Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 2003 – 2014
© TÜBİTAK
doi:10.55730/1300-0098.3248

A new approach to word standardization and some of its applications

Wesam TALAB*
Department of Mathematics, Faculty of Engineering, Ankara Yıldırım Beyazıt University, Ankara, Turkey

Received: 07.08.2021 • **Accepted/Published Online:** 21.05.2022 • **Final Version:** 20.06.2022

Abstract: In this article, we study word standardization in comparison to Young tableau standardization. We count the number of words (respectively Young tableau) standardized to a given permutation (respectively to a given standard Young tableau). We prove that both rectification and standardization applications commute and show that the standardization commutes with the insertion of Robinson–Schensted. We show that the standardizations of Knuth-equivalent two words are also Knuth equivalent. Finally, using word standardization we establish a proof for the following well-known equality:

$$\forall l \in \{0,1,\ldots,n-1\}, \quad \left\langle {n \atop l} \right\rangle = d_{n,l} = a_{n,l} = \sum_{0\le k\le l} (-1)^k \binom{n+1}{k} (l+1-k)^n.$$

Key words: RSK the correspondence of Robinson–Schensted–Knuth, Young tableaux, word standardization, Knuth equivalent of words, Eulerian number

1. Multisets

The notion of multiset is a generalization of the notion of set, in the sense that an element of multiset could be present more than once (see [3, 5]). This notion is useful in mathematics in general, for example the roots of a polynomial naturally form a multiset. This notion is particularly used in combinatorics, where it provides similar enumeration problems, different from those for sets.

Definition 1.1 *A multiset, of a set A, is a couple (A,m) where m is a function from A to the set $\mathbf{N}^*$ of positive integers, called multiplicite.*

The multiset (A,m) could be seen as a set of elements of A where an element can appear several times: in such case in the multiset (A,m), the element x appears $m(x)$ times. A finite multiset is denoted by using double braces $\{\!\{\ldots\}\!\}$ which enclose the elements, having a strictly positive multiplicity, and which are repeated as many times as their multiplicity. Thus $\{\!\{a,b,a,b,b,c\}\!\}$ represents the multiset $(\{a,b,c\},m)$ where m is the function such that $m(a)=2$, $m(b)=3$, and $m(c)=1$.

We call the number of multisets of cardinal k, with the elements choosen from a finite set of cardinal n,

*Correspondence: wtalab@ybu.edu.tr
2010 *AMS Mathematics Subject Classification:* 05A05

the multinomial coefficient $\left(\!\binom{n}{k}\!\right)$, this notation (for example used in [5]) can be given explicitly by

$$\left(\!\!\binom{n}{k}\!\!\right) = \frac{n(n+1)(n+2)\cdots(n+k-1)}{k!} = \binom{n+k-1}{k} = \binom{n+k-1}{n-1}. \tag{1.1}$$

We can define a generalized binomial coefficient

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \tag{1.2}$$

in which n is not required to be a positive integer, but may be negative and even a rational (see [2] 5.1). So the number of multisets of cardinal k choosen from a set of cardinal n is

$$\left(\!\!\binom{n}{k}\!\!\right) = (-1)^k \binom{-n}{k}, \tag{1.3}$$

because

$$\begin{aligned} \frac{n(n+1)(n+2)\cdots(n+k-1)}{k!} &= (-1)^k \frac{-n(-n-1)(-n-2)\cdots(-n-k+1)}{k!} \\ &= (-1)^k \binom{-n}{k}. \end{aligned} \tag{1.4}$$

Then we have the following formula

$$\sum_{k\geq 0} \left(\!\!\binom{n}{k}\!\!\right) x^k = (1-x)^{-n}. \tag{1.5}$$

Considering that

$$\sum_{k\geq 0} (-1)^k \binom{-n}{k} x^k = (1-x)^{-n}. \tag{1.6}$$

Lemma 1.2 *Given* $\mathcal{R}_1, \ldots, \mathcal{R}_{n-1}$ *fixed order relationships among* $\{<, \leq\}$, *such that*

$$t = \#\left\{i \in [n-1] : \mathcal{R}_i = <\right\}, \tag{1.7}$$

The number of sequences $c_1, \ldots, c_n \in \mathbf{N}$ *with*

$$1 \leq c_1 \ \mathcal{R}_1 \ c_2 \ \mathcal{R}_2 \ \cdots \mathcal{R}_{n-2} \ c_{n-1} \ \mathcal{R}_{n-1} \ c_n \leq m \tag{1.8}$$

is equal to

$$\left(\!\!\binom{n+1}{m-1-t}\!\!\right). \tag{1.9}$$

Proof We will present a demonstration based on a generalization of the bijection between the set of sequences

$$1 \leq c_1 \leq c_2 \leq \cdots \leq c_n \leq m \tag{1.10}$$

and the set of sequences

$$1 \leq a_1 < a_2 < \cdots < a_n \leq m+n-1, \tag{1.11}$$

where $\forall i \in [n-1], \quad a_i := c_i + i - 1$.

