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GL_n-invariant functions on $M_n(\mathcal{G})$

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Abstract: We describe the $GL_n(F)$-invariant functions on $M_n(\mathcal{G})$ (where $\mathcal{G}$ is the infinite dimensional Grassmann algebra) and show that not all of them are trace polynomials, if $n \geq 3$

Key words: Invariants, trace identities, matrices over Grassmann algebras

1. Introduction

This paper is dedicated to Vesselin Drensky on the occassion of his seventieth birthday. We have admired Drensky's work ever since the time we were a graduate student and discovered that Drensky had already published some of the results of our thesis (in a slightly different form), although at the time our admiration was mixed with some less generous feelings.

Throughout this paper, we will be working over a field F of characteristic zero. We will denote by $\mathcal{G}$ the infinite dimensional Grassmann algebra over F. The algebra $\mathcal{G}$ will be taken to be generated by letters e_i which anticommute and have square 0. The Grassmann algebra has a natural $\mathbb{Z}_2$-grading in which the degree zero part is spanned by products of even numbers of vectors and the degree one part is spanned by odd numbers of them.

In [3] Domokos studied two-by-two matrices over $\mathcal{G}$. For such a matrix A, let $tr(A)$ be the sum of the diagonal elements of A. Although this function does not satisfy all of the usual properties of traces it does satisfy $tr(A) = tr(gAg^{-1})$ for all $g \in GL_2(F)$. Likewise, the functions $A \mapsto tr(A^k)$ are invariant under conjugation by $GL_2(F)$, as well as all products and sums of such functions. Domokos proved that the algebra of invariant functions on $M_2(\mathcal{G})$, polynomial in the entries, is generated by these trace polynomials. He concludes the paper by mentioning that the case of $M_n(\mathcal{G})$ for $n \geq 3$ remains open. It is our intention in this paper to describe the ring of $GL_n(F)$ -invariant functions on $M_n(\mathcal{G})$, and more generally on $M_n(\mathcal{G})^k$, and to use this description to construct one which is not in the algebra of functions generated by powers of traces for $n \geq 3$.

In the next section, we describe some relevant related results in invariant theory and in the last section, we prove our theorem and construct our example.

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2. Related results

2.1. The classical case

The touchstone for all major applications of invariant theory to p.i. theory is the invariant theory of $M_n(F)$ studied by Procesi in [5]. Let $t_{ij}^{(\alpha)}$ be the functional on $M_n(F)^k$ which takes a k-tuple of matrices to the (i,j) entry of the α-th matrix in the k-tuple. Elements of the polynomial ring $F[t_{ij}^{(\alpha)}]_{i,j,\alpha}$ are then identified with functions $M_n(F)^k \to F$. These functions are said to be polynomial in the entries. Let T_α be the $n \times n$ generic matrix with entries $t_{ij}^{(\alpha)}$. There is a trace map from the algebra of generic matrices $F[T_1, \ldots, T_k] = F[T_\alpha]$ to $F[t_{ij}^{(\alpha)}]$ (from this point on we will mostly be supressing the subscripts on the right hand brackets, e.g., $F[T_\alpha]$ will mean $F[T_\alpha]_\alpha$) whose image is not a ring, but it does generate one, denoted $TR(t_{ij}^{(\alpha)})$ called the generic trace ring of $M_n(F)$. More explicitely, elements of $TR(t_{ij}^{(\alpha)})$ are linear combinations of terms of the form

$$tr(T_{i_1} \cdots T_{i_a}) \cdots tr(T_{j_1} \cdots T_{j_b}), \quad 1 \le i_1, \ldots, j_b \le k.$$

Keeping in mind the identification of elements of $F[t_{ij}^{(\alpha)}]$ with functionals on $M_n(F)^k$, it is not hard to see that elements of $TR(t_{ij}^{(\alpha)})$ are invariant under conjugation from $GL_n(F)$. Namely, if $\varphi \in TR(t_{ij}^{(\alpha)})$ is considered as a functional on $M_n(F)^k$, then $\varphi(A_1, \ldots, A_k) = \varphi(gA_1g^{-1}, \ldots, gA_kg^{-1})$ for all $A_1, \ldots, A_k \in M_n(F)$ and all $g \in GL_n(F)$. The converse is a deeper theorem, namely that $TR(t_{ij}^{(\alpha)})$ gives all invariant maps from $M_n(F)^k$ to F, polynomial in the entries.

