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Nilpotent varieties and metabelian varieties

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Abstract: We deal with varieties of nonassociative algebras having polynomial growth of codimensions. We describe some results obtained in recent years in the class of left nilpotent algebras of index two. Recently the authors established a correspondence between the growth rates for left nilpotent algebras of index two and the growth rates for commutative or anticommutative metabelian algebras that allows to transfer the results concerning varieties of left nilpotent algebras of index two to varieties of commutative or anticommutative metabelian algebras.

Key words: Varieties, codimension growth

1. Introduction

Let F be a field of characteristic zero and $F\{X\}$ the free nonassociative algebra on a countable set $X = \{x_1, x_2, \dots\}$ over F. Let A be a nonnecessarily associative algebra and $Id(A)$ the T-ideal of $F\{X\}$ of polynomial identities of A. An effective way of measuring the polynomial identities satisfied by A is provided by its sequence of codimensions $\{c_n(A)\}_{n\geq 1}$, that, in characteristic zero, gives an actual quantitative measure of the identities satisfied by a given algebra. In particular, a general strategy in the study of $Id(A)$ is that of studying the space of multilinear polynomials in n fixed variables modulo the identities of the algebra A through the representation theory of the symmetric group S_n on n symbols. Then one attaches to $Id(A)$ a sequence of S_n-modules, $n = 1, 2, \dots$, and studies the corresponding sequence of characters.

More precisely, for every $n \geq 1$, let P_n be the space of multilinear polynomials in the variables $x_1, \dots, x_n$. Since char $F = 0$, $Id(A)$ is determined by the multilinear polynomials it contains; hence the relatively free algebra $F\{X\}/Id(A)$ is determined by the sequence of subspaces $\{P_n/(P_n \cap Id(A))\}_{n\geq 1}$. The symmetric group S_n acts on P_n by permuting the variables: if $\sigma \in S_n$ and $f(x_1, \dots, x_n) \in P_n$, then

$$\sigma f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

The space $P_n \cap Id(A)$ is invariant under this action and one studies the structure of $P_n(A) = P_n/(P_n \cap Id(A))$ as an S_n-module. The S_n-character of $P_n(A)$, denoted $\chi_n(A)$, is the n-th cocharacter of A. By complete reducibility one writes

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda,$$

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where χ_λ is the irreducible S_n-character corresponding to the partition $\lambda \vdash n$ of n and $m_\lambda \geq 0$ is the corresponding multiplicity (see for example [15], [13] for the representation theory of the symmetric group). The integer

$$c_n(A) = \dim P_n/(P_n \cap Id(A))$$

is called the n-th codimension of A and the growth function determined by the sequence of integers $\{c_n(A)\}_{n\geq 1}$ is the codimension growth of the algebra A.

In the language of varieties if $\mathcal{V} = \mathrm{var}(A)$ is the variety generated by an algebra A, then the growth of $\mathcal{V}$ is the codimension growth of the algebra A. Also we write $Id(\mathcal{V}) = Id(A)$ and $c_n(\mathcal{V}) = c_n(A)$.
Let us recall that a variety $\mathcal{V} = var(A)$ has polynomial growth if there exist α, t such that $c_n(\mathcal{V}) = c_n(A) \leq \alpha n^t$, for all $n \geq 1$; moreover $\mathcal{V}$ has exponential growth if there exist $\alpha, \beta > 0$ and $a, b > 1$ such that $\alpha a^n \leq c_n(\mathcal{V}) = c_n(A) \leq \beta b^n$, for all n. The sequence $c_n(A)$, $n = 1, 2 \ldots$, in general has overexponential growth. For instance, if $F\{X\}$ is the free (nonassociative) algebra on a set X, $|X| \geq 2$ then $c_n(F\{X\}) = C_n n!$ where $C_n = \frac{1}{n}\binom{2n-2}{n-1}$ is the n-th Catalan number. Moreover, for the free associative algebra $F\langle X\rangle$ and the free Lie algebra $L\langle X\rangle$ we have $c_n(F\langle X\rangle) = n!$ and $c_n(L\langle X\rangle) = (n-1)!$, respectively. Nevertheless, there is a wide class of algebras with exponentially bounded codimension growth.