For a property $\mathcal{P}$, we will use the following notation $[\![\mathcal{P}]\!]$ which is 1 if this property is true, and 0 if this property is false. The set of sequences

$$1 \leq c_1 \ \mathcal{R}_1 \ c_2 \ \mathcal{R}_2 \ \cdots \mathcal{R}_{n-2} \ c_{n-1} \ \mathcal{R}_{n-1} \ c_n \leq m \tag{1.12}$$

is in bijection with the set of sequences

$$1 \leq d_1 \ \mathcal{R}_1 \ d_2 \ \mathcal{R}_2 \ \cdots \mathcal{R}_{n-2} \ d_{n-1} \ \mathcal{R}_{n-1} \ d_n \leq m+n-1-t \tag{1.13}$$

where

$$d_i = c_i + \sum_{k=1}^{i-1} [\![\mathcal{R}_k = \leq]\!], \quad \text{because} \tag{1.14}$$

if $\mathcal{R}_i = < \ : \ c_i < c_{i+1} \Leftrightarrow d_i < d_{i+1}$,

if $\mathcal{R}_i = \leq \ : \ c_i \leq c_{i+1} \Leftrightarrow d_i < d_{i+1}$ and

$$t = \sum_{k=1}^{n-1} [\![\mathcal{R}_k = <]\!] = n - 1 - \sum_{k=1}^{n-1} [\![\mathcal{R}_k = \leq]\!]. \tag{1.15}$$

The cardinal of this set of sequences is

$$\binom{m+n-1-t}{n} = \left(\!\!\binom{m-t}{n}\!\!\right) = \left(\!\!\binom{n+1}{m-t-1}\!\!\right), \tag{1.16}$$

if $n+m-t \geq 1$, which is true because $t \leq n-1$ (all of $\mathcal{R}_i = <$). □

2. Word standardization

We denote by $[m]^n$ the set of all words of length n whose letters are in the alphabet $[m] := \{1, 2, ..., m\}$. Let us provide the set $\mathbf{N}^2$ with the following four order relations, for each $(i,j), (i',j') \in \mathbf{N}^2$, we suppose $\prec \in \{<, >\}$:

$$(i,j) \prec_r (i',j') \ if \ i < i' \quad or \quad (i = i' and j \prec j'); \tag{2.1}$$

and

$$(i,j) \prec_c (i',j') \ if \ j < j' \quad or \quad (j = j' and i \prec i'). \tag{2.2}$$

Definition 2.1 *Let $w = w_1 w_2 \cdots w_n \in [m]^n$, and let $\pi = (i_1, i_2, \ldots, i_n)$ be the unique permutation that satisfies*

$$(w_{i_1}, i_1) \prec_r (w_{i_2}, i_2) \prec_r \cdots \prec_r (w_{i_n}, i_n) \tag{2.3}$$

We define the standardization of w (denoted by $st_{\prec}(w)$), for $\prec$, as the permutation π^{-1} satisfying $\pi_{i_j}^{-1} = j$.

Example 2.2 *Let* $w = 4\ 1\ 2\ 1\ 4\ 2\ 3\ 1\ 3\ 5\ 3 \in W_{11}([5])$*, then the unique element* $(i_1, i_2, \ldots, i_{11}) \in [11]^{11}$ *which satisfies*

$$(w_{i_1}, i_1) \prec_r (w_{i_2}, i_2) \prec_r \cdots \prec_r (w_{i_{11}}, i_{11}) \tag{2.4}$$

is

$$(2, 4, 8, 3, 6, 7, 9, 11, 1, 5, 10) \quad if \quad \prec is <, \tag{2.5}$$

$$(8, 4, 2, 6, 3, 11, 9, 7, 5, 1, 10) \quad if \quad \prec is >, \tag{2.6}$$

then to standardize w*, we replace in* w

$$(w_2, w_4, w_8, w_3, w_6, w_7, w_9, w_{11}, w_1, w_5, w_{10}) \quad if \quad \prec is <, \tag{2.7}$$

or

$$(w_8, w_4, w_2, w_6, w_3, w_{11}, w_9, w_7, w_5, w_1, w_{10}) \quad if \quad \prec is >, \tag{2.8}$$

by $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11)$ *to have*

$$st_<(w) = 9\ 1\ 4\ 2\ 10\ 5\ 6\ 3\ 7\ 11\ 8. \tag{2.9}$$

$$st_>(w) = 10\ 3\ 5\ 2\ 9\ 4\ 8\ 1\ 7\ 11\ 6. \tag{2.10}$$

Corollary 2.3 *Let* π *be a permutation in* S_n*, then we have* $st_\prec(w) = \pi$ *if the following two conditions are satisfied*

- $w_{\pi_1^{-1}} \leq w_{\pi_2^{-1}} \leq \cdots \leq w_{\pi_n^{-1}}$.
- *For each* $i \in \{1, 2, \ldots, n-1\}$*, if* $\pi_{i+1}^{-1} \prec \pi_i^{-1}$*, then* $w_{\pi_i^{-1}} < w_{\pi_{i+1}^{-1}}$.