2.2. Super traces

The infinite dimensional Grassmann algebra has a natural $\mathbb{Z}_2$-grading in which $\mathcal{G}_0$ is spanned by products of even numbers of generators and $\mathcal{G}_1$ is spanned by products of odd numbers of them. With respect to this grading $\mathcal{G}$ is supercommutative, meaning that if $a, b \in \mathcal{G}$ are homogeneous of degrees i, j, then $ab = (-1)^{ij}ba$.

The matrix algebra $M_n(\mathcal{G})$ inherits a $\mathbb{Z}_2$-grading from $\mathcal{G}$, which also permits the construction of a supertrace map. For $A \in M_n(\mathcal{G})$, let $str(A)$ be the sum of the diagonal elements. Then for all homgeneous $A, B \in M_n(\mathcal{G})$ of degrees i, j we have

$$str(AB) = (-1)^{ij} str(BA) \text{ and } str(A)str(B) = (-1)^{ij} str(B)str(A). \tag{2.1}$$

This, together with linearity is the general definition of a supertrace.

Now consider maps $M_n(\mathcal{G})^k$ to $\mathcal{G}$. Let $t_{ij}^{(\alpha)}$ be the map that takes the k-tuple $(A_1, \ldots, A_k)$ to the degree 0 part of the (i,j) entry of A_α, and let $e_{ij}^{(\alpha)}$ be the map that takes it to the degree 1 part. These maps generate a supercommutative algebra $F[t_{ij}^{(\alpha)}, e_{ij}^{(\alpha)}]$. Elements of this ring should be thought of as polynomial functions on $M_n(\mathcal{G})^k$, considered as a superalgebra; as opposed to $x_{ij}^{(\alpha)} = t_{ij}^{(\alpha)} + e_{ij}^{(\alpha)}$ which picks out the (i,j) entry of the α^{th} matrix, which is a polynomial map on $M_n(\mathcal{G})^k$ considered as an algebra only. In this section, we will be interested in superalgebraic polynomial maps invariant under conjugation by $GL_n(F)$, namely ones coming from $F[t_{ij}^{(\alpha)}, e_{ij}^{(\alpha)}]$ such that $(A_1, \ldots, A_k)$ and $(gA_1g^{-1}, \ldots, gA_kg^{-1})$ always take the same value. Again, there is a notion of supertrace polynomials and a theorem that says that they give all invariant functions $M_n(\mathcal{G}) \to \mathcal{G}$.

Analogously to the classical case, we define generic homogenous matrices $T_\alpha = (t_{ij}^{(\alpha)})$ and $E_\alpha = (e_{ij}^{(\alpha)})$. Let $F[T_\alpha, E_\alpha]$ be the algebra generated by $T_1, \dots, T_k$ and $E_1, \dots, E_k$. There is a supertrace map from $F[T_\alpha, E_\alpha]$ to $F[e_{ij}^{(\alpha)}, t_{ij}^{(\alpha)}]$. The image is not a ring, but it generates a ring $STR[t_{ij}^{(\alpha)}, e_{ij}^{(\alpha)}]$. In [1], we proved that $STR[t_{ij}^{(\alpha)}, e_{ij}^{(\alpha)}]$, identified with polynomial maps on $M_n(\mathcal{G})^k$, is precisely the set of $GL_n(F)$-invariant maps, polynomial in the components of the entries. Before stating this formally, we now turn to the general definition of supertace polynomials.

2.3. Supertrace polynomials

Let $x_1, \dots, x_k$ be even (degree 0) variables and let $y_1, \dots, y_k$ be odd (degree 1). The algebra they generate $F[x_\alpha, y_\alpha]$ is a $\mathbb{Z}_2$-graded algebra, which we now use to construct the algebra of supertrace polynomials. This is the supercommutative algebra generated by the symbols $str(u)$ for $u \in F[x_\alpha, y_\alpha]$ subject to the relations that str is linear and

$$str(uv) = (-1)^{ij} str(vu) \text{ and } str(u)str(v) = (-1)^{ij} str(v)str(u) \tag{2.2}$$

for all u, v homogeneous of degrees i and j. We denote the algebra generated by these supertraces $STR[x_\alpha, y_\alpha]$. More generally, any function on a superalgebra satisfying these properties will be said to be a supertrace. Elements of $F[x_\alpha, y_\alpha]$ are called supertrace polynomials and they satisfy the expected universal property. In particular, given $f(x_1, \dots, y_k) \in STR[x_\alpha, y_\alpha]$ we can substitute for the x_α and y_α the generic graded matrices T_α and E_α. The resulting $f(T_1, \dots, E_k)$ is an element of $F[t_{ij}^{(\alpha)}, e_{ij}^{(\alpha)}]$. Since the latter are identified with functions $M_n(\mathcal{G})^k \to \mathcal{G}$ we can state the results of the previous section in this language.