The first result on the asymptotic behavior of $c_n(A)$ was proved by Regev in [31]. He showed that if A is an associative algebra satisfying a nontrivial polynomial identity, then the sequence of codimensions is exponentially bounded. Later, Kemer in [16] proved that for such algebras, the sequence $c_n(A)$ is either polynomially bounded or grows exponentially.

In case $\mathcal{V}$ is a variety of nonassociative algebras, the sequence of codimensions has a much more involved behavior and can have overexponential growth. This was first proved by Volichenko in [32] who showed that the variety of Lie algebras satisfying the identity $[[x_1, x_2, x_3], [x_4, x_5, x_6]] \equiv 0$ has overexponential growth.

Along this line, Petrogradsky in [29] exhibited a whole scale of overexponential functions specifying the overexponential behavior of the identities of polynilpotent Lie algebras. Moreover, by results of Drensky in [5] and Giambruno and Zelmanov in [14], there exist varieties of Jordan algebras with overexponential growth.

If the sequence of codimensions $c_n(\mathcal{V}) = c_n(A)$ is exponentially bounded then one naturally defines, $\exp(\mathcal{V}) = \exp(A)$, the exponent of the variety $\mathcal{V} = \mathrm{var}(A)$. Let

$$\overline{\exp(\mathcal{V})} = \lim\sup_{n\to\infty} \sqrt[n]{c_n(\mathcal{V})}, \qquad \underline{\exp(\mathcal{V})} = \lim\inf_{n\to\infty} \sqrt[n]{c_n(\mathcal{V})}$$

the upper and lower exponent respectively of the variety $\mathcal{V}$. If $\overline{\exp(\mathcal{V}} = \underline{\exp(\mathcal{V})}$ then

$$\exp(\mathcal{V}) = \overline{\exp(\mathcal{V})} = \underline{\exp(\mathcal{V})}.$$

In 1999 Giambruno and Zaicev in [11] and [12] showed that for an associative PI-algebra A the exponent $exp(A)$ exists and is an integer called the PI-exponent of the algebra A.

It is well known that finite dimensional nonnecessarily associative algebras have exponentially bounded codimensions (see [1]) and in case of Lie algebras in [9], [10], [33] it was shown that their exponential growth is an integer. This is not an expected behavior for Lie algebras, namely, Mishchenko and Zaicev in [34] constructed a Lie algebra with exponential growth of the codimensions strictly between 3 and 4. This result was extended in [28] to a simple infinite dimensional Lie algebra of Cartan type. On the other hand no intermediate growth and no exponential growth between 1 and 2 is allowed ([18],[21]).

For general nonassociative algebras only few results have been proved so far. In [30], in the classes of absolutely free algebras, free commutative and free anticommutative nonassociative algebras the exponential generating functions are explicitly computed for solvable algebras of fixed length, completely solvable algebras of fixed length, left nilpotent algebras of fixed index. One derives results for codimension growth sequences. In particular, the exponential generating function for absolutely free metabelian algebras S^2 is $C(S^2,z) = z(1-z)/(1-2z)$ which implies that $c_n(S^2) = n!(2+o(1))^n$ and the exponential generating function for 2-step right nilpotent absolutely free algebras N^2 is $C(N^2,z) = z/(1-z)$ which implies that $c_n(N^2) = n!$.

In [8] for any real number $\alpha > 1$ it was constructed an algebra whose sequence of codimensions has exponential growth equal to α. Moreover, there exist examples of algebras with intermediate growth of the codimensions. In fact, in [7] for any real number $0 < \beta < 1$ an algebra was constructed whose sequence of codimensions grows as n^{n^β}. Anyway, in [7] it was proved that the codimensions of a finite dimensional algebra A are either polynomially bounded or grow exponentially. So, no intermediate growth is allowed in the finite dimensional case.