Proof According to the definition of $st_\prec$, we have

$$st_\prec(w) = \pi \iff (w_{\pi_1^{-1}}, \pi_1^{-1}) \prec_r (w_{\pi_2^{-1}}, \pi_2^{-1}) \prec_r \cdots \prec_r (w_{\pi_n^{-1}}, \pi_n^{-1}), \tag{2.11}$$

but according to the definition of the order relation $\prec_r$, the two expressions

$$(w_{\pi_1^{-1}}, \pi_1^{-1}) \prec_r (w_{\pi_2^{-1}}, \pi_2^{-1}) \prec_r \cdots \prec_r (w_{\pi_n^{-1}}, \pi_n^{-1}) \tag{2.12}$$

and

$$\left(w_{\pi_1^{-1}} \leq w_{\pi_2^{-1}} \leq \cdots \leq w_{\pi_n^{-1}}\right) \quad and \quad \left(w_{\pi_i^{-1}} < w_{\pi_{i+1}^{-1}} \quad if \quad \pi_{i+1}^{-1} \prec \pi_i^{-1}\right) \tag{2.13}$$

are equivalent. So

$$st_\prec(w) = \pi \iff \left(w_{\pi_1^{-1}} \leq w_{\pi_2^{-1}} \leq \cdots \leq w_{\pi_n^{-1}}\right) \quad and \quad \left(w_{\pi_i^{-1}} < w_{\pi_{i+1}^{-1}} \quad if \quad \pi_{i+1}^{-1} \prec \pi_i^{-1}\right). \tag{2.14}$$

□

Corollary 2.4 *For each permutation $\pi \in S_n$, the number of words $w \in [m]^n$ such as $st_{\prec}(w) = \pi$ is equal to*

$$\left(\!\!\binom{n+1}{m-1-d(\pi^{-1})}\!\!\right) \quad if \quad \prec \quad is \quad <, \tag{2.15}$$

and

$$\left(\!\!\binom{n+1}{m-1-a(\pi^{-1})}\!\!\right) \quad if \quad \prec \quad is \quad >. \tag{2.16}$$

Proof We fix a permutation $\pi \in \mathcal{S}_n$, and we denote $st^{-1}_{\prec,m}(\pi)$ the set of words w of length n in the alphabet $[m]$, where $st_{\prec}(w) = \pi$. According to Corollary 2.3, this set is equal to the set of words

$$\left\{ w_1 \dots w_n \;/\; 1 \le w_{\pi_1^{-1}} \le \dots \le w_{\pi_n^{-1}} \le m \;\; and \;\; \left(w_{\pi_i^{-1}} < w_{\pi_{i+1}^{-1}} \; if \; \pi_{i+1}^{-1} \prec \pi_i^{-1} \right) \right\}. \tag{2.17}$$

By writing c_i for the letter $w_{\pi_i^{-1}}$, this set is in bijection with that of the sequence $c_1, \dots, c_n$ such as

$$1 \le c_1 \; \mathcal{R}_1 \; c_2 \; \mathcal{R}_2 \; \cdots \mathcal{R}_{n-2} \; c_{n-1} \; \mathcal{R}_{n-1} \; c_n \le m, \tag{2.18}$$

where $\mathcal{R}_i$ is the relation $\le$ if $\pi_i^{-1} < \pi_{i+1}^{-1}$, and $\mathcal{R}_i$ is the strict inequality $<$ if $\pi_{i+1}^{-1} < \pi_i^{-1}$. So $d(\pi^{-1})$ inequalities are strict if $\prec$ is $<$, and if $\prec$ is $>$ this number will be $a(\pi^{-1})$. According to Lemma 1.2, the cardinal of this set of sequence is

$$\left(\!\!\binom{n+1}{m-1-d(\pi^{-1})}\!\!\right) \quad if \quad \prec \quad is \quad <, \tag{2.19}$$

and of cardinal

$$\left(\!\!\binom{n+1}{m-1-a(\pi^{-1})}\!\!\right) \quad if \quad \prec \quad is \quad >. \tag{2.20}$$

□

2.1. Word's standardization and that of Young tableaux

Let T be any semistandard Young tableau of n cases. Suppose that the distinct entries of T are taken form $i_1, i_2, \dots, i_m$ (which are in increasing order), and that the weight of T is $\mu = (\mu_1, \mu_2, \dots, \mu_m)$, i.e. that means i_j is repeated μ_j times in T, $\forall j \in [m]$.