Theorem 2.1 *The space of $GL_n(F)$-invariant functions $M_n(\mathcal{G})^k \to \mathcal{G}$, polynomial in the components of the entries equals the space of evaluations of supertrace polynomials on the generic graded matrices, i.e. the space of all $f(T_1, \dots, E_k)$, where $f(x_1, \dots, y_k)$ is a supertrace polynomial.*

Before turning to the situation studied by Domokos, we will be so self-serving as to mention that in [2] we found the $GL(\mathcal{G})$ invariants of $M_n(\mathcal{G})^k$, in case the reader is interested.

3. The counterexample

3.1. Volichenko's polynomials

Let $F[t_i, e_i]$ be a free supercommuative algebra in which the t_i are commuting, degree 0 elements and the e_i are anticommuting, degree 1 elements. The subalgebra $F[t_i + e_i]$ generated by the sums $t_i + e_i$ is a universal p.i. algebra for the Grassmann algebra $\mathcal{G}$ in the sense it satisfies all of the polynomial identities of $\mathcal{G}$ and that given any elements $a_i \in \mathcal{G}$ there is a unique homomorphism $F[t_i + e_i] \to \mathcal{G}$ that takes each $t_i + e_i$ to a_i. Volichenko characterized $F[t_i + e_i]$ as a subalgebra of $F[t_i, e_i]$ using two maps from $F[t_i, e_i]$ to itself. One is π_1, the projection to the degree one component. The other is the unique superderivation δ such that $\delta(t_i) = e_i$ and $\delta(e_i) = 0$. A superderivation is an F-linear map that satisfies $\delta(ab) = \delta(a)b + (-1)^\alpha a\delta(b)$ for all a homogeneous of degree α and for all b. Note that

$$\pi_1(t_i + e_i) = \delta(t_i + e_i) = e_i.$$

Volichenko proved that this equality characterizes elements of $F[t_i + e_i]$.

Theorem 3.1 (Volichenko) *$u \in F[t_i, e_i]$ lies in the subalgebra $F[t_i + e_i]$ if and only if $\pi_1(u) = \delta(u)$.*

Volichenko's theorem can be found in [6]. A summary can also be found in Appendix 3 of [4].

This theorem will be important for us in the case that the variables come from the alphabets $\{t_{ij}^{(\alpha)}\}$ and $\{e_{ij}^{(\alpha)}\}$. These variables generate the functions $M_n(\mathcal{G})$ polynomial in the graded components of the entries, and the subalgebra generated by the sums $t_{ij}^{(\alpha)} + e_{ij}^{(\alpha)}$, with all indices the same, generate the functions polynomial in the entries themselves, so it is important to have a critereon to tell when an element of $F[t_{ij}^{(\alpha)}, e_{ij}^{(\alpha)}]$ lies in $F[t_{ij}^{(\alpha)} + e_{ij}^{(\alpha)}]$.

The key tool is to generalize Volichenko's functions to the generic super algebras $F[x_\alpha, y_\alpha]$ and $STR[x_\alpha, y_\alpha]$. Since each is $\mathbb{Z}_2$-graded there is no problem in defining π_1 to be the projection onto the odd components. On $F[x_\alpha, y_\alpha]$, we can define δ to be the superderivation that takes each x_α to y_α and each y_α to zero. To extend this map to the supertrace algebra, we use $\delta(str(u)) = str(\delta(u))$. That they are well defined on $STR(x_\alpha, y_\alpha)$ follows from the definition of a superderivation and the relations (2.2). We let the interested reader have the fun of checking this. We also will need the fact that $\delta^2 = 0$ which we now prove.

Lemma 3.2 *The superderivation δ has square zero on both $F[x_\alpha, y_\alpha]$ and $STR(x_\alpha, y_\alpha)$.*

Proof

By linearity, it suffices to prove that δ^2 is zero on monomials in $F[x_\alpha, y_\alpha]$ and supertrace monomials in $STR(x_\alpha, y_\alpha)$, and we use induction on the degree, the case of degree one being trivial. First consider a monomial in $F[x_\alpha, y_\alpha]$. If the degree in every x_α is zero, then δ automatically sends it to zero and so we can dismiss this case. Let ux_iv be a monomial in which $u \in F[y_\alpha]$ has degree a. Then

$$\delta(\delta(ux_iv)) = \delta((-1)^a uy_iv) + \delta((-1)^a ux_i\delta(v))$$
$$= (-1)^\alpha(-1)^{a+1}uy_i\delta(v) + (-1)^a(-1)^a uy_i\delta(v) + (-1)^a(-1)^a ux_i\delta^2(v)$$

The first two terms cancel and the third will be zero by the induction hypothesis.