In this paper, first we present some results on varieties of algebras having polynomial growth. Then we classify the growth of left nilpotent varieties of index 2, that are the varieties of algebras satisfying the identity $x(yz) \equiv 0$, of at most cubic growth. Notice that modulo the identity $x(yz) \equiv 0$ all nonzero monomials of the free algebra are left normed, i.e. are of the type $(((x_1x_2)x_3)\dots)$. Since we shall be working modulo such identity we shall omit the parenthesis in left normed monomials, hence we shall write $(((x_1x_2)x_3)\dots x_n) = x_1x_2\dots x_n$, and xy^2 for xyy. Finally we recall the correspondence between varieties of left nilpotent algebras of index two and varieties of commutative or anticommutative metabelian algebras that allow us to transfer some results of left nilpotent varieties of index 2 to varieties of commutative or anticommutative metabelian algebras.

2. Polynomial growth

In this section we describe the varieties $\mathcal{V}$ of associative, Lie and Leibniz algebras such that the sequence of codimension is polynomially bounded. We recall that a Leibniz algebra over a field F is a nonassociative algebra with a bilinear multiplication, satisfying the Leibniz identity $(xy)z = (xz)y + x(yz)$.

In [16] Kemer characterized the varieties of associative algebras $\mathcal{V}$ such that $c_n(\mathcal{V})$ is polynomially bounded in terms of their cocharacter sequence. A similar result was proved for varieties of Lie algebras in [2] and for varieties of Leibniz algebras in [4].

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r) \vdash n$ be a partition on n. The characterization obtained is the following

Theorem 2.1 *For a variety of associative, Lie or Leibniz algebras $\mathcal{V}$ the following conditions are equivalent*

1) $c_n(\mathcal{V})$ *is polynomially bounded.*

2) *there exists a constant q such that*

$$\chi_n(A) = \sum_{\substack{\lambda\vdash n \\ |\lambda|-\lambda_1\le q}} m_\lambda \chi_\lambda$$

for all $n \geq 1$.

Theorem 2.2 *Let $\mathcal{V}$ be a variety of associative or Lie algebras. If $c_n(\mathcal{V}) < C(2-\varepsilon)^n$, for some constants C and $0 < \varepsilon < 1$, then $c_n(A)$ is polynomially bounded.*

Another characterization of varieties of associative algebras can be given in terms of the Grassmann algebra G and the algebra UT_2 of 2×2 upper triangular matrices.

Theorem 2.3 *(see [17]) A variety of associative algebras $\mathcal{V}$ is polynomially bounded if and only if $G, UT_2 \notin \mathcal{V}$.*

Let $\mathcal{N}_t\mathcal{A}$ denote the variety of Lie algebras defined by the identity

$$((x_1, x_2), \ldots, (x_{2t+1}, x_{2t+2})) \equiv 0 \tag{2.1}$$

and by $\widetilde{N_tA}$ the variety of Leibniz algebras determined by the same identity. Obviously, $N_tA \subset \widetilde{N_tA}$.

For varieties of Lie algebras we have the following characterization

Theorem 2.4 *(see [20]) A variety of Lie algebras $\mathcal{V}$ has polynomial growth if and only if*

$$N_2A \not\subset \mathcal{V} \subset N_tA$$

for some $t \geq 1$.

An analogous result holds for varieties of Leibniz algebras. Let us denote by $\widetilde{\mathcal{V}_1}$ the variety defined by the identity

$$x_1(x_2x_3)(x_4x_5) \equiv 0.$$

Theorem 2.5 *(see [3]) Let $\mathcal{V}$ be a variety of Leibniz algebras. Then $\mathcal{V}$ has polynomial growth if and only if there exists $t \geq 1$ such that*

$$N_2A, \widetilde{V_1} \not\subset V \subset \widetilde{N_tA}$$

3. The variety of left nilpotent algebras of index two

Let $\mathcal{V} = {}_2\mathcal{N}$ be the variety of left nilpotent algebras of index two that is the variety determined by the identity $x(yz) \equiv 0$.