We define the standardization of T, denoted by $st(T)$, in informal way as the unique standard Young tableau obtained from T by replacing his n boxes with the integers $1, 2, \dots, n$ according to the following rule.

We start with the smallest entry i_1 in T, we replace the letters i_1, repeated μ_1 times, by $1, 2, \dots, \mu_1$, by going from left to right in the tableau (these letters are never in the same column), then we replace the i_2, repeated μ_2 times, by $\mu_1+1, \mu_1+2, \dots, \mu_1+\mu_2$, and always starting from left to right, etc..., until we replace i_m by $\mu_1+\mu_2+\dots+\mu_{m-1}+1, \dots, \mu_1+\mu_2+\dots+\mu_m = n$. For example if

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 4 \\ \hline 2 & 9 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \qquad \text{then} \qquad st(T) = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 6 \\ \hline 2 & 7 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}\,. \tag{2.21}$$

Let T be a Young tableau, we denote the result of inserting the letter x by row (respectively by column) in T by $T \leftarrow x$ (resp. $x \rightarrow T$), we denote $\odot$ the empty tableau. For each word $w = w_1 w_2 \cdots w_n$ it exists Young tableaux we get by inserting the letters of w (from left to right) by row and by column, which we denote by

$$\odot \leftarrow w := (((\odot \leftarrow w_1) \leftarrow w_2) \leftarrow \cdots) \leftarrow w_n, \tag{2.22}$$

and

$$f_0(w) \rightarrow \odot := w_n \rightarrow (\cdots \rightarrow (w_2 \rightarrow (w_1 \rightarrow \odot))), \tag{2.23}$$

where $f_0(w_1 w_2 \cdots w_n) = w_n w_{n-1} \cdots w_1$.

Let S be a skew Young tableau, we denote $\mathrm{Rect}(S)$ for the rectification of S (i.e. Young tableau which we get from S by "jeu de taquin" (See [1], 1.2)). We define the standardization of a skew Young tableau similarly to that of a Young tableau. We say that i is a descent of a standard skew Young tableau T if $i+1$ is in a line lower than the line of i in T, and say i is ascent of T if i is not a descent.

Proposition 2.5 *The jeu de taquin on a skew standard Young tableau does not change the descents or ascents of this tableau.*

Proof It suffices to show that a stage of elemental taquin does not change descents or ascents. Let $a < b$ two positive integers, we have two cases If $b = a+1$, we notice that a is a descent for both tableaux

$$\begin{array}{cc} & \boxed{a} \\ \boxed{b} & \end{array} \; , \; \begin{array}{c} \boxed{a} \\ \boxed{b} \end{array} \tag{2.24}$$

on the other hand a is an ascent for the two tableaux

$$\begin{array}{cc} & \boxed{b} \\ \boxed{a} & \end{array} \; , \; \boxed{a}\boxed{b} \; . \tag{2.25}$$

□

Before giving the following Proposition, which shows the switching between standardization and "jeu de taquin", we start with a very particular case. Let T be a skew young tableau such as $sh(T) = \begin{array}{cc} & \square \\ \square & \square \end{array}$, we will verify that $st(Rect(T)) = Rect(st(T))$.

Suppose that $T = \begin{array}{cc} & \boxed{b} \\ \boxed{a} & \boxed{c} \end{array}$, with $a \leq c, b \leq c$. There are two cases:

If $a \leq b$, then

$$\begin{array}{cc} & \boxed{b} \\ \boxed{a} & \boxed{c} \end{array} \xrightarrow{Rect} \begin{array}{cc} \boxed{a} & \boxed{b} \\ \boxed{c} & \end{array} \xrightarrow{st} \begin{array}{cc} \boxed{1} & \boxed{2} \\ \boxed{3} & \end{array}, \tag{2.26}$$

$$\begin{array}{cc} & \boxed{b} \\ \boxed{a} & \boxed{c} \end{array} \xrightarrow{st} \begin{array}{cc} & \boxed{2} \\ \boxed{1} & \boxed{3} \end{array} \xrightarrow{Rect} \begin{array}{cc} \boxed{1} & \boxed{2} \\ \boxed{3} & \end{array}. \tag{2.27}$$