For the $STR(x_\alpha, y_\alpha)$, again δ will vanish on monomials unless they have at least one element x_α, and to save notation we will take $\alpha = 1$ and take the monomial to be $str(x_1u)v$ where u is a monomial in $F[x_\alpha, y_\alpha]$ of degree a in the odd variables, and v is a monomial in $STR(x_\alpha, y_\alpha)$. We first compute

$$\delta(str(x_1u)v) = str(y_1u)v + str(x_1\delta(u))v + (-1)^a str(x_1u)\delta(v),$$

and then we compute δ of each of the three terms on the right. The first term is

$$\delta(str(y_1u)v) = -str(y_1\delta(u))v + (-1)^{a+1}str(y_1u)\delta(v).$$

The second term is

$$\delta(str(x_1\delta(u))v) = str(y_1\delta(u))v + str(x_1\delta^2(u))v + (-1)^{a+1}str(x_1\delta(u))\delta(v)$$
$$= str(y_1\delta(u))v + (-1)^{a+1}str(x_1\delta(u))\delta(v)$$

And the third term is $(-1)^a\delta(str(x_1u)\delta(v))$ which equals

$$(-1)^a str(y_1u)\delta(v) + (-1)^a str(x_1\delta(u))\delta(v) + (-1)^a(-1)^a str(x_1u)\delta^2(v)$$
$$= (-1)^a str(y_1u)\delta(v) + (-1)^a str(x_1\delta(u))\delta(v)$$

It is now easy to see that all the terms cancel. □

We let $T_1,\dots,T_k,E_1,\dots,E_k$ be generic $n\times n$ matrices, the former in $t_{ij}^{(\alpha)}$ and the latter in $e_{ij}^{(\alpha)}$, as in the previous section. Also let $STR(T_\alpha,E_\alpha)\subset F[t_{ij}^{(\alpha)},e_{ij}^{(\alpha)}]$ the space of all specializations of elements of $STR(x_\alpha,y_\alpha)$ via $x_\alpha\mapsto T_\alpha$ and $y_\alpha\mapsto E_\alpha$. The following two diagrams commute

$$\begin{array}{ccc} STR(x_\alpha,y_\alpha) & \xrightarrow{\delta,\pi_1} & STR(x_\alpha,y_\alpha) \\ \downarrow & & \downarrow \\ F[t_{ij}^{(\alpha)},e_{ij}^{(\alpha)}] & \xrightarrow{\delta,\pi_1} & F[t_{ij}^{(\alpha)},e_{ij}^{(\alpha)}] \end{array}$$

Honoring Volichenko, we define $V[x_\alpha,y_\alpha]$, and $V[t_{ij}^{(\alpha)},e_{ij}^{(\alpha)}]$ to be the elements in $STR[x_\alpha,y_\alpha]$ and $F[t_{ij}^{(\alpha)},e_{ij}^{(\alpha)}]$, respectively, for which $\pi_1(u)=\delta(u)$ and call them Volichenko polynomials and supertrace Volichenko polynomials. Because of the commuting diagram, if $f(x_1,\dots,y_k)$ is in $V(x_\alpha,y_\alpha)$, then the specialization $f(T_1,\dots,E_k)$ will be in $V(t_{ij}^{(\alpha)},e_{ij}^{(\alpha)})$. Here is a partial converse.

Lemma 3.3 *Let $f(x_1,\dots,x_k,y_1,\dots,y_k)\in STR(x_\alpha,y_\alpha)$ be such that the specialization $f(T_1,\dots,E_k)$ is an element of $V(t_{ij}^{(\alpha)},e_{ij}^{(\alpha)})$. Then f equals $g(T_1,\dots,E_k)$ for some $g(x_1,\dots,y_k)$ in $V(x_\alpha,y_\alpha)$*

Proof Setting $\bar f := f(T_1,\dots,E_k)$, we have $\bar f=\bar f_0+\bar f_1$, and

$$\bar f_1=\pi(\bar f)=\delta(\bar f)=\delta(\bar f_0)+\delta(\bar f_1).$$

It follows that

$$\bar f_1=\delta(\bar f_0),$$

since $\delta(\bar f_1)$ is even or zero. Take now $g:=f_0+\delta(f_0)$. Clearly $g_0=f_0$ and $g_1=\delta(f_0)=\delta(g_0)$, implying by Lemma 3 that