The interest in this variety is motivated by the following

Remark 1 *Let $\mathcal{V}$ be the variety of algebras satisfying the identity $x(yz) \equiv \alpha(xy)z$, for some $\alpha \in \mathbb{R}$. Then either*

1. *$\mathcal{V}$ is nilpotent, or*
2. *$\mathcal{V}$ is the variety of associative algebras, or*
3. *$\mathcal{V} = {}_2\mathcal{N}$.*

Proof Clearly, if f $\alpha = 1$ we obtain the variety of associative algebras and in case $\alpha = 0$ we get the variety ${}_2\mathcal{N}$. So suppose that $\alpha \neq 0, 1$. Then working modulo $x(yz) \equiv \alpha(xy)z$, we obtain

$$(xy)(zt) \equiv \alpha((xy)z)t \equiv (x(yz))t \equiv \frac{1}{\alpha}x((yz)t) \equiv \frac{1}{\alpha^2}x(y(zt)) \equiv \frac{1}{\alpha}(xy)(zt).$$

Hence

$$(xy)(zt) \equiv 0.$$

Moreover, we have that

$$((xy)z)t \equiv \frac{1}{\alpha}(xy)(zt) \equiv 0,$$

$$t((xy)z) \equiv \frac{1}{\alpha}t(x(yz)) \equiv (tx)(yz) \equiv 0,$$

$$(z(xy))t \equiv 0,$$

$$t(z(xy)) \equiv 0.$$

It follows that $\mathcal{V}$ is a nilpotent variety and $c_n(\mathcal{V}) = 0$, for all $n \geq 4$.

□

The asymptotic behavior of the codimensions of a unitary algebra was described by Drensky in [6]. He proved the following

Theorem 3.1 *Let A be an associative algebra or a Lie algebra or a Jordan algebra whose sequence of codimension is polynomially bounded. Then*

$$c_n(A) = Cn^k + \mathrm{O}(n^{k-1}),$$

for some integer k and for some rational number C.

It follows that for classical algebras (associative algebras, Lie algebras and Jordan algebras) there are no varieties of fractional polynomial growth. This is not more true for the variety ${}_2\mathcal{N}$, in fact Mishchenko and Zaicev in [26] gave examples of varieties $\mathcal{V}_\alpha \subseteq {}_2\mathcal{N}$ with fractional polynomial growth. In particular they proved the following

Theorem 3.2 *For any real number α, $3 < \alpha < 4$, there exists a variety of algebras $\mathcal{V}_\alpha \subseteq {}_2\mathcal{N}$, such that, for sufficiently large n, the following condition holds*

$$C_1 n^\alpha < c_n(\mathcal{V}_\alpha) < C_2 n^\alpha,$$

where C_1, C_2 are positive constants.

Motivated by this result we tried to classify all possible growth of varieties $\mathcal{V}$ such that $c_n(\mathcal{V}) < Cn^\alpha$, with $0 < \alpha < 3$, for some constant C. We obtained the following (see [23], [22])

Theorem 3.3 *Let $\mathcal{V}$ be a variety of algebras. If $c_n(\mathcal{V}) \leq Cn^\alpha$ for some constants $C > 0$ and $0 < \alpha < 1$, then, for n large, $c_n(\mathcal{V}) \leq 1$.*

Theorem 3.4 *Let $\mathcal{V}$ be a variety of commutative or anticommutative (nonnecessarily associative) algebras. If $c_n(\mathcal{V}) \leq Cn^\alpha$ for some constant $C > 0$ and $1 \leq \alpha < 2$, then either, for n large, $c_n(\mathcal{V}) \leq 1$ or* $\lim_{n\to\infty} \log_n c_n(\mathcal{V}) = 1$.