If $b < a$, then

$$\begin{array}{cc} & \boxed{b} \\ \boxed{a} & \boxed{c} \end{array} \xrightarrow{Rect} \begin{array}{cc} \boxed{b} & \boxed{c} \\ \boxed{a} & \end{array} \xrightarrow{st} \begin{array}{cc} \boxed{1} & \boxed{3} \\ \boxed{2} & \end{array}, \tag{2.28}$$

$$\begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & b \\ \hline a & c \\ \hline \end{array} \xrightarrow{st} \begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 1 \\ \hline 2 & 3 \\ \hline \end{array} \xrightarrow{Rect} \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array}, \tag{2.29}$$

so we have (in both cases)

$$st\left(Rect\left(\begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & b \\ \hline a & c \\ \hline \end{array}\right)\right) = Rect\left(st\left(\begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & b \\ \hline a & c \\ \hline \end{array}\right)\right). \tag{2.30}$$

Proposition 2.6 *Both rectification and standardization applications commute, i.e.*

$$st(Rect(P)) = Rect(st(P)). \tag{2.31}$$

Proof Just consider a single sliding: $P \triangleright P'$ and to show $st(P) \triangleright st(P')$ (Because $P \triangleright \cdots \triangleright Rect(P)$ gives $st(P) \triangleright \cdots \triangleright st(Rect(P))$ which is rectified, so $st(Rect(S)) = Rect(st(S))$). Let $T = st(P)$. It exists $f : [n] \longrightarrow A$ weakly increasing with $P = f(T)$ (which designates the application of f to all the entries of T) and ($P_{i,j} = P_{i',j'}$ and $j < j'$) $\Longrightarrow T_{i,j} < T_{i',j'}$. Let T' the result of the slip on T to the same box as in $P \triangleright P'$. Then $st(P) = T \triangleright T'$ and it is about showing $T' = st(P')$. We will prove by recurrence on $k \in A$ (the alphabet of P) that

$$B'_k = \left\{(i,j) \ / \ T'_{i,j} \in f^{-1}(k)\right\} = \left\{(i,j) \ / \ P'_{i,j} = k\right\} \tag{2.32}$$

(a horizontal band) and T' is " increasing from left to right" on this band. Let s be the empty box in the slip $P \triangleright P'$ after the sliding of the values $< k$. According to the recurrence hypothesis $\left\{(i,j) \ / \ f(T'_{i,j}) < k\right\} = \bigcup_{l<k} B'_l$ so in the slip $T \triangleright T'$, the empty box after sliding of the boxes have values in $f^{-1}([1, k-1])$ is also s. In P the values k fill a horizontal band $B_k = \{(i,j) \ / \ f(T_{i,j}) = k\}$ and T is increasing from left to right on B_k.

Let $s = (i,j)$, we have two possibilities: $s^{\downarrow} \in B_k$ or $s^{\downarrow} \notin B_k$ (where $(i,j)^{\downarrow} = (i+1, j)$).
(1) If $s^{\downarrow} \in B_k$ then $B'_k = (B_k \coprod \{s\}) \setminus \left\{s^{\downarrow}\right\}$.
(2) If $s^{\downarrow} \notin B_k$ then B'_k is obtained from B_k by sliding to left all the boxes in the line i of s.

In the case (1), $T_{i+1,j} < T_{i,j+1}$ (if it exists) : If $f(T_{i,j+1}) > k$ it is by "increasing" of f, if not $f(T_{i,j+1}) = k$ (that means $(i, j+1) \in B_k$) and $j < j+1 \Longrightarrow T_{i+1,j} < T_{i,j+1}$ by the " increasing from left to right of T on B_k" so the slip operates on $T_{i+1,j}$ which slides to (i,j). In both cases, the values in B_k in T are found in B'_k in T' and they are always increasing from left to right: in the case (1) no value changes of column, and in the case (2) the sliding values do not change order between them, nor with the others (because they do not have common columns before or after, and displacement of each concerns at most one column). □

We denote

$$Skew(w_1, w_2, \ldots, w_n) = \begin{array}{ccccc} & & & & \boxed{w_n} \\ & & & \boxed{} & \\ & & \boxed{} & & \\ & \boxed{w_2} & & & \\ \boxed{w_1} & & & & \end{array}, \tag{2.33}$$

that is, the only skew Young tableau with n boxes distributed in n rows and n columns with only one box at each row and each column, and such as the i -th column contains w_i, for all $i \in [n]$.

According to [1] appendix $A2$ Proposition 1, we have

$$\odot \leftarrow w = w \rightarrow \odot, \tag{2.34}$$

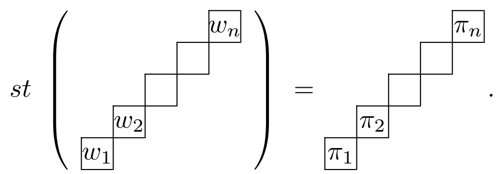
so

$$\odot \leftarrow w = Rect\left(Skew(w_1, w_2, \ldots, w_n)\right) \tag{2.35}$$

and

$$f_0(w) \rightarrow \odot = Rect\left(Skew(w_n, \ldots, w_2, w_1)\right). \tag{2.36}$$

