v)$ is the set of edges incident to v. A graph G is called antimagic if G has an antimagic labeling. Several authors studied and obtained several results based on the conjectures, every connected graph is antimagic except K_2 and all trees are antimagic. For further study see in [3–5, 9, 12]. Still, these two conjectures are open. These antimagic labeling and vertex coloring concepts are motivated to introduce a local version of an antimagic labeling.

The local vertex antimagic labeling was introduced by Arumugam et al. [1] in 2017. A local vertex antimagic is a bijection from the set of edges to $\{1, 2, 3, ..., q\}$ such that for any two adjacent vertices are not received the same weight (color), where the weight of the vertex v is $w(v) = \sum_{e \in E(v)} f(e)$, and $E(v)$ is the set of edges incident to v. The local vertex antimagic chromatic number $\chi_{la}(G)$ is the least number of colors used in any local vertex antimagic labeling of G. They proved some basic results in [1]. For more study see in [4, 6, 8, 11].

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The local edge antimagic labeling was introduced by Agustin et al. [7] in 2017. The local edge antimagic labeling is a bijection from the set of vertices to $\{1, 2, 3, ..., p\}$ such that for any two adjacent edges are not received the same edge-weight (color), where the edge-weight $w(e = uv)$ is the sum of two end vertices labels, $w(uv) = f(u) + f(v), uv \in E(G)$. The local edge antimagic chromatic number $\chi'_{lea}(G)$ is the least number of colors used in any local edge antimagic labeling of G. The following results are obtained in [7].

Theorem 1.1 [7] If $\Delta(G)$ is maximum degree of G, we have $\chi'_{lea}(G) \geq \Delta(G)$.

Theorem 1.2 [7] For a path graph P_n on $n \geq 3$ vertices, we have $\chi'_{lea}(P_n) = 2$.

Theorem 1.3 [7] For a complete graph K_n on $n \geq 3$ vertices, we have $\chi'_{lea}(K_n) = 2n - 3$.

Theorem 1.4 [7] For a wheel graph W_n on $n \geq 3$ vertices, $\chi'_{lea}(W_n) = n + 2$.

Rajkumar and Nalliah [10] found the correct local edge antimagic chromatic number for the Agustin et al.'s [7] result of wheel graph W_n. The correct result is given below. Also, they obtained local edge antimagic chromatic number for fan T_n and friendship F_n graphs as follows.

Theorem 1.5 [10] For the wheel graph W_n on $n \geq 3$ vertices, we have

$$\chi'_{lea}(W_n) = \begin{cases} 5, & \text{if } n = 3, 4 \\ n, & \text{if } n \geq 5. \end{cases}$$

Theorem 1.6 [10] For the fan graph T_n on $n + 1$ vertices, we have

$$\chi'_{lea}(T_n) = \begin{cases} n + 1 & \text{if } n = 2, 3 \\ n & \text{if } \quad n \geq 4. \end{cases}$$

Theorem 1.7 [10] For the friendship graph F_n, we have

$$\chi'_{lea}(F_n) = \begin{cases} 3 & \text{if } \quad n = 1, \\ 2n & \text{if } \quad n \geq 2. \end{cases}$$

2. Main results

The local edge antimagic labeling and the chromatic number of a graph are motivated to study a new type of labeling and a parameter.

Definition 2.1 *A (a, d)-edge local antimagic labeling is a bijection l from the set of vertices to $\{1, 2, 3, ..., p\}$ such that for any two adjacent edges are not received the same edge-weight (color), where an edge-weight $w(e = uv)$ is the sum of two end vertices labels, $w(uv) = f(u) + f(v), uv \in E(G)$ and the set of all edge-weights are formed an arithmetic progression $\{a, a + d, a + 2d, \ldots, a + (c - 1)d\}$, for some integers $a, d > 0$ and c is the number of distinct colors are used in the proper coloring. A (a, d)-edge local antimagic labeling is denoted by (a, d)-ELA labeling.*

Definition 2.2 *The (a,d)-edge local antimagic coloring number of a graph G is the least number of colors used in any (a,d)-edge local antimagic labeling of G and is denoted by $\chi'_{(a,d)\text{-}ela}(G)$.*

Example 2.3 From Theorem 1.5[10], we get the local edge antimagic chromatic number of W_5 is 5 and we proved there is no $(a,1)$-ELA labeling with 5-colors in Theorem 2.14. Hence, the graph W_5 has $(4,1)$-ELA labeling with 6-colors, which is given in Figure 1.