For the growth rates of the sequence of codimensions of the variety ${}_2\mathcal{N}$, we reached the following

Theorem 3.5 *(see [24]) Let $\mathcal{V}$ be a variety of algebras satisfying the identity $x(yz) = 0$. If $c_n(\mathcal{V}) \leq Cn^{\alpha}$ for some constant $C > 0$ and $1 \leq \alpha < 2$, then $c_n(\mathcal{V}) \leq C_1 n$ for some constant $C_1 > 0$.*

Theorem 3.6 *(see [24]) Let $\mathcal{V}$ be a variety of algebras satisfying the identity $x(yz) = 0$. If $c_n(\mathcal{V}) \leq Cn^{\alpha}$ for some constant $C > 0$ and $2 \leq \alpha < 3$, then $c_n(\mathcal{V}) \leq C_1 n^2$ for some constant $C_1 > 0$.*

Recently in [25] we found a correspondence between varieties of left nilpotent algebras of index two and varieties of commutative or anticommutative metabelian algebras that establish correlations between the growth rates of these varieties. In particular, we constructed a metabelian commutative or anticommutative algebra and a left nilpotent algebras of index two that share the same behavior of the sequence of codimensions. This allow us to transfer the above results to varieties of commutative or anticommutative metabelian algebras.

4. Metabelian commutative or anticommutative algebra defined from a special left nilpotent algebra

In this section, starting by A, a left nilpotent algebra of index two with some special condition, we construct a metabelian commutative algebra A^+ and a metabelian anticommutative algebra A^- and we will give a relation between the sequences of the codimensions $c_n(A)$ and $c_n(A^+)$ or $c_n(A^-)$.

Let A be a left nilpotent algebra of index two, that is an algebra satisfying the identity $x(yz) \equiv 0$. Let $A_0 = \{a \in A| \ ba = 0, \ \forall b \in A\}$ be the right annihilator of A and let $\{a_1, a_2, \ldots\}$ be a basis of A_0. We complete this basis to a basis $B = \{a_1, a_2, \ldots b_1, b_2, \ldots\}$ of the whole algebra A. Since $a_i \in A_0$ it follows that

$$a_i a_j = b_i a_j = 0.$$

Let us assume that in A holds "the special condition"

$$b_i b_j = 0$$

for all i, j. Hence, from the identity $x(yz) \equiv 0$, it follows that

$$a_i b_j = \sum_k \alpha_{ij}^k a_k = c_{ij}.$$

Let A^+ be the algebra with the same basis $B = \{a_1, a_2, \ldots b_1, b_2, \ldots\}$ and with the following multiplication table: for all i, j

$$a_i a_j = 0, \ \ a_i b_j = b_j a_i = \sum_k \alpha_{ij}^k a_k = c_{ij}, \quad b_i b_j = 0.$$

The algebra A^+ satisfies the identities $(xy)(zt) \equiv 0$, $xy \equiv yx$, and so is a metabelian commutative algebra. In a similar way we can construct a metabelian anticommutative algebra.

Let A^- be the algebra with basis $B = \{a_1, a_2, \ldots b_1, b_2, \ldots ...\}$ and with the following multiplication table, for all i, j

$$a_i a_j = 0, \ \ a_i b_j = -b_j a_i = \sum_k \alpha_{ij}^k a_k = c_{ij}, \quad b_i b_j = 0.$$

The algebra A^- is a metabelian anticommutative algebra since satisfies the identities $(xy)(zt) \equiv 0$ and $xy \equiv -yx$.

In [25, Theorem 1] we found a relation between the sequence of codimension of A^+ or A^- and the sequence of codimensions of A. We obtained the following

Theorem 4.1

$$\frac{1}{2}c_n(A) \leq c_n(A^{\pm}) \leq c_n(A).$$

5. Left nilpotent algebra defined from metabelian commutative or anticommutative algebra

In this section, starting from a metabelian commutative algebra A^+ (or a metabelian anticommutative algebra A^-), we construct a left nilpotent algebra of index two and we will provide a relation between the sequences of the codimensions $c_n(A^+)$ (or $c_n(A^-)$) and $c_n(A)$.