Lemma 2.7 *Let* $w = w_1 w_2 \cdots w_n$ *be a word of length* n*, such as* $st_<(w) = \pi_1\pi_2 \ldots \pi_n$*, then*



(2.37)

Lemma 2.8 *Let* f_0 *be the involution on the set of words of length* n*, which reverses the words, i.e.*

$$f_0\left(w_1 \ldots w_n\right) = w_n \ldots, w_1, \tag{2.38}$$

then

$$st_>\left(f_0(w)\right) = f_0\left(st_<(w)\right) \quad , \quad st_<\left(f_0(w)\right) = f_0\left(st_>(w)\right). \tag{2.39}$$

Proof We will prove that $st_>\left(f_0(w)\right) = f_0(st_<(w))$, the other being analogous. Suppose that $st_>\left(f_0(w)\right) = \pi_1 \ldots \pi_n$, then according to the Lemma 2.3, we have

$$f_0(w)_{\pi_1^{-1}} \leq f_0(w)_{\pi_2^{-1}} \leq \cdots \leq f_0(w)_{\pi_n^{-1}} \;\; and \;\; f_0(w)_{\pi_i^{-1}} < f_0(w)_{\pi_{i+1}^{-1}} \;\; si \;\; \pi_i^{-1} < \pi_{i+1}^{-1}, \tag{2.40}$$

but $f_0(w)_j = w_{n+1-j}, \forall j \in [n]$, so

$$w_{n+1-\pi_1^{-1}} \leq w_{n+1-\pi_2^{-1}} \leq \cdots \leq w_{n+1-\pi_n^{-1}} \tag{2.41}$$

and

$$w_{n+1-\pi_i^{-1}} < w_{n+1-\pi_{i+1}^{-1}} \;\; if \;\; n+1-\pi_i^{-1} > n+1-\pi_{i+1}^{-1}. \tag{2.42}$$

Then $st_<(w) = \sigma_1^{-1}\sigma_2^{-1} \ldots \sigma_n^{-1}$, where σ is the permutation sending $i \mapsto n+1-\pi_i^{-1}$, and so σ^{-1} is the permutation sending $i \mapsto \pi_{n+1-i}$, i.e.

$$st_<(w) = \pi_n\pi_{n-1} \ldots \pi_1 = f_0(\pi_1 \ldots \pi_n). \tag{2.43}$$

But

$$st_>\left(f_0(w)\right) = \pi_1 \ldots \pi_n, \tag{2.44}$$

so

$$st_<(w) = f_0(\pi_1 \ldots \pi_n) = f_0(st_>\left(f_0(w)\right)). \tag{2.45}$$

□

Proposition 2.9 *Let* $w = w_1 w_2 \cdots w_n$ *be a word of length* n*, such as* $st_{\prec}(w) = \pi_1^{\prec}\pi_2^{\prec}\dots\pi_n^{\prec}$*, for* $\prec \in \{<,>\}$*, then*

1.

$$st(\odot \leftarrow w_1 \leftarrow w_2 \leftarrow \cdots \leftarrow w_n) = \odot \leftarrow \pi_1^{<} \leftarrow \pi_2^{<} \leftarrow \cdots \leftarrow \pi_n^{<} \tag{2.46}$$

or

$$st(\odot \leftarrow w) = \odot \leftarrow st_{<}(w). \tag{2.47}$$

2.

$$st(w_n \rightarrow \cdots \rightarrow w_2 \rightarrow w_1 \rightarrow \odot) = \pi_n^{>} \rightarrow \cdots \rightarrow \pi_2^{>} \rightarrow \pi_1^{>} \rightarrow \odot \tag{2.48}$$

or

$$st(f_0(w) \rightarrow \odot) = f_0(st_{>}(w)) \rightarrow \odot. \tag{2.49}$$

Proof Let w be a word of length n, we will prove that $st(\odot \leftarrow w) = \odot \leftarrow st_{<}(w)$. Suppose that $st_{<}(w) = \pi_1^{<}\pi_2^{<}\dots\pi_n^{<}$, then according to the two previous lemmas, we have

$$\begin{aligned} st(\odot \leftarrow w) &= st\left(Rect\left(skew(w_1, w_2, \dots, w_n)\right)\right) \\ &= Rect\left(st\left(skew(w_1, w_2, \dots, w_n)\right)\right) \\ &= Rect\left(skew(\pi_1^{<}, \pi_2^{<}, \dots, \pi_n^{<})\right) \\ &= \odot \leftarrow \pi_1^{<} \leftarrow \pi_2^{<} \leftarrow \cdots \leftarrow \pi_n^{<} \\ &= \odot \leftarrow st_{<}(w). \end{aligned} \tag{2.50}$$

On the other hand, we have

$$\begin{aligned} st(f_0(w) \rightarrow \odot) &= st(\odot \leftarrow f_0(w)) = \odot \leftarrow st_{<} f_0(w) \\ &= \odot \leftarrow f_0\left(st_{>}(w)\right) = f_0\left(st_{>}(w)\right) \rightarrow \odot. \end{aligned} \tag{2.51}$$

□

Proposition 2.10 *We have a bijection* *, *called Schensted correspondence, between the set of words of length* n *with letters in* $[m]$*, and the set of pairs of tables* (P, Q)*, of the same form with* n *boxes, where* P *is semistandard with entries in* $[m]$*, and* Q *is standard.*

According to Propositions 2.10 and 2.9, we have the following Corollary.