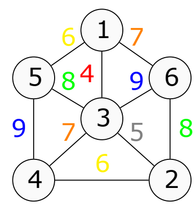

Figure 1. The $(4,1)$-ELA labeling of W_5 with 6-colors.

Observation 2.4 *If the graph G admits an (a,d)-ELA labeling f, then $\chi'_{(a,d)\text{-}ela}(G) \geq \chi'_{lea}(G) \geq \chi'(G) \geq \Delta(G)$.*

Observation 2.5 *If the graph G admits an (a,d)-ELA labeling with c-colors, then $d \leq \frac{2p-4}{c-1}$.*

Proof Let G be a graph of order p. If G admits an (a,d)-ELA labeling f with c-colors, then the edge-weights are $a, a+d, \ldots, a+(c-1)d$. Then the maximum possible edge-weight of an edge e is $w(e) \leq p+p-1$. The minimum possible edge-weight of an edge e is $a \geq 3$. Therefore, $a+(c-1)d \leq 2p-1$, which implies, we get $d \leq \frac{2p-4}{c-1}$. □

Observation 2.6 *Let G be a graph with order p and size q. If G admits an (a,d)-ELA labeling f with c-colors then*

$$\sum_{v \in V(G)} deg(v) f(v) = \sum_{i=1}^{c} a_i w_i, \quad \text{where} \quad \sum_{i=1}^{c} a_i = |E(G)| = q. \tag{2.1}$$

Now, we consider a path graph P_n. From Observation 2.5, if the (a,d)-ELA coloring number of P_n with 2-colors, then $d \leq 2n-4$. The following theorems gives the (a,d)-ELA labeling with 2-colors, when $d=1$ and 2.

Theorem 2.7 *For a path graph P_n on $n \geq 3$ vertices. Then $\chi'_{(n+1,1)\text{-}ela}(P_n) = 2$.*

Proof Let $V(P_n) = \{v_i, 1 \leq i \leq n\}$ and $E(P_n) = \{v_i v_{i+1}, 1 \leq i \leq n-1\}$. Then $|V(P_n)| = n$ and $|E(P_n)| = n-1$. Now, define a bijection $f_1 : V(P_n) \to \{1, 2, \ldots, n\}$ by

$$f_1(v_i) = \begin{cases} \frac{i+1}{2}, & \text{if } i \text{ is odd} \\ \frac{2n+2-i}{2}, & \text{if } i \text{ is even.} \end{cases}$$

The edge-weights of P_n are

$$w_1(v_iv_{i+1}) = \begin{cases} n+1, & \text{if } i \text{ is odd} \\ n+2, & \text{if } i \text{ is even.} \end{cases}$$

It is easy to identify that f_1 proves a proper edge coloring of P_n and hence $\chi'_{(n+1,1)\text{-ela}}(P_n) \leq 2$. Since $\chi'_{lea}(P_n) = 2$, it follows, we get $\chi'_{(n+1,1)\text{-ela}}(P_n) \geq 2$. Hence $\chi'_{(n+1,1)\text{-ela}}(P_n) = 2$. □

Example 2.8 *The graph P_7 has $(8,1)$-ELA labeling with 2-colors, which is given in Figure 2.*

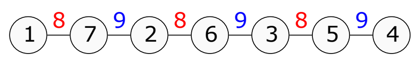

Figure 2. The $(8,1)$-ELA labeling of P_7 with 2-colors.