Let A^+ be a metabelian commutative algebra satisfying the identities

$$(xy)(zt) \equiv 0,\ xy \equiv yx.$$

Let A_0^+ be the span of all products of two elements of A^+ and $\{a_1, a_2, \ldots\}$ a basis of A_0^+. Let us complete this basis to a basis $B = \{a_1, a_2, \ldots\ b_1, b_2, \ldots\ ...\}$ of the whole algebra A^+. Since A^+ is metabelian it follows that

$$a_i a_j = 0$$

for any i, j and for the other products of the basis elements we have that

$$a_i b_j = b_j a_i = \sum_k \alpha_{i,j}^k a_k = c_{i,j}, \quad b_i b_j = b_j b_i = \sum_k \beta_{i,j}^k a_k = d_{i,j}.$$

Let A be the algebra with basis $\{a_1, a_2, \ldots, b_1, b_2, \ldots\}$ and with the following multiplication table

$$a_i a_j = 0, \ \ a_i b_j = \sum_k \alpha_{i,j}^k a_k = c_{i,j}, \ \ b_i a_j = 0, \ \ b_i b_j = b_j b_i = \sum_k \beta_{i,j}^k a_k = d_{i,j}.$$

Clearly the algebra A satisfies the identity $x(yz) \equiv 0$, and so is a left nilpotent algebra of index two.

Let us now denote by A^- a metabelian anticommutative algebra, so A^- satisfies the identities $(xy)(zt) \equiv 0$, $xy \equiv -yx$. Let A_0^- be the span of all products of two elements of A^- and $\{a_1, a_2, \ldots\}$ a basis of A_0^-. Let us complete this basis to a basis $\{a_1, a_2, \ldots\ b_1, b_2, \ldots\}$ of the whole algebra A^-. As before we have that $a_i a_j = 0$ for any i, j, and

$$a_i b_j = -b_j a_i = \sum_k \alpha_{i,j}^k a_k = c_{i,j}, \quad b_i b_j = -b_j b_i \sum_k \beta_{i,j}^k a_k = d_{i,j}.$$

We construct a left nilpotent algebra of index two denoted again by A with basis $\{a_1, a_2, \ldots, b_1, b_2, \ldots\}$ and with the following multiplication table

$$a_i a_j = 0, \ \ a_i b_j = \sum_k \alpha_{i,j}^k a_k = c_{i,j}, \ \ b_i a_j = 0,, \ \ b_i b_j = -b_j b_i = \sum_k \beta_{i,j}^k a_k = d_{i,j}.$$

Also in this case we have a relation between the sequence of codimension of A and the sequence of codimensions of $A^{\pm}$.

Theorem 5.1 *(see [25])*

$$c_n(A^{\pm}) \leq c_n(A) \leq 2c_n(A^{\pm}).$$

6. Consequences

As a consequence of the previous correspondence we obtain the following

Corollary 6.1 *1. There are no varieties of commutative (or anticommutative) metabelian algebras such that $C_1 n^{\alpha} \leq c_n(\mathcal{V}) \leq C_2 n^{\alpha}$ with C_1, C_2 constants and $1 < \alpha < 2$,*

2. There are no varieties of commutative (or anticommutative) metabelian algebras such that $C_1 n^{\alpha} \leq c_n(\mathcal{V}) \leq C_2 n^{\alpha}$ with C_1, C_2 constants and $2 < \alpha < 3$.

Proof Let us assume that there exists a metabelian commutative (or anticommutative) algebra such that $c_n(\mathcal{V}) \leq Cn^{\alpha}$ with $1 < \alpha < 2$ or $c_n(\mathcal{V}) \leq Cn^{\alpha}$ with $2 < \alpha < 3$. By Theorem 5.1 it follows that there exists a left nilpotent algebra of index two with the same behavior of codimensions. This contradicts the results of Theorems 3.5 and 3.6. □

Contrary to what happens for varieties of associative or Lie algebras where no exponential growth between 1 and 2 and no intermediate growth is allowed, for the variety ${}_2\mathcal{N}$ the situation is different. In fact, in [8], for any real number $\alpha > 1$, the authors found a subvariety of ${}_2\mathcal{N}$ whose exponential growth is equal to α and in [7] a sequence of subvarieties of ${}_2\mathcal{N}$ of intermediate growth was constructed. The results are the following