Corollary 2.11 *If the word* w *corresponds by RSK to the pair of tableaux* (P, Q)*, then the word* $st_{<}(w)$ *corresponds the pair of standard tableaux* $(st(P), Q)$*.*

If the word w' *corresponds (by Schensted correspondence with insertions by columns) to the pair of tableaux* (P', Q') *, so the word* $st_{>}(w')$ *corresponds to the standard pair of tableaux* $(st(P'), Q')$*.*

*This bijection is actually older than RSK; it was described by Schensted [4].

Definition 2.12 *(The descents of a permutation or Young tableau)*

1. We say that $i \in [n-1]$ is a descent of the permutation π if $\pi_i > \pi_{i+1}$.

2. We say that i is a descent of a standard Young tables T if $i+1$ is in a line lower than the line of i in T.

We denote $des(\pi)$ (respectively $des(T)$) the set of descents of π (respectively T).

Proposition 2.13 *Let P be a skew standard Young tableau with n cases, and let ($des P$) be the set of descents of P (cf 2.12), we denote its cardinal by $d(P)$. Then the number of semistandard skew Young tableau (with entries in $[m]$) whose standardization is P, is*

$$\left(\!\!\binom{n+1}{m-1-d(P)}\!\!\right). \tag{2.52}$$

Proof Let P be a skew standard Young tableau with n cases, and suppose that

$$P_{i_k,j_k} = k, \forall k \in [n], \tag{2.53}$$

and

$$des(P) = \{P_{i,j} \ / \ (i,j) \in D\}. \tag{2.54}$$

Let T be a skew Young tableau with n cases and entries in $[m]$, then according to the definition of the standardization of a Young tableau, we have $st(T) = P$ iff
$1 \leq T_{i_1,j_1} \leq T_{i_2,j_2} \leq \cdots \leq T_{i_n,j_n} \leq m$, with $T_{i_k,j_k} < T_{i_{k+1},j_{k+1}}$ if $(i_k,j_k) \in D$, $\forall k \in [n-1]$. So the number of semistandard skew Young tableaux with n cases and entries in $[m]$, whose standardization is P, is equal to the number of preceding sequences (with $d(P)$ lower strict fixed positions), and according to Lemma 1.2, this number is equal to

$$\left(\!\!\binom{n+1}{m-1-d(P)}\!\!\right). \tag{2.55}$$

□

Definition 2.14 *If two words w, w' correspond (by RSK) to the couple of Young tableau $(P,Q), (P',Q')$ respectively, then we will say that w, w' are Knuth equivalent if $P = P'$.*

Corollary 2.15 *If two words are Knuth equivalent, then their standardization, for $\prec$, are also Knuth equivalent.*

Proof

Suppose that the two words w, w' are Knuth equivalents, then the insertion by line (resp. Column) of these two words gives the same tableau, that means

$$\odot \leftarrow w = \odot \leftarrow w' \quad (\text{ resp. } \ w \rightarrow \odot = w' \rightarrow \odot \ ), \tag{2.56}$$

so

$$st(\odot \leftarrow w) = st(\odot \leftarrow w') \quad (\text{ resp. } \ st(w \rightarrow \odot) = st(w' \rightarrow \odot) \ ), \tag{2.57}$$

which gives us according to Proposition 2.9

$$\odot \leftarrow st_{<}(w) = \odot \leftarrow st_{<}(w') \quad (\text{ resp. } \ st_{>}(w) \rightarrow \odot = st_{>}(w') \rightarrow \odot \ ); \tag{2.58}$$

in other words, the two words $st_{<}(w), st_{<}(w')$ (resp. $st_{>}(w), st_{>}(w')$) are Knuth equivalent.

□

3. An application of word standardization

In this section, we will establish a proof of the following equality which is well known (cf.[2], 6.38 given without a proof), using the notion of standardization of words.

$$\forall l \in \{0,1,\ldots,n-1\}\,, \quad \left\langle {n \atop l} \right\rangle = d_{n,l} = a_{n,l} = \sum_{0\le k\le l} (-1)^k \binom{n+1}{k} (l+1-k)^n, \tag{3.1}$$

where $d_{n,l}$ (respectively $a_{n,l}$) denote the number of permutations in S_n with l descents (respectively l ascents), and this number is called $\left\langle {n \atop l} \right\rangle$ the Eulerian number.