$$\pi(g)=g_1=\delta(g_0)+\delta^2(g_0)=\delta(g_0+\delta(g_0))=\delta(g).$$

So g is a Volicenko element, and

$$\begin{aligned} g(T_1,\dots,E_k) &= (f_0+\delta(f_0))(T_1,\dots,E_k)\\ &= f_0(T_1,\dots,E_k)+\delta(f_0(T_1,\dots,E_k))\\ &= \bar f_0+\delta(\bar f_0)\\ &= \bar f_0+\bar f_1 \end{aligned}$$

which equals $f(T_1,\dots,E_k)$. □

Combining Theorem 2.1 with Lemma 3.3, we can identify the $GL_n(F)$-invariant polynomial functions on $M_n(\mathcal{G})^k$. The former says that all invariant functions $M_n(\mathcal{G})^k \to \mathcal{G}$ which are polynomial in the homogeneous components of the entries come from supertrace polynomials, $STR(x_\alpha, y_\alpha)$. This, of course, includes functions polynomial in the entries themselves. And Lemma 3.3 identifies which elements in $STR(t_{ij}^{(\alpha)}, e_{ij}^{(\alpha)})$ are such, namely, specializations of Volichenko polynomials. Combining, we get this description of the invariant functions on $M_n(\mathcal{G})$.

Theorem 3.4 *The $GL_n(F)$-invariant functions on $M_n(\mathcal{G})^k$, polynomial in the entries, are the polynomials of the form $f(T_1, \dots, E_k)$ where $f \in V(x_\alpha, y_\alpha)$ is a supertrace Volichenko polynomial.*

Proof By Theorem 2.1 an invariant function satisfying the weaker condition that it is polynomial in the components of the entries must be of the form $f(T_1, \dots, E_k)$, for some supertrace polynomial $f \in STR(x_{ij}^{(\alpha)}, y_{ij}^{(\alpha)})$. By the previous lemma such a function will be polynomial in the entries themselves precisely when f is a Volichenko polynomial. □

3.2. The counterexample

Using Theorem 3.4, we can construct a $GL_n(F)$-invariant function $M_n(\mathcal{G}) \to \mathcal{G}$, polynomial in the entries, but not in the trace ring, for all $n \geq 3$. Let $\Phi : M_n(\mathcal{G}) \to \mathcal{G}$ be the function $str(T_1E_1^2 + E_1^3)$. Since $\delta(T_1) = E_1$ and $\delta(E_1) = 0$ it follows that $\delta(str(T_1E_1^2 + E_1^3)) = 0$ and so by Lemma 3.3, Φ is a polynomial function in the entries. Alternately, if $A = (a_{ij}) \in M_n(\mathcal{G})$ the reader can verify that $\Phi(A) = \frac{1}{2}\sum_{i,j,k} a_{ij}[a_{jk}, a_{ki}]$ proving directly that Φ is polynomial in the entries. Finally, we prove that Φ is not a trace map.

In [3], Domokos studied the function $M_n(\mathcal{G}) \to \mathcal{G}$ which sends a matrix to the sum of its diagonal matrix. In keeping with that paper, we will call this map a trace and denote it tr although it does not satisfy $tr(ab) = tr(ba)$. It might be of interest to investigate the general properties of this map and put it in a more general context, but that is not our concern here.

Theorem 3.5 *The map $\Phi : M_n(\mathcal{G}) \to \mathcal{G}$ is polynomial in the entries, invariant under conjugation by $GL_n(F)$ but is not a trace map.*

Proof It remains only to show that Φ is not a trace map. Since it is degree 3, if it were a trace map we would have

$$\Phi(x) = Atr(x)^3 + Btr(x^2)tr(x) + Ctr(x)tr(x^2) + Dtr(x^3).$$

Taking x of degree 0, we get $\Phi(x) = 0$ and so the left hand side of the equation is a trace identity for $M_n(F)$. It follows from the Razmyslov-Procesi theory that there is no such nontrivial identity. On a more elementary level, taking $x = e_{12} + e_{23} + e_{31}$ we have $tr(x)^3 = 3$ but $tr(x) = tr(x^2) = 0$ and so $D = 0$. Also, taking x to be a matrix with nonzero trace whose square has trace 0 will give $A = 0$. One such example would be $e_{11} + 3e_{22} + 5e_{23} - e_{32}$. And, taking $x = e_{11}$ gives $B = -C$ implying that $\Phi(x) = B(tr(x^2)tr(x) - tr(x)tr(x^2))$. On the other hand, take $x = \alpha e_{12} + \beta e_{23} + \gamma e_{31}$, where α, β, γ are degree one elements of $\mathcal{G}$ with nonzero product. Then $\Phi(x) = tr(x^3) = 3\alpha\beta\gamma \neq 0$ whereas $tr(x^2)tr(x) - tr(x)tr(x^2)$ equals 0. □

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