Theorem 2.9 *For a path graph P_n on $n \geq 3$ vertices. Then $\chi'_{(a,2)\text{-}ela}(P_n) = 2$. Moreover,*

$$a = \begin{cases} n, & n \text{ is odd} \\ n+1, & n \text{ is even.} \end{cases}$$

Proof Let $V(P_n) = \{v_i, 1 \leq i \leq n\}$ and $E(P_n) = \{v_iv_{i+1}, 1 \leq i \leq n-1\}$. Then $|V(P_n)| = n$ and $|E(P_n)| = n-1$.
Case(i) n is odd.
Now, define a bijection $f_2 : V(P_n) \to \{1, 2, \ldots, n\}$ by

$$f_2(v_i) = \begin{cases} i, & \text{if } i \text{ is odd} \\ n+1-i, & \text{if } i \text{ is even.} \end{cases}$$

The edge-weights of P_n are

$$w_2(v_iv_{i+1}) = \begin{cases} n, & \text{if } i \text{ is odd} \\ n+2, & \text{if } i \text{ is even.} \end{cases}$$

Case(ii) n is even.
Now, define a bijection $f_3 : V(P_n) \to \{1, 2, \ldots, n\}$ by

$$f_3(v_i) = \begin{cases} i, & \text{if } i \text{ is odd} \\ n+2-i, & \text{if } i \text{ is even.} \end{cases}$$

The edge-weights of P_n are

$$w_3(v_iv_{i+1}) = \begin{cases} n+1, & \text{if } i \text{ is odd} \\ n+3, & \text{if } i \text{ is even.} \end{cases}$$

It is easy to identify that f_2 and f_3 proves a proper edge coloring of P_n and hence $\chi'_{(a,2)\text{-ela}}(P_n) \leq 2$. Since $\chi'_{lea}(P_n) = 2$, it follows, we get $\chi'_{(a,2)\text{-ela}}(P_n) \geq 2$. Hence $\chi'_{(a,2)\text{-ela}}(P_n) = 2$. □

Example 2.10 *The graphs P_7 and P_8 admit $(a,2)$-ELA labeling with 2-colors, where $a = 7$ and 9, which is given in Figures 3 and 4.*

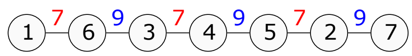

Figure 3. The $(7,2)$-ELA labeling of P_7 with 2-colors.

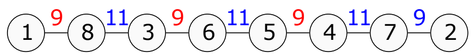

Figure 4. The $(9,2)$-ELA labeling of P_8 with 2-colors.

Theorem 2.11 *There is no (a,d)-ELA labeling of the path P_n with 2-clolors, where $d \geq 3$.*

Proof Suppose $\chi'_{(a,d)\text{-ela}}(P_n) = 2$, where $d \geq 3$. Then there exists a (a,d)-ELA labeling f with 2-colors. The minimum possible edge-weight is $a \geq 3$ and the maximum possible edge-weight is $a + (2-1)d \leq n + n - 1$. Therefore, we get $3 \leq a \leq 2n - 1 - d$. If $d \geq 3$ then the edge-weights are a and $a + d$. If $n \geq 3$ is odd then the edge-weights a and $a+d$ must be used $\frac{n-1}{2}$ times and hence an edge-weight $w = n$ or $n-1$ has only $\frac{n-1}{2}$ possibles of two elements sets with their sum is w. Therefore, there is no edge with weight $a + d, d \geq 3$, which is a contradiction.