For each permutation $\pi \in S_n$, we denote $st^{-1}_{\prec,m}(\pi_1\pi_2\ldots\pi_n)$ for the set of words $w \in [m]^n$, of length n in the alphabet $[m]$, whose standard, for $\prec$, is the word $\pi_1\pi_2\ldots\pi_n$.

So, as each word has a unique standardization (for $<$ as for $>$)

$$[m]^n = \coprod_{\pi\in S_n} st^{-1}_{<,m}(\pi_1\pi_2\ldots\pi_n) = \coprod_{\pi\in S_n} st^{-1}_{>,m}(\pi_1\pi_2\ldots\pi_n) \tag{3.2}$$

where $\coprod$ refers to the disjointed union. As

$$\# st^{-1}_{<,m}(\pi_1\pi_2\ldots\pi_n) = \left(\!\!\binom{n+1}{m-1-d(\pi^{-1})}\!\!\right), \tag{3.3}$$

and

$$\# st^{-1}_{>,m}(\pi_1\pi_2\ldots\pi_n) = \left(\!\!\binom{n+1}{m-1-a(\pi^{-1})}\!\!\right), \tag{3.4}$$

so

$$\begin{aligned} m^n = \#[m]^n &= \sum_{\pi\in S_n} \# st^{-1}_{<,m}(\pi_1\pi_2\ldots\pi_n) = \sum_{\pi\in S_n} \left(\!\!\binom{n+1}{m-1-d(\pi)}\!\!\right) \\ &= \sum_{\pi\in S_n} \# st^{-1}_{>,m}(\pi_1\pi_2\ldots\pi_n) = \sum_{\pi\in S_n} \left(\!\!\binom{n+1}{m-1-a(\pi)}\!\!\right), \end{aligned} \tag{3.5}$$

Then if we take the generating function of the sequence $(m^n)_{m\ge1}$, we will have

$$\begin{aligned} \sum_{m\ge1} m^n X^m &= \sum_{\substack{m\ge1 \\ \pi\in S_n}} \# st^{-1}_{<,m}(\pi_1\pi_2\ldots\pi_n) X^m = \sum_{\substack{m\ge1 \\ \pi\in S_n}} \# st^{-1}_{>,m}(\pi_1\pi_2\ldots\pi_n) X^m \\ &= \sum_{\substack{m\ge1 \\ \pi\in S_n}} \left(\!\!\binom{n+1}{m-1-d(\pi)}\!\!\right) X^m = \sum_{\substack{m\ge1 \\ \pi\in S_n}} \left(\!\!\binom{n+1}{m-1-a(\pi)}\!\!\right) X^m \\ &= \frac{X}{(1-X)^{n+1}} \sum_{\pi\in S_n} X^{d(\pi)} = \frac{X}{(1-X)^{n+1}} \sum_{\pi\in S_n} X^{a(\pi)}. \end{aligned} \tag{3.6}$$

By writing this result in the form

$$(1-X)^{n+1}.\sum_{m\geq 1} m^n X^{m-1} = \sum_{i=0}^{n-1} d_{n,i} X^i = \sum_{i=0}^{n-1} a_{n,i} X^i, \tag{3.7}$$

so

$$\left(\sum_{k=0}^{n+1}(-1)^k \binom{n+1}{k} X^k\right)\left(\sum_{m\geq 1} m^n X^{m-1}\right) = \sum_{i=0}^{n-1} d_{n,i} X^i, \tag{3.8}$$

then by taking the coefficient of X^l:

$$\left\langle {n \atop l} \right\rangle = \sum_{k=0}^{l}(-1)^k \binom{n+1}{k}(l+1-k)^n, \tag{3.9}$$

which is the searched formula.

Acknowledgment

I would like to express my sincere appreciation to Prof. Dr. Marc van Leeuwen for his valuable and constructive suggestions during the planning and development of this research work. I extend my thanks and gratitude to him for his precious time and the feedback he provided. I would like to express also my sincere thanks to the editor and the anonymous referees for their helpful comments and suggestions.

References

[1] Fulton W. Young tableaux, volume 35 of London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry.

[2] Graham RL, Knuth DE, Patashnik O. Concrete mathematics. Addison-Wesley Publishing Company, Reading, MA, second edition, 1994. A foundation for computer science.

[3] Knuth DE. Permutations, matrices, and generalized Young tableaux. Pacific Journal of Mathematics, 1970; 34: 709-727.

[4] Schensted C. Longest increasing and decreasing subsequences. Canadian Journal of Mathematics, 1961; 13: 179-191.

[5] Stanley RP. Enumerative combinatorics. Vol. 1, volume 49 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1997.