Theorem 6.2 *(see [8]). For any real number $\alpha > 1$, there exists a variety $\mathcal{V}_\alpha \subseteq {}_2\mathcal{N}$ such that $\exp(\mathcal{V}_\alpha) = \alpha$.*

Theorem 6.3 *(see [7]). For any real number α, $0 < \alpha < 1$, there exists a variety $\mathcal{V}_\alpha \subseteq {}_2\mathcal{N}$ such that*

$$\lim_{n\to\infty} \log_n \log_n c_n(\mathcal{V}_\alpha) = \alpha,$$

i.e. the sequence $c_n(\mathcal{V}_\alpha)$ behaves like $n^{n^{\alpha}}$, $n = 1, 2, \ldots$.

In [27] the same results have been proved for varieties of commutative or anticommutative metabelian algebras. Since in the construction of the previous varieties were considered left nilpotent algebras of index two satisfying the special condition of section 4 the results proved in [27] can be obtained as a consequence of our correspondence. In fact, from the relation between $c_n(A^{\pm})$ and $c_n(A)$ proved in Theorem 4.1 and from standard arguments it follows that $\exp(A^{\pm}) = \exp(A)$. Hence we obtain the following

Corollary 6.4 *For any real number $\alpha > 1$, there exists a variety $\mathcal{V}_\alpha$ of commutative (or anticommutative) metabelian algebras such that $\exp(\mathcal{V}_\alpha) = \alpha$.*

Corollary 6.5 *For any real number α, $0 < \alpha < 1$, there exists a variety $\mathcal{V}_\alpha$ of commutative (or anticommutative) metabelian algebras such that*

$$\lim_{n\to\infty} \log_n \log_n c_n(\mathcal{V}_\alpha) = \alpha,$$

i.e. the sequence $c_n(\mathcal{V}_\alpha)$ behaves like $n^{n^{\alpha}}$, $n = 1, 2, \ldots$.

Let us finish by giving an example of variety of commutative or anticommutative metabelian algebras with fractional polynomial growth α, $3 < \alpha < 4$.

Recall that in case of associative algebras, or Lie algebras or Jordan algebras if a variety has polynomial growth, then the sequence of codimensions asymptotically behaves like Cn^k, for some constant C and for some integer k, (see [6]). Anyway in [26] examples of varieties with fractional polynomial growth were constructed.

Using this result and Theorem 4.1 we can give the following

Example. Let $w = w_1 w_2 \cdots w_m$ be an associative word over the alphabet $\{0, 1\}$. Let $A(w)$ denote the algebra with basis $\{a, b, z_1, z_2, \ldots, z_{m+1}\}$ satisfying the following relations:

1. $z_i a = \pm a z_i = (1 - w_i) z_{i+1}, \;\; i = 1, 2, \ldots, m;$
2. $z_i b = \pm b z_i = w_i z_{i+1}, \;\; i = 1, 2, \ldots, m;$
3. $a^2 = b^2 = ab = ba = z_i z_j = 0, \;\; \forall i, j.$

For any m and s positive integers, with $1 \le s \le \sqrt{m+1}$, let $w(m, s)$ be the word of length m such that its s-th and m-th letters are units and all other letters are zeros. Let $\mathcal{V}_m = \mathrm{var}(A_m)$ be the variety generated by the algebra $A(m) = A(w(m,1)) \oplus A(w(m,2)) \oplus \cdots \oplus A(w(m, [\sqrt{m+1}]))$. Let $\mathcal{V} = \bigcup_{m>1} \mathcal{V}_m$.

This variety is a variety of commutative or anticommutative metabelian algebras and, as proved in [26], it is possible to show that for any $n \ge 25$

$$\frac{1}{2}([\sqrt{n}\,] - 2)\frac{n(n-1)(n-5)}{6} \le c_n(\mathcal{V}) \le n^3\sqrt{n} + n^2(2n + 3\sqrt{n}) + n^2.$$

In other words, the variety $\mathcal{V}$ has fractional polynomial growth between 3 and 4, more precisely $\lim_{n\to\infty} \log_n c_n(\mathcal{V}) = \frac{7}{2}$.

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