If n is even then the edge-weights a and $a+d$ must be used $\frac{n}{2}$ and $\frac{n}{2}-1$ times and hence an edge-weight $w = n+1$ has only $\frac{n}{2}$ possibles of two elements sets with their sum is w. Similarly, an edge weight $w' = n-2$ or $n-1$ or n or $n+2$ or $n+3$ is only have $\frac{n}{2}-1$ possibles of two elements sets with their sum is w'. Therefore, there is no edge with weight $a + d, d \geq 3$, which is a contradiction. □

Theorem 2.12 *Let G be the graph of wheel graph $W_n, n \geq 5$, fan graph $T_n, n \geq 4$ and friendship graph $F_n, n \geq 2$ with p vertices. If the graph G admits an (a,d)-ELA labeling with $c = \chi'_{lea}(G)$-colors,then $d \leq 2$.*

Proof Let $G \cong W_n$ and T_n be the wheel and fan graph on $p = n+1$ vertices. From Theorem 1.5 [10] and Theorem 1.6 [10], we get $c = \chi'_{lea}(G) = n$ and Observation 2.5, we get $d \leq \frac{2(n+1)-4}{n-1} = 2$. Let G be the friendship graph F_n with $p = 2n+1$. From Theorem 1.7 [10], we get $c = \chi'_{lea}(F_n) = 2n$, and Observation 2.5, we get $d \leq \frac{2(2n+1)-4}{2n-1} = 2$. □

Theorem 2.13 *For a wheel graph W_n on $n+1$ vertices, where $n = 3, 4$. Then $\chi'_{(a,1)\text{-ela}}(W_n) = 5$, where*

$$a = \begin{cases} 3, & n = 3 \\ 4, & n = 4. \end{cases}$$

Proof For $n = 3$, clearly, $W_3 \cong K_4$, it follows from Theorem 1.3 [7], we get $\chi'_{(3,1)\text{-ela}}(W_3) = 5$. For $n = 4$, let $V(W_4) = \{c, v_1, v_2, v_3, v_4\}$. Now, define a labeling $f_4 : V(W_4) \to \{1,2,3,4,5\}$ by $f_4(c) = 3, f_4(v_1) = 1, f_4(v_2) = 5, f_4(v_3) = 2$, and $f_4(v_4) = 4$. Then the edge-weights are $w_4(cv_1) = 4, w_4(cv_2) = 8, w_4(cv_3) = w_4(v_4v_1) = 5, w_4(cv_4) = w_4(v_2v_3) = 7$, and $w_4(v_1v_2) = w_4(v_3v_4) = 6$. It is easy to identify that f_4 proves a proper edge coloring of W_n and hence $\chi'_{(4,1)\text{-ela}}(W_4) \leq 5$. From Theorem 1.5 [10], we get $\chi'_{(4,1)\text{-ela}}(W_4) \geq 5$. Thus $\chi'_{(4,1)\text{-ela}}(W_4) = 5$. □

Theorem 2.14 *Let W_5 be the wheel graph on 6 vertices. Then there is no $(a,1)$-ELA labeling of W_5 with 5-colors.*

Proof Suppose the graph W_5 has $\chi'_{(a,1)\text{-ela}}(W_5) = 5$. Then there exists an $(a,1)$-ELA labeling f with 5-colors. The minimum possible edge-weight is $a \geq 3$ and the maximum possible edge-weight is $a+(5-1)1 \leq 11$, which implies $a \leq 7$. Hence $3 \leq a \leq 7$. From Equation (2.1), we get

$$3\left[\frac{(6)(7)}{2} - i\right] + 5i = \sum_{i=1}^{5} a_i w_i, \quad \text{where} \quad f(c) = i \quad \text{and} \quad \sum_{i=1}^{5} a_i = 10.$$

This implies, we get the equation

$$63 + 2i = \sum_{i=1}^{5} a_i w_i, \quad \text{where} \quad f(c) = i \quad \text{and} \quad \sum_{i=1}^{5} a_i = 10. \tag{2.2}$$

Since $3 \leq a \leq 7$, it follows that, there are five possible edge-weight sets: $W_1' = \{3,4,5,6,7\}$, $W_2' = \{4,5,6,7,8\}$, $W_3' = \{5,6,7,8,9\}$, $W_4' = \{6,7,8,9,10\}$ and $W_5' = \{7,8,9,10,11\}$. Since the edge-weights $3,4,10$ and 11 are only one possible set of two elements set, the edge-weights $5,6,8$ and 9 are two possibles sets of two elements sets, and an edge-weight 7 is three possibles sets of two elements sets.

Case(i) $a = 3$ and 7

If $a = 3$ then the edge-weight set is $W_1' = \{3,4,5,6,7\}$ and hence $a_1 = a_2 = 1$, $a_3 = a_4 = 2$, $a_5 = 3$. Therefore, we get $\sum_{i=1}^{5} a_i = 9$, which is a contradiction. A similar contradiction arise for the case $a = 7$.

Case(ii) $a = 4$

Then the edge-weight set is $W_2' = \{4,5,6,7,8\}$. Now, we substitute the values $w_1 = 4, w_2 = 5, w_3 = 6, w_4 = 7, w_5 = 8$ and $a_1 = 1$, $a_2 = a_3 = a_5 = 2$, $a_4 = 3$ in Equation (2.2), we get $63+2i = 1(4)+2(5)+2(6)+3(7)+2(8)$ with $\sum_{i=1}^{5} a_i = 10$. This implies, we get $i = 0$, which is a contradiction.

Case(iii) $a = 5$

Then the edge-weight set is $W_3' = \{5,6,7,8,9\}$. Since $\sum_{i=1}^{5} a_i = 10$, it follows that, the possible 5-tuples (a_1,a_2,a_3,a_4,a_5) are $(1,2,3,2,2)$, $(2,1,3,2,2)$, $(2,2,2,2,2)$, $(2,2,3,1,2)$ and $(2,2,3,2,1)$. If $(a_1,a_2,a_3,a_4,a_5) = (1,2,3,2,2)$, $(2,2,2,2,2)$ and $(2,2,3,2,1)$ then we substitute the values $w_1 = 5, w_2 = 6, w_3 = 7, w_4 = 8, w_5 = 9$ and the corresponding 5-tuples values of a_i in Equation (2.2), we get i is not an integer, which is a contradiction.

If $(a_1,a_2,a_3,a_4,a_5) = (2,1,3,2,2)$ then we substitute the values $w_1 = 5, w_2 = 6, w_3 = 7, w_4 = 8, w_5 = 9$ and the corresponding 5-tuples values of a_i in Equation (2.2), we get $i = 4$ and hence the central vertex label $f(c) = 4$ and other vertices are received the labels except 4. Let $e = cv$. If $f(v) = 6$ then $w(e) = 10 \notin W_3'$, which is a contradiction. A similar contradiction arise for the case $(a_1,a_2,a_3,a_4,a_5) = (2,2,3,1,2)$. □

Problem 2.15 *Does there exist a $(a,1)$-ELA coloring number with n-colors for the wheel graph $W_n, n \geq 6$?*

Theorem 2.16 *For a wheel graph W_n on $n+1$ vertices, where $n \geq 5$. Then $\chi^{'}_{(a,1)\text{-}ela}(W_n) \leq n+1$, where*

$$a = \begin{cases} \frac{n+3}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}+2, & \text{if } n \text{ is even.} \end{cases}$$

Proof Let $V(W_n) = \{c, v_i, 1 \leq i \leq n\}$ and $E(W_n) = \{cv_i, 1 \leq i \leq n\} \cup \{v_iv_{i+1}, 1 \leq i \leq n-1\} \cup \{v_nv_1\}$. Then $|V(W_n)| = n+1$ and $|E(W_n)| = 2n$. Now, define a bijection $f_5 : V(W_n) \to \{1, 2, \ldots, n+1\}$ by

$$f_5(c) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n+2}{2} & \text{if } n \text{ is even} \end{cases}$$

$$f_5(v_i) = \begin{cases} \frac{i+1}{2}, & \text{if } n \text{ is odd}, i \text{ is odd}, 1 \leq i \leq n-2 \\ n+1 & \text{if } n \text{ is odd}, \ i\text{=}2 \\ \frac{2n+2-i}{2}, & \text{if } n \text{ is odd}, i \text{ is even}, 4 \leq i \leq n-1 \\ n, & \text{if } n \text{ is odd}, \ i\text{=}n \\ \frac{i+1}{2}, & \text{if } n \text{ is even}, i \text{ is odd}, 1 \leq i \leq n-1 \\ n+1 & \text{if } n \text{ is even}, \ i\text{=}2 \\ \frac{2n+2-i}{2}, & \text{if } n \text{ is even}, i \text{ is even}, 4 \leq i \leq n-2 \\ n, & \text{if } n \text{ is even}, \ i\text{=}n. \end{cases}$$

The edge-weights of W_n are

$$w_5(cv_i) = \begin{cases} \frac{n+2+i}{2}, & \text{if } n \text{ is odd}, i \text{ is odd}, 1 \leq i \leq n-2 \\ \frac{3n+3}{2} & \text{if } n \text{ is odd}, \ i\text{=}2 \\ \frac{3n+3-i}{2}, & \text{if } n \text{ is odd}, i \text{ is even}, 4 \leq i \leq n-1 \\ \frac{3n+1}{2}, & \text{if } n \text{ is odd}, \ i\text{=}n \\ \frac{n+3+i}{2}, & \text{if } n \text{ is even}, i \text{ is odd}, 1 \leq i \leq n-1 \\ \frac{3n+4}{2} & \text{if } n \text{ is even}, \ i\text{=}2 \\ \frac{3n+4-i}{2}, & \text{if } n \text{ is even}, i \text{ is even}, 4 \leq i \leq n-2 \\ \frac{3n+2}{2}, & \text{if } n \text{ is even}, \ i\text{=}n \end{cases}$$

$$w_5(v_iv_{i+1}) = \begin{cases} n+2, & \text{if } n \text{ is odd}, i\text{=}1, i \text{ is even}, 4 \leq i \leq n-3 \\ n+3, & \text{if } n \text{ is odd}, \ i\text{=}2 \\ n+1, & \text{if } n \text{ is odd}, \ i \text{ is odd}, 3 \leq i \leq n-2 \\ n+2, & \text{if } n \text{ is even}, \ i\text{=}1, i \text{ is even}, 4 \leq i \leq n-2 \\ n+3, & \text{if } n \text{ is even}, \ i\text{=}2 \\ n+1, & \text{if } n \text{ is even}, \ i \text{ is odd}, 3 \leq i \leq n-3 \end{cases}$$

$$w_5(v_{n-1}v_n) = \begin{cases} \frac{3n+3}{2}, & \text{if } n \text{ is odd} \\ \frac{3n}{2}, & \text{if } n \text{ is even} \end{cases}$$

$$w_5(v_n v_1) = n + 1.$$

It is easy to identify that f_5 proves a proper edge coloring of W_n and hence $\chi'_{(a,1)\text{-ela}}(W_n) \leq n+1$, where

$$a = \begin{cases} \frac{n+3}{2}, & n \text{ is odd} \\ \frac{n}{2} + 2, & n \text{ is even.} \end{cases}$$

□

Example 2.17 The graphs W_5 and W_6 admit $(a,1)$-ELA labeling with 6-colors and 7-colors, where $a = 4$ and 5, which is given in Figures 5 and 6.

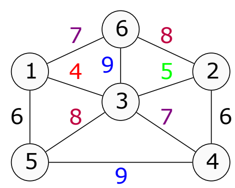

Figure 5. The $(4,1)$-ELA labeling of W_5 with 6-colors.

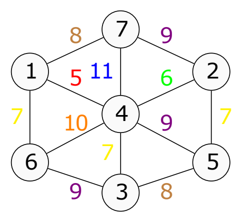

Figure 6. The $(5,1)$-ELA labeling of W_6 with 7-colors.

3. Conclusion

In this paper, we have introduced the new type of (a,d)-ELA labeling and a parameter (a,d)-ELA coloring number of G. We obtained the (a,d)-ELA coloring number for paths when $d = 1$ and 2, and wheel graph $W_n, n = 3,4,5$ when $d = 1$. The problem of determining the (a,d)-ELA coloring number for the remaining graphs is still open.

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